This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.
+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.
+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.
+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

PROPERTY OF THE
University of Michigan Libraries
1817
ARTES SCIENTIA VERITAS

Digitized by Google

Digitized by Google

Digitized by Google

Q
41
.C183

Digitized by Google

Digitized by Google

Q
41
.C183

Digitized by Google

Digitized by Google

TRANSACTIONS

OF THE

CAMBRIDGE

PHILOSOPHICAL SOCIETY.

VOLUME XIV.

CAMBRIDGE:
Printed at the University Press;
AND SOLD BY
DEIGHTON, BELL AND CO. AND MACMILLAN AND BOWES, CAMBRIDGE;
BELL AND SONS, LONDON.

M.DCCC.LXXXIX.

Digitized by Google

Cambridge:
PRINTED BY C. J. CLAY, M.A. & SONS,
AT THE UNIVERSITY PRESS.

Digitized by Google

CONTENTS.

PAGE

I. *On some General Equations which include the Equations of Hydrodynamics. By* M. J. M. HILL, M.A., Professor of Mathematics at the Mason Science College, Birmingham 1

II. *On the measurement of Temperature by Water-vapour pressure. By* W. N. SHAW, M.A., Emmanuel College (Plate I.) 30

III. *On the pulsations of spheres in an elastic medium. By* A. H. LEAHY, M.A. 45

IV. *On the curves of constant intensity of homogeneous polarized light seen in a uniaxal crystal cut at right angles to the optic axis. By* C. SPURGE, B.A. (Plate II.)... 63

V. *On a certain Atomic Hypothesis. By* PROFESSOR KARL PEARSON, M.A. Communicated by H. T. STEARN, M.A. 71

VI. *Some Applications of Generalized Space-Coordinates to Differential Analysis:—Potentials and Isotropic Elasticity. By* J. LARMOR, M.A., St John's College 121

VII. *Observations and Statistics. An Essay on the Theory of Errors of Observation and the First Principles of Statistics. By* F. Y. EDGEWORTH, M.A. Communicated by J. W. L. GLAISHER, M.A. 138

VIII. *On a new method of obtaining interference-fringes, and on their application to determine whether a displacement-current of electricity involves a motion of translation of the electromagnetic medium. By* L. R. WILBERFORCE, B.A., Scholar of Trinity College, Cambridge. (Plates III. IV. V.) 170

IX. *On the mutual action of oscillatory twists in an elastic medium, as applied to a vibratory theory of electricity. By* A. H. LEAHY, M.A., Pembroke College 188

X. *On a class of Spherical Harmonics of complex degree with application to physical problems. By* E. W. HOBSON, M.A., Fellow of Christ's College, Cambridge 211

XI. *Table of the Exponential Function e^x to twelve places of Decimals. By* F. W. NEWMAN, Emeritus Professor of University College, London 237

XII. *The Equations of an Isotropic Elastic Solid in Polar and Cylindrical Co-ordinates, their Solution and Application. By* C. CHREE, M.A., Fellow of King's College, Cambridge 250

XIII. *On Solution and Crystallization. By* G. D. LIVEING, M.A., Professor of Chemistry in the University of Cambridge 370

XIV. *On Solution and Crystallization.* No. II. *By* G. D. LIVEING, M.A., Professor of Chemistry in the University of Cambridge 394

XV. *Systems of Quaternariants that are algebraically complete. By* A. R. FORSYTH, M.A., F.R.S., Fellow of Trinity College, Cambridge 409

XVI. *On the stresses in rotating spherical shells. By* C. CHREE, M.A., Fellow of King's College, Cambridge 467

XVII. *On the Binodal Quartic and the Graphical Representation of the Elliptic Functions. By* PROF. CAYLEY 484

Digitized by Google

ADVERTISEMENT.

The Society as a body is not to be considered responsible for any facts and opinions advanced in the several Papers, which must rest entirely on the credit of their respective Authors.

The Society takes this opportunity of expressing its grateful acknowledgments to the Syndics of the University Press for their liberality in taking upon themselves the expense of printing this Volume of the Transactions.

Digitized by Google

I. *On some General Equations which include the Equations of Hydrodynamics.* By M. J. M. Hill, M.A., *Professor of Mathematics at the Mason Science College, Birmingham.*

[Read *October* 29, 1883.]

In a paper written by Clebsch, "Ueber die Integration der hydrodynamischen Gleichungen" (*Crelle*, Bd. LVI.), he has shown that if the symbol ρ of equation (II) of this paper be constant, and if n be an odd number, then $u_1, u_2 \ldots u_n$ can be expressed in the form of Art. 2, where however $\frac{n+1}{2}$ of the symbols $f_1, f_2 \ldots f_n$ vanish.

The object of this paper is to obtain results which apply to the case of n even as well as n odd, to remove the restriction regarding ρ; and also to enquire what equations correspond in space of n dimensions to Helmholtz's Equations of Fluid Motion, and what corresponds to a vortex line.

The following is an abstract:

1. Let $x_1, x_2 \ldots x_n t$ be $(n+1)$ independent variables,

$u_1, u_2 \ldots u_n \rho$ be $(n+1)$ dependent variables,

and let q stand for $V + \int \frac{dp}{\rho}$, where ρ is a known function of p, and V is a known function of the independent variables.

Let the following equations be satisfied

$$\left.\begin{array}{l} \frac{du_1}{dt} + u_1 \frac{du_1}{dx_1} + u_2 \frac{du_1}{dx_2} + \ldots + u_n \frac{du_1}{dx_n} = -\frac{dq}{dx_1} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{du_n}{dt} + u_1 \frac{du_n}{dx_1} + u_2 \frac{du_n}{dx_2} + \ldots + u_n \frac{du_n}{dx_n} = -\frac{dq}{dx_n} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\text{(I)},$$

$$\frac{d\rho}{dt} + \frac{d}{dx_1}(\rho u_1) + \frac{d}{dx_2}(\rho u_2) + \ldots + \frac{d}{dx_n}(\rho u_n) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\text{(II)},$$

$u_1, u_2 \ldots u_n$ being symbols for any definite functions of the independent variables satisfying (I) and (II), let their values be substituted in the equation

$$\frac{df}{dt} + u_1 \frac{df}{dx_1} + u_2 \frac{df}{dx_2} + \ldots\ldots + u_n \frac{df}{dx_n} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\text{(III)}.$$

Digitized by Google

Let $P_1, P_2 \ldots P_n$ be n independent integrals of this equation, definitely selected. Then if A stand for the determinant

$$\begin{vmatrix} \frac{dP_1}{dx_1} & \frac{dP_1}{dx_2} & \ldots\ldots & \frac{dP_1}{dx_n} \\ \frac{dP_2}{dx_1} & \frac{dP_2}{dx_2} & \ldots\ldots & \frac{dP_2}{dx_n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \frac{dP_n}{dx_1} & \frac{dP_n}{dx_2} & \ldots\ldots & \frac{dP_n}{dx_n} \end{vmatrix}$$

it can be shown by means of (II) and (III) that

$$\left(\frac{d}{dt} + u_1 \frac{d}{dx_1} + u_2 \frac{d}{dx_2} + \ldots\ldots + u_n \frac{d}{dx_n}\right)\left(\frac{A}{\rho}\right) = 0.$$

2. It is shown that if K be an arbitrary function of the independent variables, and $f_1, f_2 \ldots f_n$ arbitrary functions of $P_1, P_2 \ldots P_n$ (not t), then $u_1, u_2 \ldots u_n$ can always be expressed thus:—

$$u_1 = \frac{dK}{dx_1} + f_1 \frac{dP_1}{dx_1} + f_2 \frac{dP_2}{dx_1} + \ldots\ldots + f_n \frac{dP_n}{dx_1},$$

$$u_2 = \frac{dK}{dx_2} + f_1 \frac{dP_1}{dx_2} + f_2 \frac{dP_2}{dx_2} + \ldots\ldots + f_n \frac{dP_n}{dx_2},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$u_n = \frac{dK}{dx_n} + f_1 \frac{dP_1}{dx_n} + f_2 \frac{dP_2}{dx_n} + \ldots\ldots + f_n \frac{dP_n}{dx_n}.$$

3. It is shown that if n be an odd integer, if $\xi_{rs} = \frac{du_r}{dx_s} - \frac{du_s}{dx_r}$, and if $\xi_1, \xi_2 \ldots \xi_n$ be the square roots of the coefficients of $\xi_{11}, \xi_{22} \ldots \xi_{nn}$ in the determinant Ξ which is equal to

$$\begin{vmatrix} \xi_{11}, & \xi_{12} \ldots\ldots & \xi_{1n} \\ \xi_{21}, & \xi_{22} \ldots\ldots & \xi_{2n} \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \xi_{n1}, & \xi_{n2} \ldots\ldots & \xi_{nn} \end{vmatrix}$$

the signs of the roots being properly chosen, then

$$\frac{\delta}{\delta t}\left(\frac{\xi_1}{\rho}\right) = \frac{\xi_1}{\rho}\frac{du_1}{dx_1} + \frac{\xi_2}{\rho}\frac{du_1}{dx_2} + \ldots + \frac{\xi_n}{\rho}\frac{du_1}{dx_n} = \frac{\xi_1}{\rho}\frac{du_1}{dx_1} + \frac{\xi_2}{\rho}\frac{du_2}{dx_1} + \ldots + \frac{\xi_n}{\rho}\frac{du_n}{dx_1},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{\delta}{\delta t}\left(\frac{\xi_n}{\rho}\right) = \frac{\xi_1}{\rho}\frac{du_n}{dx_1} + \frac{\xi_2}{\rho}\frac{du_n}{dx_2} + \ldots + \frac{\xi_n}{\rho}\frac{du_n}{dx_n} = \frac{\xi_1}{\rho}\frac{du_1}{dx_n} + \frac{\xi_2}{\rho}\frac{du_2}{dx_n} + \ldots + \frac{\xi_n}{\rho}\frac{du_n}{dx_n},$$

$$\frac{d\xi_1}{dx_1} + \frac{d\xi_2}{dx_2} + \ldots + \frac{d\xi_n}{dx_n} = 0.$$

But when n is even, $\frac{\delta}{\delta t}\left\{\frac{\sqrt{\Xi}}{\rho}\right\} = 0$; and this is the sole representative of Helmholtz's Equations in this case.

Digitized by Google

4. In this Article it is shown that if α_{rs} be the coefficient of $\frac{dP_r}{dx_s}$ in the determinant A, and if $\lambda_1, \lambda_2 \dots \lambda_n$ are functions of $P_1, P_2 \dots P_n$ (not t), subject to the condition $\frac{\delta\lambda_1}{\delta P_1} + \frac{\delta\lambda_2}{\delta P_2} + \dots + \frac{\delta\lambda_n}{\delta P_n} = 0$, the following set of equations similar to Helmholtz's Equations hold good, whether n be even or odd.

$$\frac{\delta}{\delta t}\left(\frac{\lambda_1\alpha_{11} + \lambda_2\alpha_{21} + \dots + \lambda_n\alpha_{n1}}{\rho}\right) = \frac{\lambda_1\alpha_{11} + \lambda_2\alpha_{21} + \dots + \lambda_n\alpha_{n1}}{\rho}\frac{du_1}{dx_1} + \dots + \frac{\lambda_1\alpha_{1n} + \lambda_2\alpha_{2n} + \dots + \lambda_n\alpha_{nn}}{\rho}\frac{du_1}{dx_n},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\frac{\delta}{\delta t}\left(\frac{\lambda_1\alpha_{1s} + \lambda_2\alpha_{2s} + \dots + \lambda_n\alpha_{ns}}{\rho}\right) = \frac{\lambda_1\alpha_{11} + \lambda_2\alpha_{21} + \dots + \lambda_n\alpha_{n1}}{\rho}\frac{du_s}{dx_1} + \dots + \frac{\lambda_1\alpha_{1n} + \lambda_2\alpha_{2n} + \dots + \lambda_n\alpha_{nn}}{\rho}\frac{du_s}{dx_n},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\frac{\delta}{\delta t}\left(\frac{\lambda_1\alpha_{1n} + \lambda_2\alpha_{2n} + \dots + \lambda_n\alpha_{nn}}{\rho}\right) = \frac{\lambda_1\alpha_{11} + \lambda_2\alpha_{21} + \dots + \lambda_n\alpha_{n1}}{\rho}\frac{du_n}{dx_1} + \dots + \frac{\lambda_1\alpha_{1n} + \lambda_2\alpha_{2n} + \dots + \lambda_n\alpha_{nn}}{\rho}\frac{du_n}{dx_n},$$

$$\frac{d}{dx_1}(\lambda_1\alpha_{11} + \lambda_2\alpha_{21} + \dots + \lambda_n\alpha_{n1}) + \dots + \frac{d}{dx_s}(\lambda_1\alpha_{1s} + \lambda_2\alpha_{2s} + \dots + \lambda_n\alpha_{ns}) + \dots$$
$$+ \frac{d}{dx_n}(\lambda_1\alpha_{1n} + \lambda_2\alpha_{2n} + \dots + \lambda_n\alpha_{nn}) = 0.$$

5. It is shown that if n be odd and $\lambda_1, \lambda_2 \dots \lambda_n$ be the square roots (with properly chosen signs) of the coefficients of $\lambda_{11}, \lambda_{22} \dots \lambda_{nn}$ in the determinant Λ which is equal to

$$\dots \begin{vmatrix} \lambda_{11}, \lambda_{12} \dots\dots \lambda_{1n} \\ \lambda_{21}, \lambda_{22} \dots\dots \lambda_{2n} \\ \dots\dots\dots\dots \\ \dots\dots\dots\dots \\ \dots\dots\dots\dots \\ \lambda_{n1}, \lambda_{n2} \dots\dots \lambda_{nn} \end{vmatrix}$$

where
$$\lambda_{rs} = \frac{\delta f_r}{\delta P_s} - \frac{\delta f_s}{\delta P_r};$$

then $\xi_1 = \lambda_1\alpha_{11} + \lambda_2\alpha_{21} + \dots + \lambda_n\alpha_{n1}$, $\xi_2 = \lambda_1\alpha_{12} + \lambda_2\alpha_{22} + \dots + \lambda_n\alpha_{n2}, \dots \xi_n = \lambda_1\alpha_{1n} + \lambda_2\alpha_{2n} + \dots + \lambda_n\alpha_{nn}$.

In the case when n is even, it is shown that $\sqrt{\Xi} = A \,.\, \sqrt{\Lambda}$.

6. Adopting the language of ordinary Fluid Motion, it is shown that the vortex lines

$$\frac{dx_1}{\xi_1} = \frac{dx_2}{\xi_2} = \dots = \frac{dx_n}{\xi_n},$$

always contain the same particles, being the intersections of $(n-1)$ loci whose equations each satisfy the equation (II).

The same argument shows that the loci

$$\frac{dx_1}{\lambda_1\alpha_{11} + \lambda_2\alpha_{21} + \dots + \lambda_n\alpha_{n1}} = \frac{dx_2}{\lambda_1\alpha_{12} + \lambda_2\alpha_{22} + \dots + \lambda_n\alpha_{n2}} = \dots = \frac{dx_n}{\lambda_1\alpha_{1n} + \lambda_2\alpha_{2n} + \dots + \lambda_n\alpha_{nn}},$$

are always the intersections of $(n-1)$ loci whose equations satisfy the equation (II).

Digitized by Google

7. It is shown that (n being an odd number) the forms for $u_1, u_2 \dots u_n$ can be reduced to Clebsch's forms, so that

$$u_1 dx_1 + \dots + u_n dx_n = dK + P_{\frac{n+1}{2}} dP_1 + P_{\frac{n+3}{2}} dP_2 + \dots\dots + P_{n-1} dP_{\frac{n-1}{2}}$$

and that the lines $$\frac{dx_1}{\xi_1} = \frac{dx_2}{\xi_2} = \dots = \frac{dx_n}{\xi_n},$$

lie on the loci $P_1 = \text{constant}$, $P_2 = \text{constant}, \dots P_{\frac{n-1}{2}} = \text{constant}$, $P_{\frac{n+1}{2}} = \text{constant}, \dots P_{n-1} = \text{constant}$; the equations of which satisfy the equation (III).

---

1. Taking the notation in the abstract of this Article, let the reciprocal determinant of A be the determinant

$$\begin{vmatrix} \alpha_{11}, & \alpha_{12} \dots\dots & \alpha_{1n} \\ \alpha_{21}, & \alpha_{22} \dots\dots & \alpha_{2n} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \alpha_{n1}, & \alpha_{n2} \dots\dots & \alpha_{nn} \end{vmatrix}.$$

Putting for brevity $\frac{D}{Dt}$ for $\frac{d}{dt} + u_1 \frac{d}{dx_1} + u_2 \frac{d}{dx_2} + \dots + u_n \frac{d}{dx_n}$, it follows that

$$\begin{aligned} \frac{DA}{Dt} = {} & \alpha_{11} \frac{D}{Dt}\left(\frac{dP_1}{dx_1}\right) + \alpha_{12} \frac{D}{Dt}\left(\frac{dP_1}{dx_2}\right) + \dots + \alpha_{1n} \frac{D}{Dt}\left(\frac{dP_1}{dx_n}\right) \\ & + \alpha_{21} \frac{D}{Dt}\left(\frac{dP_2}{dx_1}\right) + \alpha_{22} \frac{D}{Dt}\left(\frac{dP_2}{dx_2}\right) + \dots + \alpha_{2n} \frac{D}{Dt}\left(\frac{dP_2}{dx_n}\right) \\ & + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & + \alpha_{n1} \frac{D}{Dt}\left(\frac{dP_n}{dx_1}\right) + \alpha_{n2} \frac{D}{Dt}\left(\frac{dP_n}{dx_2}\right) + \dots + \alpha_{nn} \frac{D}{Dt}\left(\frac{dP_n}{dx_n}\right). \end{aligned}$$

Now $$\frac{D}{Dt}\left(\frac{dP_r}{dx_s}\right) = \frac{d^2 P_r}{dt\, dx_s} + u_1 \frac{d^2 P_r}{dx_1 dx_s} + u_2 \frac{d^2 P_r}{dx_2 dx_s} + \dots + u_n \frac{d^2 P_r}{dx_n dx_s}.$$

But since P_r satisfies (III),

$$\frac{dP_r}{dt} + u_1 \frac{dP_r}{dx_1} + u_2 \frac{dP_r}{dx_2} + \dots\dots + u_n \frac{dP_r}{dx_n} = 0.$$

Differentiating this with regard to x_s,

$$\frac{d^2 P_r}{dt\, dx_s} + \frac{du_1}{dx_s} \frac{dP_r}{dx_1} + u_1 \frac{d^2 P_r}{dx_1 dx_s} + \frac{du_2}{dx_s} \frac{dP_r}{dx_2} + u_2 \frac{d^2 P_r}{dx_2 dx_s} + \dots\dots + \frac{du_n}{dx_s} \frac{dP_r}{dx_n} + u_n \frac{d^2 P_r}{dx_n dx_s} = 0;$$

$$\therefore -\frac{D}{Dt}\left(\frac{dP_r}{dx_s}\right) = \frac{du_1}{dx_s} \cdot \frac{dP_r}{dx_1} + \frac{du_2}{dx_s} \frac{dP_r}{dx_2} + \dots\dots + \frac{du_n}{dx_s} \cdot \frac{dP_r}{dx_n};$$

Digitized by Google

$$\therefore -\frac{DA}{Dt} = \left\{\alpha_{11}\left(\frac{du_1}{dx_1}\frac{dP_1}{dx_1} + \frac{du_2}{dx_1}\frac{dP_1}{dx_2} + \ldots\ldots + \frac{du_n}{dx_1}\frac{dP_1}{dx_n}\right)\right.$$

$$+ \alpha_{12}\left(\frac{du_1}{dx_2}\frac{dP_1}{dx_1} + \frac{du_2}{dx_2}\frac{dP_1}{dx_2} + \ldots\ldots + \frac{du_n}{dx_2}\frac{dP_1}{dx_n}\right)$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\left. + \alpha_{1n}\left(\frac{du_1}{dx_n}\frac{dP_1}{dx_1} + \frac{du_2}{dx_n}\frac{dP_1}{dx_2} + \ldots\ldots + \frac{du_n}{dx_n}\frac{dP_1}{dx_n}\right)\right\}$$

$$+ \left\{\alpha_{21}\left(\frac{du_1}{dx_1}\frac{dP_2}{dx_1} + \frac{du_2}{dx_1}\frac{dP_2}{dx_2} + \ldots\ldots + \frac{du_n}{dx_1}\frac{dP_2}{dx_n}\right)\right.$$

$$+ \alpha_{22}\left(\frac{du_1}{dx_2}\frac{dP_2}{dx_1} + \frac{du_2}{dx_2}\frac{dP_2}{dx_2} + \ldots\ldots + \frac{du_n}{dx_2}\frac{dP_2}{dx_n}\right)$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\left. + \alpha_{2n}\left(\frac{du_1}{dx_n}\frac{dP_2}{dx_1} + \frac{du_2}{dx_n}\frac{dP_2}{dx_2} + \ldots\ldots + \frac{du_n}{dx_n}\frac{dP_2}{dx_n}\right)\right\}$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ \left\{\alpha_{n1}\left(\frac{du_1}{dx_1}\frac{dP_n}{dx_1} + \frac{du_2}{dx_1}\frac{dP_n}{dx_2} + \ldots\ldots + \frac{du_n}{dx_1}\frac{dP_n}{dx_n}\right)\right.$$

$$+ \alpha_{n2}\left(\frac{du_1}{dx_2}\frac{dP_n}{dx_1} + \frac{du_2}{dx_2}\frac{dP_n}{dx_2} + \ldots\ldots + \frac{du_n}{dx_2}\frac{dP_n}{dx_n}\right)$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\left. + \alpha_{nn}\left(\frac{du_1}{dx_n}\frac{dP_n}{dx_1} + \frac{du_2}{dx_n}\frac{dP_n}{dx_2} + \ldots\ldots + \frac{du_n}{dx_n}\frac{dP_n}{dx_n}\right)\right\};$$

$$\therefore -\frac{DA}{Dt} = \frac{du_1}{dx_1}\,.\,A + \frac{du_2}{dx_1}\,.\,0 + \ldots\ldots + \frac{du_n}{dx_1}\,.\,0$$

$$+ \frac{du_1}{dx_2}\,.\,0 + \frac{du_2}{dx_2}\,.\,A + \ldots\ldots + \frac{du_n}{dx_2}\,.\,0$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ \frac{du_1}{dx_n}\,.\,0 + \frac{du_2}{dx_n}\,.\,0 + \ldots\ldots + \frac{du_n}{dx_n}\,.\,A;$$

$$\therefore -\frac{DA}{Dt} = A\left(\frac{du_1}{dx_1} + \frac{du_2}{dx_2} + \ldots\ldots + \frac{du_n}{dx_n}\right).$$

But equation (II) in the abstract may be written

$$\frac{D\rho}{Dt} + \rho\left(\frac{du_1}{dx_1} + \frac{du_2}{dx_2} + \ldots\ldots + \frac{du_n}{dx_n}\right) = 0.$$

Hence

$$\frac{D}{Dt}\left(\frac{A}{\rho}\right) = 0.$$

Digitized by Google

It will be shown in the next article that this equation may be written

$$\frac{\delta}{\delta t}\left(\frac{A}{\rho}\right) = 0 \quad \text{.....} \quad \text{(IV)}.$$

2. Now suppose that instead of $x_1, x_2 \ldots x_n t$ being the independent variables $P_1, P_2 \ldots P_n t$ are taken as such. Let differential coefficients according to the new system be distinguished by the symbol δ.

$$\text{Then } \frac{d}{dt} = \frac{dP_1}{dt} \cdot \frac{\delta}{\delta P_1} + \frac{dP_2}{dt}\frac{\delta}{\delta P_2} + \ldots\ldots + \frac{dP_n}{dt}\frac{\delta}{\delta P_n} + \frac{\delta}{\delta t}.$$

But since P_r satisfies (III)

$$\frac{dP_r}{dt} = -u_1\frac{dP_r}{dx_1} - u_2\frac{dP_r}{dx_2} - \ldots\ldots - u_n\frac{dP_r}{dx_n};$$

$$\begin{aligned}
\therefore \frac{\delta}{\delta t} &= \frac{d}{dt} + \left(u_1\frac{dP_1}{dx_1} + u_2\frac{dP_1}{dx_2} + \ldots\ldots + u_n\frac{dP_1}{dx_n}\right)\frac{\delta}{\delta P_1} \\
&\quad + \left(u_1\frac{dP_2}{dx_1} + u_2\frac{dP_2}{dx_2} + \ldots\ldots + u_n\frac{dP_2}{dx_n}\right)\frac{\delta}{\delta P_2} \\
&\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\quad + \left(u_1\frac{dP_n}{dx_1} + u_2\frac{dP_n}{dx_2} + \ldots\ldots + u_n\frac{dP_n}{dx_n}\right)\frac{\delta}{\delta P_n} \\
&= \frac{d}{dt} + u_1\left(\frac{dP_1}{dx_1}\frac{\delta}{\delta P_1} + \frac{dP_2}{dx_1}\frac{\delta}{\delta P_2} + \ldots\ldots + \frac{dP_n}{dx_1}\frac{\delta}{\delta P_n}\right) \\
&\quad + u_2\left(\frac{dP_1}{dx_2}\frac{\delta}{\delta P_1} + \frac{dP_2}{dx_2}\frac{\delta}{\delta P_2} + \ldots\ldots + \frac{dP_n}{dx_2}\frac{\delta}{\delta P_n}\right) \\
&\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\quad + u_n\left(\frac{dP_1}{dx_n}\frac{\delta}{\delta P_1} + \frac{dP_2}{dx_n}\frac{\delta}{\delta P_2} + \ldots\ldots + \frac{dP_n}{dx_n}\frac{\delta}{\delta P_n}\right) \\
&= \frac{d}{dt} + u_1\frac{d}{dx_1} + u_2\frac{d}{dx_2} + \ldots\ldots + u_n\frac{d}{dx_n};
\end{aligned}$$

$\therefore$ what has been denoted by $\frac{D}{Dt}$ may now be written $\frac{\delta}{\delta t}$.

Also equations (I) of the abstract may be written

$$\frac{\delta u_1}{\delta t} = -\frac{dq}{dx_1} = -\left(\frac{dP_1}{dx_1}\frac{\delta q}{\delta P_1} + \frac{dP_2}{dx_1}\frac{\delta q}{\delta P_2} + \ldots + \frac{dP_n}{dx_1}\frac{\delta q}{\delta P_n}\right),$$

$$\frac{\delta u_2}{\delta t} = -\frac{dq}{dx_2} = -\left(\frac{dP_1}{dx_2}\frac{\delta q}{\delta P_1} + \frac{dP_2}{dx_2}\frac{\delta q}{\delta P_2} + \ldots + \frac{dP_n}{dx_2}\frac{\delta q}{\delta P_n}\right),$$

.....

Digitized by Google

$$\frac{\delta u_n}{\delta t} = -\frac{dq}{dx_n} = -\left(\frac{dP_1}{dx_n}\frac{\delta q}{\delta P_1} + \frac{dP_2}{dx_n}\frac{\delta q}{\delta P_2} + \dots + \frac{dP_n}{dx_n}\frac{\delta q}{\delta P_n}\right);$$

$$\left.\begin{aligned} \therefore\ \alpha_{11}\frac{\delta u_1}{\delta t} + \alpha_{12}\frac{\delta u_2}{\delta t} + \dots + \alpha_{1n}\frac{\delta u_n}{\delta t} &= -A\frac{\delta q}{\delta P_1} \\ \alpha_{21}\frac{\delta u_1}{\delta t} + \alpha_{22}\frac{\delta u_2}{\delta t} + \dots + \alpha_{2n}\frac{\delta u_n}{\delta t} &= -A\frac{\delta q}{\delta P_2} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \alpha_{n1}\frac{\delta u_1}{\delta t} + \alpha_{n2}\frac{\delta u_2}{\delta t} + \dots + \alpha_{nn}\frac{\delta u_n}{\delta t} &= -A\frac{\delta q}{\delta P_n} \end{aligned}\right\} \dots\dots\dots\dots\dots\dots\dots\dots\text{(V)}.$$

It is now necessary to find the value of $\frac{\delta \alpha_{rs}}{\delta t}$.

$$(-1)^{r+s}\alpha_{rs} = \begin{vmatrix} \frac{dP_1}{dx_1} & \dots\dots & \frac{dP_1}{dx_{s-1}} & \frac{dP_1}{dx_{s+1}} & \dots\dots & \frac{dP_1}{dx_n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{dP_{r-1}}{dx_1} & \dots\dots & \frac{dP_{r-1}}{dx_{s-1}} & \frac{dP_{r-1}}{dx_{s+1}} & \dots\dots & \frac{dP_{r-1}}{dx_n} \\ \frac{dP_{r+1}}{dx_1} & \dots\dots & \frac{dP_{r+1}}{dx_{s-1}} & \frac{dP_{r+1}}{dx_{s+1}} & \dots\dots & \frac{dP_{r+1}}{dx_n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{dP_n}{dx_1} & \dots\dots & \frac{dP_n}{dx_{s-1}} & \frac{dP_n}{dx_{s+1}} & \dots\dots & \frac{dP_n}{dx_n} \end{vmatrix}.$$

Let the reciprocal determinant of $(-1)^{r+s}\alpha_{rs}$ be

$$\begin{vmatrix} \beta_{11} & \dots\dots & \beta_{1,s-1} & \beta_{1,s+1} & \dots\dots & \beta_{1n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \beta_{r-1,1} & \dots\dots & \beta_{r-1,s-1} & \beta_{r-1,s+1} & \dots\dots & \beta_{r-1,n} \\ \beta_{r+1,1} & \dots\dots & \beta_{r+1,s-1} & \beta_{r+1,s+1} & \dots\dots & \beta_{r+1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \beta_{n1} & \dots\dots & \beta_{n,s-1} & \beta_{n,s+1} & \dots\dots & \beta_{nn} \end{vmatrix};$$

$$\therefore\ -\frac{\delta}{\delta t}\{(-1)^{r+s}\alpha_{rs}\}$$

$$= \left\{\beta_{11}\left(\frac{du_1}{dx_1}\frac{dP_1}{dx_1} + \frac{du_2}{dx_1}\frac{dP_1}{dx_2} + \dots + \frac{du_n}{dx_1}\frac{dP_1}{dx_n}\right) + \dots + \beta_{1,s-1}\left(\frac{du_1}{dx_{s-1}}\frac{dP_1}{dx_1} + \frac{du_2}{dx_{s-1}}\frac{dP_1}{dx_2} + \dots + \frac{du_n}{dx_{s-1}}\frac{dP_1}{dx_n}\right)\right.$$

$$\left. + \beta_{1,s+1}\left(\frac{du_1}{dx_{s+1}}\frac{dP_1}{dx_1} + \frac{du_2}{dx_{s+1}}\frac{dP_1}{dx_2} + \dots + \frac{du_n}{dx_{s+1}}\frac{dP_1}{dx_n}\right) + \dots + \beta_{1n}\left(\frac{du_1}{dx_n}\frac{dP_1}{dx_1} + \frac{du_2}{dx_n}\frac{dP_1}{dx_2} + \dots + \frac{du_n}{dx_n}\frac{dP_1}{dx_n}\right)\right\}$$

$$+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$+\left\{\beta_{r-1,1}\left(\frac{du_1}{dx_1}\frac{dP_{r-1}}{dx_1} + \frac{du_2}{dx_1}\frac{dP_{r-1}}{dx_2} + \dots + \frac{du_n}{dx_1}\frac{dP_{r-1}}{dx_n}\right) + \dots + \beta_{r-1,s-1}\left(\frac{du_1}{dx_{s-1}}\frac{dP_{r-1}}{dx_1} + \frac{du_2}{dx_{s-1}}\frac{dP_{r-1}}{dx_2} + \dots + \frac{du_n}{dx_{s-1}}\frac{dP_{r-1}}{dx_n}\right)\right.$$

$$\left. + \beta_{r-1,s+1}\left(\frac{du_1}{dx_{s+1}}\frac{dP_{r-1}}{dx_1} + \frac{du_2}{dx_{s+1}}\frac{dP_{r-1}}{dx_2} + \dots + \frac{du_n}{dx_{s+1}}\frac{dP_{r-1}}{dx_n}\right) + \dots + \beta_{r-1,n}\left(\frac{du_1}{dx_n}\frac{dP_{r-1}}{dx_1} + \frac{du_2}{dx_n}\frac{dP_{r-1}}{dx_2} + \dots + \frac{du_n}{dx_n}\frac{dP_{r-1}}{dx_n}\right)\right\}$$

Digitized by Google

$$+\left\{\beta_{r+1,1}\left(\frac{du_1}{dx_1}\frac{dP_{r+1}}{dx_1}+\frac{du_2}{dx_1}\frac{dP_{r+1}}{dx_2}+\ldots+\frac{du_n}{dx_1}\frac{dP_{r+1}}{dx_n}\right)+\ldots+\beta_{r+1,s-1}\left(\frac{du_1}{dx_{s-1}}\frac{dP_{r+1}}{dx_1}+\frac{du_2}{dx_{s-1}}\frac{dP_{r+1}}{dx_2}+\ldots+\frac{du_n}{dx_{s-1}}\frac{dP_{r+1}}{dx_n}\right)\right.$$

$$\left.+\beta_{r+1,s+1}\left(\frac{du_1}{dx_{s+1}}\frac{dP_{r+1}}{dx_1}+\frac{du_2}{dx_{s+1}}\frac{dP_{r+1}}{dx_2}+\ldots+\frac{du_n}{dx_{s+1}}\frac{dP_{r+1}}{dx_n}\right)+\ldots+\beta_{r+1,n}\left(\frac{du_1}{dx_n}\frac{dP_{r+1}}{dx_1}+\frac{du_2}{dx_n}\frac{dP_{r+1}}{dx_2}+\ldots+\frac{du_n}{dx_n}\frac{dP_{r+1}}{dx_n}\right)\right\}$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\beta_{n1}\left(\frac{du_1}{dx_1}\frac{dP_n}{dx_1}+\frac{du_2}{dx_1}\frac{dP_n}{dx_2}+\ldots+\frac{du_n}{dx_1}\frac{dP_n}{dx_n}\right)+\ldots+\beta_{n,s-1}\left(\frac{du_1}{dx_{s-1}}\frac{dP_n}{dx_1}+\frac{du_2}{dx_{s-1}}\frac{dP_n}{dx_2}+\ldots+\frac{du_n}{dx_{s-1}}\frac{dP_n}{dx_n}\right)$$

$$+\beta_{n,s+1}\left(\frac{du_1}{dx_{s+1}}\frac{dP_n}{dx_1}+\frac{du_2}{dx_{s+1}}\frac{dP_n}{dx_2}+\ldots+\frac{du_n}{dx_{s+1}}\frac{dP_n}{dx_n}\right)+\ldots+\beta_{nn}\left(\frac{du_1}{dx_n}\frac{dP_n}{dx_1}+\frac{du_2}{dx_n}\frac{dP_n}{dx_2}+\ldots+\frac{du_n}{dx_n}\frac{dP_n}{dx_n}\right);$$

$$\therefore\ -\frac{\delta}{\delta t}\{(-1)^{r+s}\alpha_{rs}\}$$

$$=\frac{du_1}{dx_1}\{(-1)^{r+s}\alpha_{rs}\}+\frac{du_2}{dx_1}.0+\ldots+\frac{du_s}{dx_1}\left(\beta_{11}\frac{dP_1}{dx_s}+\ldots+\beta_{r-1,1}\frac{dP_{r-1}}{dx_s}+\beta_{r+1,1}\frac{dP_{r+1}}{dx_s}+\ldots+\beta_{n1}\frac{dP_n}{dx_s}\right)+\ldots+\frac{du_n}{dx_1}.0$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\frac{du_1}{dx_{s-1}}.0+\frac{du_2}{dx_{s-1}}.0+\ldots+\frac{du_{s-1}}{dx_{s-1}}\{(-1)^{r+s}\alpha_{rs}\}+\frac{du_s}{dx_{s-1}}\left(\beta_{1,s-1}\frac{dP_1}{dx_s}+\ldots+\beta_{r-1,s-1}\frac{dP_{r-1}}{dx_s}+\beta_{r+1,s-1}\frac{dP_{r+1}}{dx_s}+\ldots+\beta_{n,s-1}\frac{dP_n}{dx_s}\right)+\ldots+\frac{du_n}{dx_{s-1}}.0$$

$$+\frac{du_1}{dx_{s+1}}.0+\frac{du_2}{dx_{s+1}}.0+\ldots+\frac{du_s}{dx_{s+1}}\left(\beta_{1,s+1}\frac{dP_1}{dx_s}+\ldots+\beta_{r-1,s+1}\frac{dP_{r-1}}{dx_s}+\beta_{r+1,s+1}\frac{dP_{r+1}}{dx_s}+\ldots+\beta_{n,s+1}\frac{dP_n}{dx_s}\right)+\frac{du_{s+1}}{dx_{s+1}}\{(-1)^{r+s}\alpha_{rs}\}+\ldots+\frac{du_n}{dx_{s+1}}.0$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\frac{du_1}{dx_n}.0+\frac{du_2}{dx_n}.0+\ldots+\frac{du_s}{dx_n}\left(\beta_{1n}\frac{dP_1}{dx_s}+\ldots+\beta_{r-1,n}\frac{dP_{r-1}}{dx_s}+\beta_{r+1,n}\frac{dP_{r+1}}{dx_s}+\ldots+\beta_{nn}\frac{dP_n}{dx_s}\right)+\ldots+\frac{du_n}{dx_n}\{(-1)^{r+s}\alpha_{rs}\};$$

$$\therefore\ -\frac{\delta}{\delta t}\{(-1)^{r+s}\alpha_{rs}\}=(-1)^{r+s}\alpha_{rs}\left(\frac{du_1}{dx_1}+\ldots+\frac{du_{s-1}}{dx_{s-1}}+\frac{du_{s+1}}{dx_{s+1}}+\ldots+\frac{du_n}{dx_n}\right)$$

$$+(-1)^{r+s-1}\alpha_{r1}\frac{du_s}{dx_1}+\ldots+(-1)^{r+s-1}\alpha_{r,s-1}\frac{du_s}{dx_{s-1}}+(-1)^{r+s-1}\alpha_{r,s+1}\frac{du_s}{dx_{s+1}}+\ldots+(-1)^{r+s-1}\alpha_{rn}\frac{du_s}{dx_n};$$

$$\therefore\ \frac{\delta\alpha_{rs}}{\delta t}=-\alpha_{rs}\left(\frac{du_1}{dx_1}+\ldots+\frac{du_{s-1}}{dx_{s-1}}+\frac{du_s}{dx_s}+\frac{du_{s+1}}{dx_{s+1}}+\ldots+\frac{du_n}{dx_n}\right)$$

$$+\left(\alpha_{r1}\frac{du_s}{dx_1}+\ldots+\alpha_{r,s-1}\frac{du_s}{dx_{s-1}}+\alpha_{r,s}\frac{du_s}{dx_s}+\alpha_{r,s+1}\frac{du_s}{dx_{s+1}}+\ldots+\alpha_{rn}\frac{du_s}{dx_n}\right).$$

But $$\because\ \frac{d}{dx_1}=\frac{dP_1}{dx_1}\frac{\delta}{\delta P_1}+\frac{dP_2}{dx_1}\frac{\delta}{\delta P_2}+\ldots\ldots+\frac{dP_n}{dx_1}\frac{\delta}{\delta P_n},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{d}{dx_n}=\frac{dP_1}{dx_n}\frac{\delta}{\delta P_1}+\frac{dP_2}{dx_n}\frac{\delta}{\delta P_2}+\ldots\ldots+\frac{dP_n}{dx_n}\frac{\delta}{\delta P_n};$$

$$\therefore\ \alpha_{r1}\frac{d}{dx_1}+\ldots+\alpha_{r,s-1}\frac{d}{dx_{s-1}}+\alpha_{r,s}\frac{d}{dx_s}+\alpha_{r,s+1}\frac{d}{dx_{s+1}}+\ldots+\alpha_{r,n}\frac{d}{dx_n}=A\frac{\delta}{\delta P_r}.$$

Digitized by Google

By means of this, and equation (II)

$$\frac{\delta \alpha_{rs}}{\delta t} = \alpha_{rs}\left(\frac{1}{\rho}\frac{\delta \rho}{\delta t}\right) + A\frac{\delta u_s}{\delta P_r};$$

therefore

$$\frac{\delta}{\delta t}\left(\frac{\alpha_{rs}}{\rho}\right) = \frac{A}{\rho}\frac{\delta u_s}{\delta P_r} \quad \text{.....(VI).}$$

But from (V)

$$\frac{\alpha_{r1}}{\rho}\frac{\delta u_1}{\delta t} + \frac{\alpha_{r2}}{\rho}\frac{\delta u_2}{\delta t} + \ldots\ldots + \frac{\alpha_{rn}}{\rho}\frac{\delta u_n}{\delta t} = -\frac{A}{\rho}\frac{\delta q}{\delta P_r},$$

and from (VI)

$$u_1\frac{\delta}{\delta t}\left(\frac{\alpha_{r1}}{\rho}\right) + u_2\frac{\delta}{\delta t}\left(\frac{\alpha_{r2}}{\rho}\right) + \ldots + u_n\frac{\delta}{\delta t}\left(\frac{\alpha_{rn}}{\rho}\right) = \frac{A}{\rho}\frac{\delta}{\delta P_r}\left(\frac{u_1^2 + u_2^2 + \ldots + u_n^2}{2}\right).$$

From these two equations

$$\frac{\delta}{\delta t}\left(\frac{\alpha_{r1}u_1 + \alpha_{r2}u_2 + \ldots + \alpha_{rn}u_n}{\rho}\right) = \frac{A}{\rho}\frac{\delta}{\delta P_r}\left(-q + \frac{u_1^2 + u_2^2 + \ldots + u_n^2}{2}\right).$$

Since $\frac{\delta}{\delta t}\left(\frac{A}{\rho}\right) = 0$, this becomes

$$\frac{\delta}{\delta t}\left(\frac{\alpha_{r1}u_1 + \alpha_{r2}u_2 + \ldots + \alpha_{rn}u_n}{A}\right) = \frac{\delta}{\delta P_r}\left(-q + \frac{u_1^2 + u_2^2 + \ldots + u_n^2}{2}\right).$$

Putting $-q + \frac{u_1^2 + u_2^2 + \ldots + u_n^2}{2} = \frac{\delta K}{\delta t}$, and integrating with regard to t,

$$\frac{\alpha_{r1}u_1 + \alpha_{r2}u_2 + \ldots + \alpha_{rn}u_n}{A} = \frac{\delta K}{\delta P_r} + f_r \quad \text{.....(VII),}$$

where f_r is an arbitrary function of $P_1, P_2 \ldots P_n$.

Solving the n equations which this represents for $u_1, u_2 \ldots u_n$,

$$\left.\begin{aligned} u_1 &= \frac{dK}{dx_1} + f_1\frac{dP_1}{dx_1} + \ldots + f_r\frac{dP_r}{dx_1} + \ldots + f_n\frac{dP_n}{dx_1} \\ u_2 &= \frac{dK}{dx_2} + f_1\frac{dP_1}{dx_2} + \ldots + f_r\frac{dP_r}{dx_2} + \ldots + f_n\frac{dP_n}{dx_2} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ u_n &= \frac{dK}{dx_n} + f_1\frac{dP_1}{dx_n} + \ldots + f_r\frac{dP_r}{dx_n} + \ldots + f_n\frac{dP_n}{dx_n} \end{aligned}\right\} \quad \text{.....(VIII),}$$

whence

$$u_1dx_1 + u_2dx_2 + \ldots + u_ndx_n = dK + f_1dP_1 + \ldots + f_rdP_r + \ldots + f_ndP_n,$$

where $dK, dP_1, \ldots dP_r \ldots dP_n$ do not contain differentials with regard to t; and $f_1, f_2 \ldots f_n$ are arbitrary functions of $P_1, P_2 \ldots P_n$.

These expressions for $u_1, u_2 \ldots u_n$ include the equations given by Weber (*Crelle*, t. 68) for ordinary fluid motion.

Digitized by Google

For, when $n=3$,
$$u_1 = \frac{dK}{dx_1} + f_1 \frac{dP_1}{dx_1} + f_2 \frac{dP_2}{dx_1} + f_3 \frac{dP_3}{dx_1},$$
$$u_2 = \frac{dK}{dx_2} + f_1 \frac{dP_1}{dx_2} + f_2 \frac{dP_2}{dx_2} + f_3 \frac{dP_3}{dx_2},$$
$$u_3 = \frac{dK}{dx_3} + f_1 \frac{dP_1}{dx_3} + f_2 \frac{dP_2}{dx_3} + f_3 \frac{dP_3}{dx_3}.$$

Hence
$$u_1 \frac{dx_1}{dP_1} + u_2 \frac{dx_2}{dP_1} + u_3 \frac{dx_3}{dP_1} = \frac{dK}{dP_1} + f_1,$$
$$u_1 \frac{dx_1}{dP_2} + u_2 \frac{dx_2}{dP_2} + u_3 \frac{dx_3}{dP_2} = \frac{dK}{dP_2} + f_2,$$
$$u_1 \frac{dx_1}{dP_3} + u_2 \frac{dx_2}{dP_3} + u_3 \frac{dx_3}{dP_3} = \frac{dK}{dP_3} + f_3.$$

These are the equations, 26, Chapter I, Art. 18, of Lamb's *Hydrodynamics;* for it is pointed out in Art. 20 of that treatise that a, b, c (the initial co-ordinates of the particle of fluid which at time t is at x, y, z) may be generalised so as to be any three quantities which serve to identify a particle, and which vary continuously from one particle to another. But if the values of P_1, P_2, P_3 be given p_1, p_2, p_3 respectively, then the surfaces $P_1=p_1$, $P_2=p_2$, $P_3=p_3$ serve to identify the particle which lies always on all three. Therefore it is possible to suppose that $a=P_1$, $b=P_2$, $c=P_3$. And if P_1 be put $=a$, $P_2=b$, $P_3=c$; then the K here becomes the χ of Weber's equations; and f_1, f_2, f_3 respectively u_0, v_0, w_0. Thus the f_1, f_2, f_3 have in this case physical meanings.

It may be readily shown that the forms given here satisfy equations (I) and (II) from which they have been deduced.

Substituting for $u_1, u_2, \ldots u_n q$ in the rth of equations (I), it becomes
$$\frac{\delta f_1}{\delta t} \cdot \frac{dP_1}{dx_r} + \ldots + \frac{\delta f_n}{\delta t} \frac{dP_n}{dx_r} + f_1 \frac{d}{dx_r}\left(\frac{\delta P_1}{\delta t}\right) + \ldots + f_n \frac{d}{dx_r}\left(\frac{\delta P_n}{\delta t}\right) = 0.$$

This is identically satisfied in virtue of the equations
$$\frac{\delta P_1}{\delta t} = 0 \ldots \frac{\delta P_n}{\delta t} = 0, \ldots \frac{\delta f_1}{\delta t} = 0 \ldots \frac{\delta f_n}{\delta t} = 0.$$

Since $f_1, f_2 \ldots f_n$ are each functions of $P_1, P_2 \ldots P_n$, these last equations are really equivalent to $\frac{\delta P_1}{\delta t} = 0 \ldots \frac{\delta P_n}{\delta t} = 0$. Writing out $\frac{\delta P_1}{\delta t} = 0 \ldots \frac{\delta P_n}{\delta t} = 0$ in full and substituting for $u_1, u_2 \ldots u_n$ their values, there will be n equations containing K, P_1, $P_2, \ldots P_n$ and functions and differential coefficients of them. These n equations together with that obtained from (II) by substituting their values for $u_1, u_2, \ldots u_n$ and with that obtained in like manner from the equation which gives q (and therefore ρ) are $(n+2)$ equations to determine K, ρ, P_1, $P_2, \ldots P_n$.

Digitized by Google

3. To determine whether any quantities exist which have the same relation to the equations here considered as the components of molecular rotation have to the equations of Hydrodynamics, it is necessary to examine the manner in which these are obtained.

If $u_1 u_2 u_3$ are the velocities parallel to the axes at the point whose co-ordinates are $x_1 x_2 x_3$; and $u_1' u_2' u_3'$ are the velocities at the point $x_1 + dx_1$, $x_2 + dx_2$, $x_3 + dx_3$.

Then

$$u_1' = u_1 + \frac{du_1}{dx_1} dx_1 + \tfrac{1}{2}\left(\frac{du_1}{dx_2} + \frac{du_2}{dx_1}\right) dx_2 + \tfrac{1}{2}\left(\frac{du_1}{dx_3} + \frac{du_3}{dx_1}\right) dx_3 + \tfrac{1}{2}(\xi_{11} dx_1 + \xi_{12} dx_2 + \xi_{13} dx_3) + \dots$$

$$u_2' = u_2 + \tfrac{1}{2}\left(\frac{du_2}{dx_1} + \frac{du_1}{dx_2}\right) dx_1 + \frac{du_2}{dx_2} dx_2 + \tfrac{1}{2}\left(\frac{du_2}{dx_3} + \frac{du_3}{dx_2}\right) dx_3 + \tfrac{1}{2}(\xi_{21} dx_1 + \xi_{22} dx_2 + \xi_{23} dx_3) + \dots$$

$$u_3' = u_3 + \tfrac{1}{2}\left(\frac{du_3}{dx_1} + \frac{du_1}{dx_3}\right) dx_1 + \tfrac{1}{2}\left(\frac{du_3}{dx_2} + \frac{du_2}{dx_3}\right) dx_2 + \frac{du_3}{dx_3} dx_3 + \tfrac{1}{2}(\xi_{31} dx_1 + \xi_{32} dx_2 + \xi_{33} dx_3) + \dots$$

where $\xi_{rs} = \dfrac{du_r}{dx_s} - \dfrac{du_s}{dx_r}$.

Now it will be observed that if $dx_1 dx_2 dx_3$ be the projections of an element of a vortex line on the axes, then the system of equations

$$\begin{aligned} \xi_{11} dx_1 + \xi_{12} dx_2 + \xi_{13} dx_3 &= 0, \\ \xi_{21} dx_1 + \xi_{22} dx_2 + \xi_{23} dx_3 &= 0, \\ \xi_{31} dx_1 + \xi_{32} dx_2 + \xi_{33} dx_3 &= 0, \end{aligned}$$

is a consistent system of equations, satisfied by putting

$$\frac{dx_1}{\xi_1} = \frac{dx_2}{\xi_2} = \frac{dx_3}{\xi_3}$$

where ξ_1, ξ_2, ξ_3 are square roots of the coefficients of the diagonal terms of the determinant

$$\begin{vmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \\ \xi_{31} & \xi_{32} & \xi_{33} \end{vmatrix}.$$

To extend the analogy to the equations of this paper it is necessary to observe first that n must be an odd number, for if n be even, the quantities corresponding to ξ_1, ξ_2, ξ_3 vanish.

Let then $\xi_1, \xi_2, \dots \xi_n$ be square roots of the coefficients of the diagonal terms of the determinant Ξ

$$\begin{vmatrix} \xi_{11} & \xi_{12} & \xi_{13} & \cdots & \xi_{1n} \\ \xi_{21} & \xi_{22} & \xi_{23} & \cdots & \xi_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \xi_{n1} & \xi_{n2} & \xi_{n3} & \cdots & \xi_{nn} \end{vmatrix}.$$

Digitized by Google

It will be shown that

$$\frac{\delta}{\delta t}\left(\frac{\xi_1}{\rho}\right)=\frac{\xi_1}{\rho}\frac{du_1}{dx_1}+\frac{\xi_2}{\rho}\frac{du_1}{dx_2}+\ldots+\frac{\xi_n}{\rho}\frac{du_1}{dx_n}=\frac{\xi_1}{\rho}\frac{du_1}{dx_1}+\frac{\xi_2}{\rho}\frac{du_2}{dx_1}+\ldots+\frac{\xi_n}{\rho}\frac{du_n}{dx_1}$$

.....

$$\frac{\delta}{\delta t}\left(\frac{\xi_s}{\rho}\right)=\frac{\xi_1}{\rho}\frac{du_s}{dx_1}+\frac{\xi_2}{\rho}\frac{du_s}{dx_2}+\ldots+\frac{\xi_n}{\rho}\frac{du_s}{dx_n}=\frac{\xi_1}{\rho}\frac{du_1}{dx_s}+\frac{\xi_2}{\rho}\frac{du_2}{dx_s}+\ldots+\frac{\xi_n}{\rho}\frac{du_n}{dx_s}$$

.....

$$\frac{\delta}{\delta t}\left(\frac{\xi_n}{\rho}\right)=\frac{\xi_1}{\rho}\frac{du_n}{dx_1}+\frac{\xi_2}{\rho}\frac{du_n}{dx_2}+\ldots+\frac{\xi_n}{\rho}\frac{du_n}{dx_n}=\frac{\xi_1}{\rho}\frac{du_1}{dx_n}+\frac{\xi_2}{\rho}\frac{du_2}{dx_n}+\ldots+\frac{\xi_n}{\rho}\frac{du_n}{dx_n}$$

and
$$\frac{d\xi_1}{dx_1}+\ldots+\frac{d\xi_s}{dx_s}+\ldots+\frac{d\xi_n}{dx_n}=0.$$

If n be an even number the above $n+1$ equations are replaced by

$$\frac{\delta}{\delta t}\left(\frac{\sqrt{\Xi}}{\rho}\right)=0.$$

Since
$$\frac{du_r}{dt}+u_1\frac{du_r}{dx_1}+u_2\frac{du_r}{dx_2}+\ldots+u_n\frac{du_r}{dx_n}=-\frac{dq}{dx_r},$$

and
$$\frac{du_s}{dt}+u_1\frac{du_s}{dx_1}+u_2\frac{du_s}{dx_2}+\ldots+u_n\frac{du_s}{dx_n}=-\frac{dq}{dx_s};$$

$$\therefore\ \frac{d}{dx_s}\left(\frac{du_r}{dt}+u_1\frac{du_r}{dx_1}+u_2\frac{du_r}{dx_2}+\ldots+u_n\frac{du_r}{dx_n}\right)=\frac{d}{dx_r}\left(\frac{du_s}{dt}+u_1\frac{du_s}{dx_1}+u_2\frac{du_s}{dx_2}+\ldots+u_n\frac{du_s}{dx_n}\right).$$

But
$$\frac{\delta\xi_{rs}}{\delta t}=\frac{\delta}{\delta t}\left(\frac{du_r}{dx_s}-\frac{du_s}{dx_r}\right)=\left(\frac{d}{dt}+u_1\frac{d}{dx_1}+u_2\frac{d}{dx_2}+\ldots+u_n\frac{d}{dx_n}\right)\left(\frac{du_r}{dx_s}-\frac{du_s}{dx_r}\right).$$

Hence

$$\frac{\delta\xi_{rs}}{\delta t}+\frac{du_1}{dx_s}\frac{du_r}{dx_1}+\frac{du_2}{dx_s}\frac{du_r}{dx_2}+\ldots+\frac{du_n}{dx_s}\frac{du_r}{dx_n}-\frac{du_1}{dx_r}\frac{du_s}{dx_1}-\frac{du_2}{dx_r}\frac{du_s}{dx_2}-\ldots-\frac{du_n}{dx_r}\frac{du_s}{dx_n}=0,$$

i.e.
$$\left.\begin{aligned}\frac{\delta\xi_{rs}}{\delta t}&+\frac{du_r}{dx_1}\left(\frac{du_1}{dx_s}-\frac{du_s}{dx_1}\right)+\frac{du_r}{dx_2}\left(\frac{du_2}{dx_s}-\frac{du_s}{dx_2}\right)+\ldots+\frac{du_r}{dx_n}\left(\frac{du_n}{dx_s}-\frac{du_s}{dx_n}\right)\\&+\frac{du_s}{dx_1}\left(\frac{du_r}{dx_1}-\frac{du_1}{dx_r}\right)+\frac{du_s}{dx_2}\left(\frac{du_r}{dx_2}-\frac{du_2}{dx_r}\right)+\ldots+\frac{du_s}{dx_n}\left(\frac{du_r}{dx_n}-\frac{du_n}{dx_r}\right)\end{aligned}\right\}=0,$$

i.e.
$$\frac{\delta\xi_{rs}}{\delta t}+\left(\xi_{1s}\frac{du_r}{dx_1}+\xi_{2s}\frac{du_r}{dx_2}+\ldots+\xi_{ns}\frac{du_r}{dx_n}\right)+\left(\xi_{r1}\frac{du_s}{dx_1}+\xi_{r2}\frac{du_s}{dx_2}+\ldots+\xi_{rn}\frac{du_s}{dx_n}\right)=0.$$

Consider now
$$\frac{\delta}{\delta t}\begin{vmatrix}\xi_{22} & \xi_{23} & \xi_{24} & \cdots & \xi_{2n}\\ \xi_{32} & \xi_{33} & \xi_{34} & \cdots & \xi_{3n}\\ \xi_{42} & \xi_{43} & \xi_{44} & \cdots & \xi_{4n}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ \xi_{n2} & \xi_{n3} & \xi_{n4} & \cdots & \xi_{nn}\end{vmatrix}.$$

Digitized by Google

It is

$$
=-\begin{vmatrix}
\left.\begin{matrix}\frac{du_2}{dx_1}\xi_{12}+\frac{du_2}{dx_2}\xi_{22}+\frac{du_2}{dx_3}\xi_{23}+\dots\\ +\frac{du_3}{dx_1}\xi_{21}+\frac{du_3}{dx_2}\xi_{22}+\frac{du_3}{dx_3}\xi_{23}+\dots\end{matrix}\right\} & \left.\begin{matrix}\frac{du_2}{dx_1}\xi_{13}+\frac{du_2}{dx_2}\xi_{23}+\frac{du_2}{dx_3}\xi_{33}+\dots\\ +\frac{du_3}{dx_1}\xi_{31}+\frac{du_3}{dx_2}\xi_{32}+\frac{du_3}{dx_3}\xi_{33}+\dots\end{matrix}\right\} & \left.\begin{matrix}\frac{du_2}{dx_1}\xi_{14}+\frac{du_2}{dx_2}\xi_{24}+\frac{du_2}{dx_3}\xi_{34}+\dots\\ +\frac{du_4}{dx_1}\xi_{21}+\frac{du_4}{dx_2}\xi_{22}+\frac{du_4}{dx_3}\xi_{23}+\dots\end{matrix}\right\} & \dots \\
\xi_{22} & \xi_{23} & \xi_{24} & \dots \\
\xi_{42} & \xi_{43} & \xi_{44} & \dots \\
\dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots
\end{vmatrix}
$$

$$
-\begin{vmatrix}
\xi_{22} & \xi_{23} & \xi_{24} & \dots \\
\left.\begin{matrix}\frac{du_2}{dx_1}\xi_{12}+\frac{du_2}{dx_2}\xi_{22}+\frac{du_2}{dx_3}\xi_{23}+\dots\\ +\frac{du_3}{dx_1}\xi_{21}+\frac{du_3}{dx_2}\xi_{22}+\frac{du_3}{dx_3}\xi_{23}+\dots\end{matrix}\right\} & \left.\begin{matrix}\frac{du_2}{dx_1}\xi_{13}+\frac{du_2}{dx_2}\xi_{23}+\frac{du_2}{dx_3}\xi_{33}+\dots\\ +\frac{du_3}{dx_1}\xi_{31}+\frac{du_3}{dx_2}\xi_{32}+\frac{du_3}{dx_3}\xi_{33}+\dots\end{matrix}\right\} & \left.\begin{matrix}\frac{du_2}{dx_1}\xi_{14}+\frac{du_2}{dx_2}\xi_{24}+\frac{du_2}{dx_3}\xi_{34}+\dots\\ +\frac{du_4}{dx_1}\xi_{21}+\frac{du_4}{dx_2}\xi_{22}+\frac{du_4}{dx_3}\xi_{23}+\dots\end{matrix}\right\} & \dots \\
\xi_{42} & \xi_{43} & \xi_{44} & \dots \\
\dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots
\end{vmatrix}
$$

$$
-\dots\dots\dots

$$-\begin{vmatrix} \xi_{22} & \xi_{23} & \xi_{24} & \cdots \\ \left.\frac{du_2}{dx_1}\xi_{21}+\frac{du_2}{dx_2}\xi_{22}+\frac{du_2}{dx_3}\xi_{23}+\cdots\right\} & \left.\frac{du_3}{dx_1}\xi_{21}+\frac{du_3}{dx_2}\xi_{22}+\frac{du_3}{dx_3}\xi_{23}+\cdots\right\} & \left.\frac{du_4}{dx_1}\xi_{21}+\frac{du_4}{dx_2}\xi_{22}+\frac{du_4}{dx_3}\xi_{23}+\cdots\right\} & \cdots \\ \xi_{42} & \xi_{43} & \xi_{44} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

$$-\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

The first determinant written down is

$$=-\frac{du_2}{dx_1}\begin{vmatrix} \xi_{12}, & \xi_{13}, & \xi_{14}\cdots \\ \xi_{32}, & \xi_{33}, & \xi_{34}\cdots \\ \xi_{42}, & \xi_{43}, & \xi_{44}\cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix} -\frac{du_2}{dx_2}\begin{vmatrix} \xi_{22}, & \xi_{23}, & \xi_{24}\cdots \\ \xi_{32}, & \xi_{33}, & \xi_{34}\cdots \\ \xi_{42}, & \xi_{43}, & \xi_{44}\cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix}$$

The third determinant is

$$=-\frac{du_3}{dx_1}\begin{vmatrix} \xi_{22}, & \xi_{23}, & \xi_{24}\cdots \\ \xi_{12}, & \xi_{13}, & \xi_{14}\cdots \\ \xi_{42}, & \xi_{43}, & \xi_{44}\cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix} -\frac{du_3}{dx_3}\begin{vmatrix} \xi_{22}, & \xi_{23}, & \xi_{24}\cdots \\ \xi_{32}, & \xi_{33}, & \xi_{34}\cdots \\ \xi_{42}, & \xi_{43}, & \xi_{44}\cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix}$$

and so on, for the first of every pair of unexpressed determinants.

Also picking out the coefficient of $\frac{du_r}{dx_s}$, where r is any integer between 2 and n inclusive, and s any integer between 1 and n inclusive, in the second of each pair of determinants, it is found to be

$$-\begin{vmatrix} \xi_{22}\xi_{23}\cdots\xi_{2,\,r-1}, & \xi_{2s}\xi_{2,\,r+1}\cdots\xi_{2n} \\ \xi_{32}\xi_{33}\cdots\xi_{3,\,r-1}, & \xi_{3s}\xi_{3,\,r+1}\cdots\xi_{3n} \\ \xi_{42}\xi_{43}\cdots\xi_{4,\,r-1}, & \xi_{4s}\xi_{4,\,r+1}\cdots\xi_{4n} \\ \cdots & \cdots \\ \cdots & \cdots \\ \xi_{n2}\xi_{n3}\cdots\xi_{n,\,r-1}, & \xi_{ns}\xi_{n,\,r+1}\cdots\xi_{nn} \end{vmatrix}$$

But this is zero, unless $s=1$ or $s=r$, therefore the second determinants of each pair reduce to

$$-\left(\sum_2^n \frac{du_r}{dx_r}\right)\begin{vmatrix} \xi_{22}, & \xi_{23}\cdots\xi_{2n} \\ \xi_{32}, & \xi_{33}\cdots\xi_{3n} \\ \cdots & \cdots \\ \cdots & \cdots \\ \xi_{n2}, & \xi_{n3}\cdots\xi_{nn} \end{vmatrix} -\sum_2^n\left\{\frac{du_r}{dx_1}\begin{vmatrix} \xi_{22}\xi_{23}\cdots\xi_{2,\,r-1}, & \xi_{21}\xi_{2,\,r+1}\cdots\xi_{2n} \\ \xi_{32}\xi_{33}\cdots\xi_{3,\,r-1}, & \xi_{31}\xi_{3,\,r+1}\cdots\xi_{3n} \\ \cdots & \cdots \\ \cdots & \cdots \\ \xi_{n2}\xi_{n3}\cdots\xi_{n,\,r-1}, & \xi_{n1}\xi_{n,\,r+1}\cdots\xi_{nn} \end{vmatrix}\right\}.$$

But the first determinants are equivalent to

$$-\left(\sum_2^n \frac{du_r}{dx_r}\right)\begin{vmatrix} \xi_{22}\xi_{23}\cdots & \xi_{2n} \\ \xi_{32}\xi_{33}\cdots & \xi_{3n} \\ \cdots & \cdots \\ \cdots & \cdots \\ \xi_{n2}\xi_{n3}\cdots & \xi_{nn} \end{vmatrix} -\sum_2^n\left\{\frac{du_r}{dx_1}\begin{vmatrix} \xi_{22} & \xi_{23} & \cdots\xi_{2n} \\ \xi_{32} & \xi_{33} & \cdots\xi_{3n} \\ \cdots & \cdots & \cdots \\ \xi_{r-1,\,2} & \xi_{r-1,\,3} & \cdots\xi_{r-1,\,n} \\ \xi_{12} & \xi_{13} & \cdots\xi_{1n} \\ \xi_{r+1,\,2} & \xi_{r+1,\,3} & \cdots\xi_{r+1,\,n} \\ \cdots & \cdots & \cdots \\ \xi_{n2} & \xi_{n3} & \cdots\xi_{nn} \end{vmatrix}\right\}.$$

Digitized by Google

Since n is an odd number, and $\xi_{rs} = -\xi_{sr}$

$$\begin{vmatrix} \xi_{22}\xi_{23} \cdots \xi_{2,r-1}, & \xi_{21}\xi_{2,r+1} \cdots \xi_{2n} \\ \xi_{32}\xi_{33} \cdots \xi_{3,r-1}, & \xi_{31}\xi_{3,r+1} \cdots \xi_{3n} \\ \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \\ \xi_{n2}\xi_{n3} \cdots \xi_{n,r-1}, & \xi_{n1}\xi_{n,r+1} \cdots \xi_{nn} \end{vmatrix} = \begin{vmatrix} \xi_{22} & \xi_{23} & \cdots & \xi_{2n} \\ \xi_{32} & \xi_{33} & \cdots & \xi_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ \xi_{r-1,2} & \xi_{r-1,3} & \cdots & \xi_{r-1,n} \\ \xi_{12} & \xi_{13} & \cdots & \xi_{1n} \\ \xi_{r+1,2} & \xi_{r+1,3} & \cdots & \xi_{r+1,n} \\ \cdots & \cdots & \cdots & \cdots \\ \xi_{n2} & \xi_{n3} & \cdots & \xi_{nn} \end{vmatrix};$$

therefore

$$\frac{\delta}{\delta t}\begin{vmatrix} \xi_{22}\xi_{23} \cdots \xi_{2n} \\ \xi_{32}\xi_{33} \cdots \xi_{3n} \\ \cdots\cdots \\ \cdots\cdots \\ \cdots\cdots \\ \xi_{n2}\xi_{n3} \cdots \xi_{nn} \end{vmatrix} = -2\left(\sum_2^n \frac{du_r}{dx_r}\right)\begin{vmatrix} \xi_{22}\xi_{23} \cdots \xi_{2n} \\ \xi_{32}\xi_{33} \cdots \xi_{3n} \\ \cdots\cdots \\ \cdots\cdots \\ \cdots\cdots \\ \xi_{n2}\xi_{n3} \cdots \xi_{nn} \end{vmatrix} - 2\sum_2^n \left\{\frac{du_r}{dx_1}\begin{vmatrix} \xi_{22}\xi_{23} \cdots \xi_{2,r-1}, & \xi_{21}\xi_{2,r+1} \cdots \xi_{2n} \\ \xi_{32}\xi_{33} \cdots \xi_{3,r-1}, & \xi_{31}\xi_{3,r+1} \cdots \xi_{3n} \\ \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \\ \xi_{n2}\xi_{n3} \cdots \xi_{n,r-1}, & \xi_{n1}\xi_{n,r+1} \cdots \xi_{nn} \end{vmatrix}\right\}.$$

Now if Ξ_{rs} be the coefficient of ξ_{rs} in the skew symmetrical determinant of odd order Ξ, and if $\xi_1, \xi_2 \ldots \xi_n$ be the square roots of the coefficients of the diagonal terms in Ξ, chosen with proper signs, then $\Xi_{rs} = \xi_r\xi_s = \Xi_{sr}$.

ξ_1 is the Pfaffian $\{2, 3, 4, 5 \ldots (n-1), n\}$,

ξ_r, having the same relation to the suffixes $r+1, r+2, \ldots n, 1, 2, 3 \ldots (r-1)$ that ξ_1 has to the suffixes $2, 3, \ldots (n-1), n$ may be denoted by the Pfaffian

$$\{r+1, r+2, \ldots n, 1, 2, 3 \ldots (r-1)\}.$$

Observing further that

$$\xi_1\xi_r = \Xi_{1r} = (-1)^{r-1}\begin{vmatrix} \xi_{21} & \xi_{22} & \cdots & \xi_{2,r-1}, & \xi_{2,r+1} & \cdots & \xi_{2n} \\ \xi_{31} & \xi_{32} & \cdots & \xi_{3,r-1}, & \xi_{3,r+1} & \cdots & \xi_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \xi_{n1} & \xi_{n2} & \cdots & \xi_{n,r-1}, & \xi_{n,r+1} & \cdots & \xi_{nn} \end{vmatrix} = -\begin{vmatrix} \xi_{22} \cdots \xi_{2,r-1}, & \xi_{21} & \xi_{2,r+1} \cdots \xi_{2n} \\ \xi_{32} \cdots \xi_{3,r-1}, & \xi_{31} & \xi_{3,r+1} \cdots \xi_{3n} \\ \cdots\cdots & \cdots & \cdots\cdots \\ \cdots\cdots & \cdots & \cdots\cdots \\ \xi_{n2} \cdots \xi_{n,r-1}, & \xi_{n1} & \xi_{n,r+1} \cdots \xi_{nn} \end{vmatrix},$$

it follows that

$$\frac{\delta}{\delta t}(\xi_1^2) = 2\left(\frac{1}{\rho}\frac{\delta\rho}{\delta t} + \frac{du_1}{dx_1}\right)\xi_1^2 + 2\sum_2^n\left(\frac{du_r}{dx_1}\xi_1\xi_r\right);$$

$$\therefore \frac{\delta\xi_1}{\delta t} = \left(\frac{1}{\rho}\frac{\delta\rho}{\delta t} + \frac{du_1}{dx_1}\right)\xi_1 + \sum_2^n\left(\frac{du_r}{dx_1}\xi_r\right);$$

$$\therefore \frac{\delta}{\delta t}\left(\frac{\xi_1}{\rho}\right) = \sum_1^n \frac{du_r}{dx_1}\cdot\frac{\xi_r}{\rho} = \frac{\xi_1}{\rho}\frac{du_1}{dx_1} + \frac{\xi_2}{\rho}\frac{du_2}{dx_1} + \ldots + \frac{\xi_n}{\rho}\frac{du_n}{dx_1}.$$

Digitized by Google

But
$$\xi_{11}\Xi_{11}+\xi_{12}\Xi_{12}+\dots+\xi_{1n}\Xi_{1n}=0;$$
$$\therefore\ \xi_{11}\xi_1+\xi_{12}\xi_2\ +\dots+\xi_{1n}\xi_n\ =0.$$

But
$$\xi_{11}=0\ \therefore\ \xi_{12}\frac{\xi_2}{\rho}+\dots+\xi_{1n}\frac{\xi_n}{\rho}=0;$$
$$\therefore\ \frac{\delta}{\delta t}\left(\frac{\xi_1}{\rho}\right)=\frac{\xi_1}{\rho}\frac{du_1}{dx_1}+\frac{\xi_2}{\rho}\left(\frac{du_2}{dx_1}+\xi_{12}\right)+\dots+\frac{\xi_n}{\rho}\left(\frac{du_n}{dx_1}+\xi_{1n}\right)=\frac{\xi_1}{\rho}\frac{du_1}{dx_1}+\frac{\xi_2}{\rho}\frac{du_1}{dx_2}+\dots+\frac{\xi_n}{\rho}\frac{du_1}{dx_n};$$
$$\therefore\ \frac{\delta}{\delta t}\left(\frac{\xi_1}{\rho}\right)=\frac{\xi_1}{\rho}\frac{du_1}{dx_1}+\frac{\xi_2}{\rho}\frac{du_1}{dx_2}+\dots+\frac{\xi_n}{\rho}\frac{du_1}{dx_n}=\frac{\xi_1}{\rho}\frac{du_1}{dx_1}+\frac{\xi_2}{\rho}\frac{du_2}{dx_1}+\dots+\frac{\xi_n}{\rho}\frac{du_n}{dx_1}.$$

Similarly, $\frac{\delta}{\delta t}\left(\frac{\xi_r}{\rho}\right)=\frac{\xi_1}{\rho}\frac{du_r}{dx_1}+\frac{\xi_2}{\rho}\frac{du_r}{dx_2}+\dots+\frac{\xi_n}{\rho}\frac{du_r}{dx_n}=\frac{\xi_1}{\rho}\frac{du_1}{dx_r}+\frac{\xi_2}{\rho}\frac{du_2}{dx_r}+\dots+\frac{\xi_n}{\rho}\frac{du_n}{dx_r},$

where r is any integer from 2 to n.

Thus there are n equations analogous to Helmholtz's equations of Fluid Motion (n being an odd number).

But if n be an even number then $\xi_1, \xi_2, \dots \xi_n$, being square roots of skew symmetrical determinants of odd order, all vanish.

In this case consider the value of

$$\frac{\delta\Xi}{\delta t}=\frac{\delta}{\delta t}\begin{vmatrix}\xi_{11} & \xi_{12} & \xi_{13} & \cdots & \xi_{1n}\\ \xi_{21} & \xi_{22} & \xi_{23} & \cdots & \xi_{2n}\\ \xi_{31} & \xi_{32} & \xi_{33} & \cdots & \xi_{3n}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ \xi_{n1} & \xi_{n2} & \xi_{n3} & \cdots & \xi_{nn}\end{vmatrix}.$$

It is $=-$

$$\begin{vmatrix}\left.\begin{matrix}\xi_{11}\frac{du_1}{dx_1}+\xi_{21}\frac{du_1}{dx_2}+\xi_{31}\frac{du_1}{dx_3}+\dots\\ +\xi_{11}\frac{du_1}{dx_1}+\xi_{12}\frac{du_1}{dx_2}+\xi_{13}\frac{du_1}{dx_3}+\dots\end{matrix}\right\} & \left.\begin{matrix}\xi_{12}\frac{du_1}{dx_1}+\xi_{22}\frac{du_1}{dx_2}+\xi_{32}\frac{du_1}{dx_3}+\dots\\ +\xi_{11}\frac{du_2}{dx_1}+\xi_{12}\frac{du_2}{dx_2}+\xi_{13}\frac{du_2}{dx_3}+\dots\end{matrix}\right\} & \left.\begin{matrix}\xi_{13}\frac{du_1}{dx_1}+\xi_{23}\frac{du_1}{dx_2}+\xi_{33}\frac{du_1}{dx_3}+\dots\\ +\xi_{11}\frac{du_3}{dx_1}+\xi_{12}\frac{du_3}{dx_2}+\xi_{13}\frac{du_3}{dx_3}+\dots\end{matrix}\right\} & \cdots\\ \xi_{21} & \xi_{22} & \xi_{23} & \cdots\\ \xi_{31} & \xi_{32} & \xi_{33} & \cdots\\ \cdots & \cdots & \cdots & \cdots\\ \xi_{n1} & \xi_{n2} & \xi_{n3} & \cdots\end{vmatrix}$$

$-$ similar determinants.

In the determinant which is expressed, multiply the constituents of the second row by $-\frac{du_1}{dx_2}$, of the third row by $-\frac{du_1}{dx_3}$, and so on, and add to the corresponding constituents of the first row. Treat the unexpressed determinants in a corresponding manner.

Digitized by Google

The result is

$$-\frac{du_1}{dx_1}\begin{vmatrix} \xi_{11} & \xi_{12} & \xi_{13} & \cdots\cdots \\ \xi_{21} & \xi_{22} & \xi_{23} & \cdots\cdots \\ \xi_{31} & \xi_{32} & \xi_{33} & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \xi_{n1} & \xi_{n2} & \xi_{n3} & \cdots\cdots \end{vmatrix}$$

$$-\begin{vmatrix} \xi_{11}\frac{du_1}{dx_1}+\xi_{12}\frac{du_1}{dx_2}+\xi_{13}\frac{du_1}{dx_3}+\cdots\Big\} & \xi_{11}\frac{du_2}{dx_1}+\xi_{12}\frac{du_2}{dx_2}+\xi_{13}\frac{du_2}{dx_3}+\cdots\Big\} & \xi_{11}\frac{du_3}{dx_1}+\xi_{12}\frac{du_3}{dx_2}+\xi_{13}\frac{du_3}{dx_3}+\cdots\Big\} & \cdots \\ \xi_{21} & \xi_{22} & \xi_{23} & \cdots \\ \xi_{31} & \xi_{32} & \xi_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \xi_{n1} & \xi_{n2} & \xi_{n3} & \cdots \end{vmatrix}$$

$-$ similar pairs of determinants.

The first determinants of each pair are together equal to $-\left(\frac{du_1}{dx_1}+\ldots+\frac{du_n}{dx_n}\right)(\Xi)$.

The coefficient of $\frac{du_r}{dx_s}$ in the second determinants of each pair is

$$-\begin{vmatrix} \xi_{11} \cdots \xi_{1,r-1} & \xi_{1s} & \xi_{1,r+1} \cdots \xi_{1n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \xi_{n1} \cdots \xi_{n,r-1} & \xi_{ns} & \xi_{n,r+1} \cdots \xi_{nn} \end{vmatrix},$$

and this vanishes unless $r=s$, when it is $=-\Xi$;

$$\therefore \frac{\delta\Xi}{\delta t}=-2\left(\frac{du_1}{dx_1}+\ldots+\frac{du_n}{dx_n}\right)(\Xi)=\frac{2}{\rho}\frac{\delta\rho}{\delta t}(\Xi);$$

$$\therefore \frac{\delta}{\delta t}\left(\frac{\sqrt{\Xi}}{\rho}\right)=0.$$

This is the sole representative of Helmholtz's Equations when n is even.

It still remains to show that $\frac{d\xi_1}{dx_1}+\frac{d\xi_2}{dx_2}+\ldots\ldots+\frac{d\xi_n}{dx_n}=0$.

It is known that if m be an even number, and $\begin{vmatrix} a_{11} & \cdots\cdots & a_{1m} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ a_{m1} & \cdots\cdots & a_{mm} \end{vmatrix}$ a skew symmetrical determinant, then its square root which is the Pfaffian $\{1, 2, 3, \ldots\ldots (m-1), m\}$

$= a_{12}\{3, 4, \ldots (m-1), m\}+\ldots\ldots+a_{1r_1}\{\overline{r_1+1}, \ldots \overline{m-1}, m, 2, 3, \ldots \overline{r_1-1}\}+\ldots\ldots+a_{1m}\{2, 3 \ldots \overline{m-1}\}$.

Digitized by Google

Further
$$\begin{aligned}
\xi_1 &= (2,\ 3, \ldots\ldots \overline{n-1},\ n),\\
\xi_2 &= (3,\ 4, \ldots\ldots \overline{n-1},\ n,\ 1),\\
\xi_3 &= (4,\ 5, \ldots\ldots \overline{n-1},\ n,\ 1,\ 2),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\xi_{r_1} &= (\overline{r_1+1} \ldots\ldots n,\ 1, \ldots\ldots \overline{r_1-1}),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\xi_{r_2} &= (\overline{r_2+1} \ldots\ldots n,\ 1, \ldots\ldots \overline{r_2-1}),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\xi_{r_3} &= (\overline{r_3+1} \ldots\ldots n,\ 1, \ldots\ldots \overline{r_3-1}),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\xi_n &= (1,\ 2, \ldots\ldots\ldots\ldots\ldots\ldots \overline{n-1}).
\end{aligned}$$

It being supposed that r_1, r_2, r_3 are any three integers between 1 and n inclusive, in ascending order of magnitude.

Further if two of the suffixes in any Pfaffian be interchanged the sign is changed,

$$i.e.\ (1,\ 2, \ldots r, \ldots s \ldots m) = -(1,\ 2, \ldots s \ldots r \ldots m);$$

$$\therefore\ \xi_{r_1} = (-1)^{r_3-r_1-1}\,(r_2,\ \overline{r_1+1} \ldots\ldots \overline{r_2-1},\ \overline{r_2+1} \ldots\ldots n,\ 1,\ 2, \ldots\ldots \overline{r_1-1}),$$

$$\xi_{r_2} = (-1)^{r_3-r_2-1}\,(r_3,\ \overline{r_2+1} \ldots\ldots \overline{r_3-1},\ \overline{r_3+1} \ldots\ldots n,\ 1,\ 2, \ldots\ldots \overline{r_2-1}),$$

$$\xi_{r_3} = (-1)^{(n-r_3+r_1-1)}\,(r_1,\ \overline{r_3+1} \ldots\ldots n,\ 1,\ 2, \ldots\ldots \overline{r_1-1},\ \overline{r_1+1} \ldots\ldots \overline{r_3-1});$$

therefore the term which contains

$\xi_{r_2r_3}$ in ξ_{r_1} is $(-1)^{r_3-r_1-1}\,\xi_{r_2r_3}(\overline{r_3+1} \ldots n,\ 1,\ 2 \ldots \overline{r_1-1},\ \overline{r_1+1} \ldots \overline{r_2-1},\ \overline{r_2+1} \ldots \overline{r_3-1})$,

$\xi_{r_3r_1}$ in ξ_{r_2} is $(-1)^{r_3-r_2-1}\,\xi_{r_3r_1}(\overline{r_1+1} \ldots \overline{r_2-1},\ \overline{r_2+1} \ldots \overline{r_3-1},\ \overline{r_3+1} \ldots n,\ 1,\ 2, \ldots \overline{r_1-1})$,

$\xi_{r_1r_2}$ in ξ_{r_3} is $(-1)^{(n-r_3+r_1-1)}\,\xi_{r_1r_2}(\overline{r_2+1} \ldots \overline{r_3-1},\ \overline{r_3+1} \ldots n,\ 1,\ 2, \ldots \overline{r_1-1},\ \overline{r_1+1} \ldots \overline{r_2-1})$;

therefore the term which contains

$\xi_{r_2r_3}$ in ξ_{r_1} is $(-1)^{r_3-r_1-1+(n-r_3)(r_3-3)}\,\xi_{r_2r_3}(1,\ 2 \ldots \overline{r_1-1},\ \overline{r_1+1} \ldots \overline{r_2-1},\ \overline{r_2+1} \ldots \overline{r_3-1},\ \overline{r_3+1} \ldots n)$,

$\xi_{r_3r_1}$ in ξ_{r_2} is $(-1)^{r_3-r_2-1+(n-r_1-2)(r_1-1)}\,\xi_{r_3r_1}(1,\ 2 \ldots \overline{r_1-1},\ \overline{r_1+1} \ldots \overline{r_2-1},\ \overline{r_2+1} \ldots \overline{r_3-1},\ \overline{r_3+1} \ldots n)$,

$\xi_{r_1r_2}$ in ξ_{r_3} is $(-1)^{n-r_3+r_1-1+(n-r_2-1)(r_2-2)}\,\xi_{r_1r_2}(1,\ 2 \ldots \overline{r_1-1},\ \overline{r_1+1} \ldots \overline{r_2-1},\ \overline{r_2+1} \ldots \overline{r_3-1},\ \overline{r_3+1} \ldots n)$.

Now since n is odd

$$(-1)^{r_3-r_1-1+(n-r_3)(r_3-3)} = (-1)^{r_3-r_1-1-(r_3-1)^2} = (-1)^{r_3-r_1-r_3^2} = (-1)^{r_3+r_1+r_3^2} = (-1)^{r_1+r_2+r_3},$$

$$(-1)^{r_3-r_2-1+(n-r_1-2)(r_1-1)} = (-1)^{r_3-r_2-1-(r_1-1)^2} = (-1)^{r_3-r_2-r_1^2} = (-1)^{r_3+r_2+r_1^2} = (-1)^{r_1+r_2+r_3},$$

$$(-1)^{(n-r_3+r_1-1)+(n-r_2-1)(r_2-2)} = (-1)^{r_1-r_3-r_2^2} = (-1)^{r_1+r_3+r_2^2} = (-1)^{r_1+r_2+r_3}.$$

Digitized by Google

Now every term in $\frac{d\xi_1}{dx_1}+\ldots\ldots+\frac{d\xi_r}{dx_r}+\ldots\ldots+\frac{d\xi_n}{dx_n}$ contains one first differential coefficient of one of the quantities ξ_{rs}.

The only place where $\frac{d\xi_{r_3r_2}}{dx_{r_1}}$ can occur is in $\frac{d\xi_{r_1}}{dx_{r_1}}$, and there its coefficient is

$$(-1)^{r_1+r_2+r_3}(1, 2, \ldots \overline{r_1-1}, \overline{r_1+1} \ldots \overline{r_2-1}, \overline{r_2+1}, \ldots \overline{r_3-1}, \overline{r_3+1} \ldots n).$$

The only place where $\frac{d\xi_{r_2r_1}}{dx_{r_3}}$ can occur is in $\frac{d\xi_{r_3}}{dx_{r_3}}$, and there its coefficient is the same.

The only place where $\frac{d\xi_{r_1r_3}}{dx_{r_2}}$ can occur is in $\frac{d\xi_{r_2}}{dx_{r_2}}$, and there its coefficient is the same.

But
$$\frac{d\xi_{r_3r_2}}{dx_{r_1}}+\frac{d\xi_{r_2r_1}}{dx_{r_3}}+\frac{d\xi_{r_1r_3}}{dx_{r_2}}=0;$$

therefore
$$\frac{d\xi_1}{dx_1}+\ldots\ldots+\frac{d\xi_r}{dx_r}+\ldots\ldots+\frac{d\xi_n}{dx_n}=0.$$

4. It will now be shown that equations analogous to Helmholtz's Equations of Fluid Motion are satisfied if instead of $\xi_1, \xi_2, \ldots\ldots \xi_n$ there be substituted the quantities $\lambda_1 a_{11}+\lambda_2 a_{21}+\ldots\ldots+\lambda_n a_{n1}$, $\lambda_1 a_{12}+\lambda_2 a_{22}+\ldots\ldots+\lambda_n a_{n2}$, $\ldots\ldots \lambda_1 a_{1n}+\lambda_2 a_{2n}+\ldots\ldots+\lambda_n a_{nn}$, where $\lambda_1, \lambda_2 \ldots\ldots \lambda_n$ are functions of $P_1, P_2 \ldots\ldots P_n$ not t, subject only to the condition

$$\frac{\delta\lambda_1}{\delta P_1}+\frac{\delta\lambda_2}{\delta P_2}+\ldots\ldots+\frac{\delta\lambda_n}{\delta P_n}=0.$$

These equations will not reduce to the identity $0=0$ when n is an even number in general, but they may do so for special values of $\lambda_1, \lambda_2 \ldots \lambda_n$ satisfying the condition to which they are subject.

Since
$$a_{rs}(-1)^{r+s}=\begin{vmatrix} \frac{dP_1}{dx_1} & \ldots\ldots & \frac{dP_1}{dx_{s-1}} & \frac{dP_1}{dx_{s+1}} & \ldots\ldots & \frac{dP_1}{dx_n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{dP_{r-1}}{dx_1} & \ldots\ldots & \frac{dP_{r-1}}{dx_{s-1}} & \frac{dP_{r-1}}{dx_{s+1}} & \ldots\ldots & \frac{dP_{r-1}}{dx_n} \\ \frac{dP_{r+1}}{dx_1} & \ldots\ldots & \frac{dP_{r+1}}{dx_{s-1}} & \frac{dP_{r+1}}{dx_{s+1}} & \ldots\ldots & \frac{dP_{r+1}}{dx_n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{dP_n}{dx_1} & \ldots\ldots & \frac{dP_n}{dx_{s-1}} & \frac{dP_n}{dx_{s+1}} & \ldots\ldots & \frac{dP_n}{dx_n} \end{vmatrix}.$$

Digitized by Google

Therefore $\dfrac{d}{dx_s}\alpha_{rs}(-1)^{r+s}$

$$= \begin{vmatrix} \dfrac{d^2P_1}{dx_1dx_s} & \cdots\cdots & \dfrac{d^2P_1}{dx_{s-1}dx_s} & \dfrac{d^2P_1}{dx_{s+1}dx_s} & \cdots\cdots & \dfrac{d^2P_1}{dx_ndx_s} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{dP_{r-1}}{dx_1} & \cdots\cdots & \dfrac{dP_{r-1}}{dx_{s-1}} & \dfrac{dP_{r-1}}{dx_{s+1}} & \cdots\cdots & \dfrac{dP_{r-1}}{dx_n} \\ \dfrac{dP_{r+1}}{dx_1} & \cdots\cdots & \dfrac{dP_{r+1}}{dx_{s-1}} & \dfrac{dP_{r+1}}{dx_{s+1}} & \cdots\cdots & \dfrac{dP_{r+1}}{dx_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{dP_n}{dx_1} & \cdots\cdots & \dfrac{dP_n}{dx_{s-1}} & \dfrac{dP_n}{dx_{s+1}} & \cdots\cdots & \dfrac{dP_n}{dx_n} \end{vmatrix} + \cdots\cdots + \begin{vmatrix} \dfrac{dP_1}{dx_1} & \cdots\cdots & \dfrac{dP_1}{dx_{s-1}} & \dfrac{dP_1}{dx_{s+1}} & \cdots\cdots & \dfrac{dP_1}{dx_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{d^2P_{r-1}}{dx_1dx_s} & \cdots\cdots & \dfrac{d^2P_{r-1}}{dx_{s-1}dx_s} & \dfrac{d^2P_{r-1}}{dx_{s+1}dx_s} & \cdots\cdots & \dfrac{d^2P_{r-1}}{dx_ndx_s} \\ \dfrac{dP_{r+1}}{dx_1} & \cdots\cdots & \dfrac{dP_{r+1}}{dx_{s-1}} & \dfrac{dP_{r+1}}{dx_{s+1}} & \cdots\cdots & \dfrac{dP_{r+1}}{dx_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{dP_n}{dx_1} & \cdots\cdots & \dfrac{dP_n}{dx_{s-1}} & \dfrac{dP_n}{dx_{s+1}} & \cdots\cdots & \dfrac{dP_n}{dx_n} \end{vmatrix}$$

$$+ \begin{vmatrix} \dfrac{dP_1}{dx_1} & \cdots\cdots & \dfrac{dP_1}{dx_{s-1}} & \dfrac{dP_1}{dx_{s+1}} & \cdots\cdots & \dfrac{dP_1}{dx_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{dP_{r-1}}{dx_1} & \cdots\cdots & \dfrac{dP_{r-1}}{dx_{s-1}} & \dfrac{dP_{r-1}}{dx_{s+1}} & \cdots\cdots & \dfrac{dP_{r-1}}{dx_n} \\ \dfrac{d^2P_{r+1}}{dx_1dx_s} & \cdots\cdots & \dfrac{d^2P_{r+1}}{dx_{s-1}dx_s} & \dfrac{d^2P_{r+1}}{dx_{s+1}dx_s} & \cdots\cdots & \dfrac{d^2P_{r+1}}{dx_ndx_s} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{dP_n}{dx_1} & \cdots\cdots & \dfrac{dP_n}{dx_{s-1}} & \dfrac{dP_n}{dx_{s+1}} & \cdots\cdots & \dfrac{dP_n}{dx_n} \end{vmatrix} + \cdots\cdots + \begin{vmatrix} \dfrac{dP_1}{dx_1} & \cdots\cdots & \dfrac{dP_1}{dx_{s-1}} & \dfrac{dP_1}{dx_{s+1}} & \cdots\cdots & \dfrac{dP_1}{dx_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{dP_{r-1}}{dx_1} & \cdots\cdots & \dfrac{dP_{r-1}}{dx_{s-1}} & \dfrac{dP_{r-1}}{dx_{s+1}} & \cdots\cdots & \dfrac{dP_{r-1}}{dx_n} \\ \dfrac{dP_{r+1}}{dx_1} & \cdots\cdots & \dfrac{dP_{r+1}}{dx_{s-1}} & \dfrac{dP_{r+1}}{dx_{s+1}} & \cdots\cdots & \dfrac{dP_{r+1}}{dx_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{d^2P_n}{dx_1dx_s} & \cdots\cdots & \dfrac{d^2P_n}{dx_{s-1}dx_s} & \dfrac{d^2P_n}{dx_{s+1}dx_s} & \cdots\cdots & \dfrac{d^2P_n}{dx_ndx_s} \end{vmatrix}.$$

Consider now $$\frac{d}{dx_1}\alpha_{r1} + \frac{d}{dx_2}\alpha_{r2} + \ldots + \frac{d}{dx_n}\alpha_{rn}.$$

Calculate in it the coefficient of $\dfrac{d^2P_t}{dx_udx_v}$.

Since $\dfrac{d}{dx_s}\alpha_{rs}(-1)^{r+s}$ contains only second differential coefficients, in which one of the differentiations has been effected with regard to x_s, the terms containing $\dfrac{d^2P_t}{dx_udx_v}$ occur only in $\dfrac{d}{dx_u}\alpha_{ru}$ and $\dfrac{d}{dx_v}\alpha_{rv}$.

The determinant in the first of these which contains it is

$$(-1)^{r+u}\begin{vmatrix} \dfrac{dP_1}{dx_1} & \cdots\cdots & \dfrac{dP_1}{dx_{u-1}} & \dfrac{dP_1}{dx_{u+1}} & \cdots\cdots & \dfrac{dP_1}{dx_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{dP_{r-1}}{dx_1} & \cdots\cdots & \dfrac{dP_{r-1}}{dx_{u-1}} & \dfrac{dP_{r-1}}{dx_{u+1}} & \cdots\cdots & \dfrac{dP_{r-1}}{dx_n} \\ \dfrac{dP_{r+1}}{dx_1} & \cdots\cdots & \dfrac{dP_{r+1}}{dx_{u-1}} & \dfrac{dP_{r+1}}{dx_{u+1}} & \cdots\cdots & \dfrac{dP_{r+1}}{dx_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{d^2P_t}{dx_1dx_u} & \cdots\cdots & \dfrac{d^2P_t}{dx_{u-1}dx_u} & \dfrac{d^2P_t}{dx_{u+1}dx_u} & \cdots\ \dfrac{d^2P_t}{dx_vdx_u} & \cdots\ \dfrac{d^2P_t}{dx_ndx_u} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{dP_n}{dx_1} & \cdots\cdots & \dfrac{dP_n}{dx_{u-1}} & \dfrac{dP_n}{dx_{u+1}} & \cdots\cdots & \dfrac{dP_n}{dx_n} \end{vmatrix},$$

Digitized by Google

and the determinant which contains it in the second of these is

$$(-1)^{r+v}\begin{vmatrix} \frac{dP_1}{dx_1} & \cdots\cdots & \frac{dP_1}{dx_{v-1}} & \frac{dP_1}{dx_{v+1}} & \cdots\cdots & \frac{dP_1}{dx_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{dP_{r-1}}{dx_1} & \cdots\cdots & \frac{dP_{r-1}}{dx_{v-1}} & \frac{dP_{r-1}}{dx_{v+1}} & \cdots\cdots & \frac{dP_{r-1}}{dx_n} \\ \frac{dP_{r+1}}{dx_1} & \cdots\cdots & \frac{dP_{r+1}}{dx_{v-1}} & \frac{dP_{r+1}}{dx_{v+1}} & \cdots\cdots & \frac{dP_{r+1}}{dx_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{d^2P_t}{dx_1dx_v} & \cdots\cdots & \frac{d^2P_t}{dx_udx_v} & \cdots\cdots & \frac{d^2P_t}{dx_{v-1}dx_v} & \frac{d^2P_t}{dx_{v+1}dx_v} & \cdots\cdots & \frac{d^2P_t}{dx_ndx_v} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{dP_n}{dx_1} & \cdots\cdots & \frac{dP_n}{dx_{v-1}} & \frac{dP_n}{dx_{v+1}} & \cdots\cdots & \frac{dP_n}{dx_n} \end{vmatrix}.$$

Therefore the coefficient of $\frac{d^2P_t}{dx_udx_v}$ is

$$(-1)^{t-2+v-2+r+u}\begin{vmatrix} \frac{dP_1}{dx_1} & \cdots\cdots & \frac{dP_1}{dx_{u-1}} & \frac{dP_1}{dx_{u+1}} & \cdots\cdots & \frac{dP_1}{dx_{v-1}} & \frac{dP_1}{dx_{v+1}} & \cdots\cdots & \frac{dP_1}{dx_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{dP_{r-1}}{dx_1} & \cdots\cdots & \frac{dP_{r-1}}{dx_{u-1}} & \frac{dP_{r-1}}{dx_{u+1}} & \cdots\cdots & \frac{dP_{r-1}}{dx_{v-1}} & \frac{dP_{r-1}}{dx_{v+1}} & \cdots\cdots & \frac{dP_{r-1}}{dx_n} \\ \frac{dP_{r+1}}{dx_1} & \cdots\cdots & \frac{dP_{r+1}}{dx_{u-1}} & \frac{dP_{r+1}}{dx_{u+1}} & \cdots\cdots & \frac{dP_{r+1}}{dx_{v-1}} & \frac{dP_{r+1}}{dx_{v+1}} & \cdots\cdots & \frac{dP_{r+1}}{dx_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{dP_{t-1}}{dx_1} & \cdots\cdots & \frac{dP_{t-1}}{dx_{u-1}} & \frac{dP_{t-1}}{dx_{u+1}} & \cdots\cdots & \frac{dP_{t-1}}{dx_{v-1}} & \frac{dP_{t-1}}{dx_{v+1}} & \cdots\cdots & \frac{dP_{t-1}}{dx_n} \\ \frac{dP_{t+1}}{dx_1} & \cdots\cdots & \frac{dP_{t+1}}{dx_{u-1}} & \frac{dP_{t+1}}{dx_{u+1}} & \cdots\cdots & \frac{dP_{t+1}}{dx_{v-1}} & \frac{dP_{t+1}}{dx_{v+1}} & \cdots\cdots & \frac{dP_{t+1}}{dx_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{dP_n}{dx_1} & \cdots\cdots & \frac{dP_n}{dx_{u-1}} & \frac{dP_n}{dx_{u+1}} & \cdots\cdots & \frac{dP_n}{dx_{v-1}} & \frac{dP_n}{dx_{v+1}} & \cdots\cdots & \frac{dP_n}{dx_n} \end{vmatrix} + (-1)^{t-2+u-1+r+v} \text{(same determinant)}.$$

Therefore the coefficient of $\frac{d^2P_t}{dx_udx_v}$ is zero.

Moreover no differential coefficient such as $\frac{d^2P_t}{dx_u^2}$ can occur in any quantity of the form $\frac{d}{dx_v}\alpha_{rs}$.

Therefore
$$\frac{d\alpha_{r1}}{dx_1}+\frac{d\alpha_{r2}}{dx_2}+\ldots+\frac{d\alpha_{rn}}{dx_n}=0 \quad\text{(IX)}.$$

Digitized by Google

Now put successively $s = 1, 2, \ldots n$ in (VI); and substitute $\alpha_{r1}\dfrac{d}{dx_1} + \alpha_{r2}\dfrac{d}{dx_2} + \ldots + \alpha_{rn}\dfrac{d}{dx_n}$ for $A\dfrac{\delta}{\delta P_r}$, and it follows that

$$\left.\begin{array}{l}
\dfrac{\delta}{\delta t}\left(\dfrac{\alpha_{r1}}{\rho}\right) = \dfrac{\alpha_{r1}}{\rho}\dfrac{du_1}{dx_1} + \dfrac{\alpha_{r2}}{\rho}\dfrac{du_1}{dx_2} + \ldots + \dfrac{\alpha_{rn}}{\rho}\dfrac{du_1}{dx_n} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\dfrac{\delta}{\delta t}\left(\dfrac{\alpha_{rs}}{\rho}\right) = \dfrac{\alpha_{r1}}{\rho}\dfrac{du_s}{dx_1} + \dfrac{\alpha_{r2}}{\rho}\dfrac{du_s}{dx_2} + \ldots + \dfrac{\alpha_{rn}}{\rho}\dfrac{du_s}{dx_n} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\dfrac{\delta}{\delta t}\left(\dfrac{\alpha_{rn}}{\rho}\right) = \dfrac{\alpha_{r1}}{\rho}\dfrac{du_n}{dx_1} + \dfrac{\alpha_{r2}}{\rho}\dfrac{du_n}{dx_2} + \ldots + \dfrac{\alpha_{rn}}{\rho}\dfrac{du_n}{dx_n}
\end{array}\right\} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \text{(X)}.$$

The equation (IX) and each of the equations (X) is typical of n equations which can be obtained from it, by substituting for r the numbers $1, 2, \ldots n$ successively.

Multiply the first of those derived from the first equation of (X) by λ_1, the second of those derived from the same equation by λ_2, and so on, the last by λ_n; where $\lambda_1, \lambda_2 \ldots \lambda_n$ are each functions of $P_1, P_2 \ldots P_n$ but not t, and add; treat each of the other equations (X) in like manner, then

$$\frac{\delta}{\delta t}\left(\frac{\lambda_1\alpha_{11} + \lambda_2\alpha_{21} + \ldots + \lambda_n\alpha_{n1}}{\rho}\right) = \frac{\lambda_1\alpha_{11} + \lambda_2\alpha_{21} + \ldots + \lambda_n\alpha_{n1}}{\rho}\frac{du_1}{dx_1} + \frac{\lambda_1\alpha_{12} + \lambda_2\alpha_{22} + \ldots + \lambda_n\alpha_{n2}}{\rho}\frac{du_1}{dx_2}$$
$$+ \ldots + \frac{\lambda_1\alpha_{1n} + \lambda_2\alpha_{2n} + \ldots + \lambda_n\alpha_{nn}}{\rho}\frac{du_1}{dx_n}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\delta}{\delta t}\left(\frac{\lambda_1\alpha_{1s} + \lambda_2\alpha_{2s} + \ldots + \lambda_n\alpha_{ns}}{\rho}\right) = \frac{\lambda_1\alpha_{11} + \lambda_2\alpha_{21} + \ldots + \lambda_n\alpha_{n1}}{\rho}\frac{du_s}{dx_1} + \frac{\lambda_1\alpha_{12} + \lambda_2\alpha_{22} + \ldots + \lambda_n\alpha_{n2}}{\rho}\frac{du_s}{dx_2}$$
$$+ \ldots + \frac{\lambda_1\alpha_{1n} + \lambda_2\alpha_{2n} + \ldots + \lambda_n\alpha_{nn}}{\rho}\frac{du_s}{dx_n}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\delta}{\delta t}\left(\frac{\lambda_1\alpha_{1n} + \lambda_2\alpha_{2n} + \ldots + \lambda_n\alpha_{nn}}{\rho}\right) = \frac{\lambda_1\alpha_{11} + \lambda_2\alpha_{21} + \ldots + \lambda_n\alpha_{n1}}{\rho}\frac{du_n}{dx_1} + \frac{\lambda_1\alpha_{12} + \lambda_2\alpha_{22} + \ldots + \lambda_n\alpha_{n2}}{\rho}\frac{du_n}{dx_2}$$
$$+ \ldots + \frac{\lambda_1\alpha_{1n} + \lambda_2\alpha_{2n} + \ldots + \lambda_n\alpha_{nn}}{\rho}\frac{du_n}{dx_n}$$

and from the equations typified by (IX) can be derived

$$\left(\lambda_1\frac{d\alpha_{11}}{dx_1} + \lambda_2\frac{d\alpha_{21}}{dx_1} + \ldots + \lambda_n\frac{d\alpha_{n1}}{dx_1}\right) + \ldots\ldots + \left(\lambda_1\frac{d\alpha_{1s}}{dx_s} + \lambda_2\frac{d\alpha_{2s}}{dx_s} + \ldots + \lambda_n\frac{d\alpha_{ns}}{dx_s}\right)$$
$$+ \ldots + \left(\lambda_1\frac{d\alpha_{1n}}{dx_n} + \lambda_2\frac{d\alpha_{2n}}{dx_n} + \ldots + \lambda_n\frac{d\alpha_{nn}}{dx_n}\right) = 0,$$

Digitized by Google

which can be written

$$\frac{d}{dx_1}(\lambda_1\alpha_{11}+\lambda_2\alpha_{21}+\ldots+\lambda_n\alpha_{n1})+\ldots\ldots+\frac{d}{dx_s}(\lambda_1\alpha_{1s}+\lambda_2\alpha_{2s}+\ldots+\lambda_n\alpha_{ns})$$
$$+\ldots+\frac{d}{dx_n}(\lambda_1\alpha_{1n}+\lambda_2\alpha_{2n}+\ldots+\lambda_n\alpha_{nn})=0,$$

if $\left(\alpha_{11}\frac{d\lambda_1}{dx_1}+\ldots+\alpha_{1s}\frac{d\lambda_1}{dx_s}+\ldots\ldots+\alpha_{1n}\frac{d\lambda_1}{dx_n}\right)+\left(\alpha_{21}\frac{d\lambda_2}{dx_1}+\ldots+\alpha_{2s}\frac{d\lambda_2}{dx_s}+\ldots+\alpha_{2n}\frac{d\lambda_2}{dx_n}\right)$

$$+\ldots+\left(\alpha_{n1}\frac{d\lambda_n}{dx_1}+\ldots+\alpha_{ns}\frac{d\lambda_n}{dx_s}+\ldots+\alpha_{nn}\frac{d\lambda_n}{dx_n}\right)=0.$$

But $$\alpha_{r1}\frac{d}{dx_1}+\ldots+\alpha_{rs}\frac{d}{dx_s}+\ldots+\alpha_{rn}\frac{d}{dx_n}=A\frac{\delta}{\delta P_r}.$$

Therefore the condition to which $\lambda_1, \lambda_2 \ldots \lambda_n$ are subject becomes $A\left(\frac{\delta\lambda_1}{\delta P_1}+\frac{\delta\lambda_2}{\delta P_2}+\ldots\frac{\delta\lambda_n}{\delta P_n}\right)=0.$

The quantities $\lambda_1, \lambda_2 \ldots \lambda_n$ can be chosen in an infinite number of ways so as to satisfy this condition.

Thus there are an infinitely great number of groups of quantities, such as

$$\lambda_1\alpha_{11}+\lambda_2\alpha_{21}+\ldots+\lambda_n\alpha_{n1},\ldots\ldots\ \lambda_1\alpha_{1s}+\lambda_2\alpha_{2s}+\ldots+\lambda_n\alpha_{ns},\ldots\ldots\ \lambda_1\alpha_{1n}+\lambda_2\alpha_{2n}+\ldots+\lambda_n\alpha_{nn},$$

which satisfy equations analogous to Helmholtz's Equations of Fluid Motion; and these Equations do not in general reduce to identities when n is even; as do those analogous to Helmholtz's Equations for that case.

5. The next point is to express $\xi_1, \xi_2, \ldots \xi_n$ in the forms

$$\lambda_1\alpha_{11}+\lambda_2\alpha_{21}+\lambda_3\alpha_{31}+\ldots+\lambda_n\alpha_{n1},\quad \lambda_1\alpha_{12}+\lambda_2\alpha_{22}+\lambda_3\alpha_{32}+\ldots+\lambda_n\alpha_{n2},\ldots\ldots\lambda_1\alpha_{1n}+\lambda_2\alpha_{2n}+\lambda_3\alpha_{3n}+\ldots+\lambda_n\alpha_{nn}.$$

ξ_1 is the Pfaffian $(2, 3, 4, 5 \ldots \overline{n-1}, n)$ and is

$$=\xi_{23}\xi_{45}\ldots\xi_{n-1n}+\ldots\pm\xi_{t_2t_3}\xi_{t_4t_5}\ldots\xi_{t_{n-1}t_n}+\ldots,$$

where $t_2t_3,\ t_4t_5, \ldots t_{n-1}t_n$ is any permutation of $23,\ 45,\ \ldots\ \overline{n-1}n$ such that every pair in it is not the same as every pair in $23,\ 45,\ \ldots\ \overline{n-1}n$; and $\pm$ denotes that the sign appropriate to that permutation has to be affixed. The number of such permutations is $\dfrac{\lfloor n-1}{2^{\frac{n-1}{2}}\left\lfloor\frac{n-1}{2}\right.}$.

$$\xi_{\rho\sigma}=\frac{du_\rho}{dx_\sigma}-\frac{du_\sigma}{dx_\rho}=\left(\frac{df_1}{dx_\sigma}\frac{dP_1}{dx_\rho}+\frac{df_2}{dx_\sigma}\frac{dP_2}{dx_\rho}+\ldots+\frac{df_n}{dx_\sigma}\frac{dP_n}{dx_\rho}\right)-\left(\frac{df_1}{dx_\rho}\frac{dP_1}{dx_\sigma}+\frac{df_2}{dx_\rho}\frac{dP_2}{dx_\sigma}+\ldots+\frac{df_n}{dx_\rho}\frac{dP_n}{dx_\sigma}\right)$$

$$=\left(\frac{\delta f_1}{\delta P_2}-\frac{\delta f_2}{\delta P_1}\right)\left(\frac{dP_1}{dx_\rho}\frac{dP_2}{dx_\sigma}-\frac{dP_2}{dx_\rho}\frac{dP_1}{dx_\sigma}\right)+\left(\frac{\delta f_1}{\delta P_3}-\frac{\delta f_3}{\delta P_1}\right)\left(\frac{dP_1}{dx_\rho}\frac{dP_3}{dx_\sigma}-\frac{dP_3}{dx_\rho}\frac{dP_1}{dx_\sigma}\right)+\ldots+\left(\frac{\delta f_1}{\delta P_n}-\frac{\delta f_n}{\delta P_1}\right)\left(\frac{dP_1}{dx_\rho}\frac{dP_n}{dx_\sigma}-\frac{dP_n}{dx_\rho}\frac{dP_1}{dx_\sigma}\right)$$
$$+\left(\frac{\delta f_2}{\delta P_3}-\frac{\delta f_3}{\delta P_2}\right)\left(\frac{dP_2}{dx_\rho}\frac{dP_3}{dx_\sigma}-\frac{dP_3}{dx_\rho}\frac{dP_2}{dx_\sigma}\right)+\ldots+\left(\frac{\delta f_2}{\delta P_n}-\frac{\delta f_n}{\delta P_2}\right)\left(\frac{dP_2}{dx_\rho}\frac{dP_n}{dx_\sigma}-\frac{dP_n}{dx_\rho}\frac{dP_2}{dx_\sigma}\right)$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+\left(\frac{\delta f_{n-1}}{\delta P_n}-\frac{\delta f_n}{\delta P_{n-1}}\right)\left(\frac{dP_{n-1}}{dx_\rho}\frac{dP_n}{dx_\sigma}-\frac{dP_n}{dx_\rho}\frac{dP_{n-1}}{dx_\sigma}\right).$$

Digitized by Google

If therefore λ_{rs} stand for $\frac{\delta f_r}{\delta P_s} - \frac{\delta f_s}{\delta P_r}$, and $P^{rs}_{\rho\sigma}$ for $\frac{dP_r}{dx_\rho}\frac{dP_s}{dx_\sigma} - \frac{dP_s}{dx_\rho}\frac{dP_r}{dx_\sigma}$,

$\xi_{\rho\sigma} = \Sigma\lambda_{rs}P^{rs}_{\rho\sigma}$, the summation being taken for all possible values of r, s between 1 and n inclusive, such that r is less than s.

In the expression $P^{rs}_{\rho\sigma}$, ρ and σ will be called in this investigation the suffixes.

Substituting for every term $\xi_{\rho\sigma}$ its value $\Sigma\lambda_{rs}P^{rs}_{\rho\sigma}$ in ξ_1, it is required to know the coefficient of $\lambda_{r_1s_1}\lambda_{r_2s_2}\ldots\lambda_{r_{\frac{n-1}{2}}s_{\frac{n-1}{2}}}$.

It occurs in each term. If in the term $\xi_{23}\xi_{45}\ldots\xi_{n-1,n}$, $\lambda_{r_1s_1}$ be taken from ξ_{23}, $\lambda_{r_2s_2}$ from ξ_{45}, and so on; its coefficient will be $P^{r_1s_1}_{23}P^{r_2s_2}_{45}\ldots P^{r_{\frac{n-1}{2}}s_{\frac{n-1}{2}}}_{n-1,n}$; but as $\lambda_{r_1s_1}$ may be taken from any factor of the product, the coefficient is

$P^{r_1s_1}_{23}P^{r_2s_2}_{45}\ldots P^{r_{\frac{n-1}{2}}s_{\frac{n-1}{2}}}_{n-1,n}$ + all such products as can be obtained from this by interchanging pairs of suffixes.

Hence the coefficient of $\lambda_{r_1s_1}\lambda_{r_2s_2}\ldots\lambda_{r_{\frac{n-1}{2}}s_{\frac{n-1}{2}}}$ in ξ_1 is

$P^{r_1s_1}_{23}P^{r_2s_2}_{45}\ldots P^{r_{\frac{n-1}{2}}s_{\frac{n-1}{2}}}_{n-1,n}$ + all terms which can be obtained from this by interchanging pairs of suffixes

+ .....

$\pm$ ($P^{r_1s_1}_{t_2t_3}P^{r_2s_2}_{t_4t_5}\ldots P^{r_{\frac{n-1}{2}}s_{\frac{n-1}{2}}}_{t_{n-1}t_n}$ + all terms which can be obtained from this by interchanging pairs of suffixes)

+ .....

Now the sum of all these products

$$= \begin{vmatrix} \frac{dP_{r_1}}{dx_2}, & \frac{dP_{r_1}}{dx_3} \cdots\cdots & \frac{dP_{r_1}}{dx_n} \\ \frac{dP_{s_1}}{dx_2}, & \frac{dP_{s_1}}{dx_3} \cdots\cdots & \frac{dP_{s_1}}{dx_n} \\ \frac{dP_{r_2}}{dx_2}, & \frac{dP_{r_2}}{dx_3} \cdots\cdots & \frac{dP_{r_2}}{dx_n} \\ \frac{dP_{s_2}}{dx_2}, & \frac{dP_{s_2}}{dx_3} \cdots\cdots & \frac{dP_{s_2}}{dx_n} \\ \cdots & \cdots & \cdots \\ \frac{dP_{r_{\frac{n-1}{2}}}}{dx_2}, & \frac{dP_{r_{\frac{n-1}{2}}}}{dx_3} \cdots\cdots & \frac{dP_{r_{\frac{n-1}{2}}}}{dx_n} \\ \frac{dP_{s_{\frac{n-1}{2}}}}{dx_2}, & \frac{dP_{s_{\frac{n-1}{2}}}}{dx_3} \cdots\cdots & \frac{dP_{s_{\frac{n-1}{2}}}}{dx_n} \end{vmatrix}.$$

For consider the products; each product is the product of $\frac{n-1}{2}$ terms of form $P^{rs}_{\rho\sigma}$, which is itself the difference of two products.

Digitized by Google

Each of the last products is the product of two constituents of the determinant. Thus any term of the form $\pm P_{t_2t_3}^{r_1s_1} P_{t_4t_5}^{r_2s_2} \dots P_{t_{n-1}t_n}^{r_{\frac{n-1}{2}}s_{\frac{n-1}{2}}}$ contains $2^{\frac{n-1}{2}}$ products, each of which is an element of the determinant in question with the right sign. Moreover there are $\dfrac{\lfloor n-1}{2^{\frac{n-1}{2}}}$ such terms. Therefore the coefficient of $\lambda_{r_1s_1}\lambda_{r_2s_2}\dots\lambda_{r_{\frac{n-1}{2}}s_{\frac{n-1}{2}}}$ contains $\lfloor n-1$ elements of the determinant; and since all the terms in that coefficient are different, the coefficient must be equal to the determinant.

Hence in the expression for ξ_1 there is no term $\lambda_{r_1s_1}\lambda_{r_2s_2}\dots\lambda_{r_{\frac{n-1}{2}}s_{\frac{n-1}{2}}}$ unless all the numbers $r_1s_1\, r_2s_2 \dots r_{\frac{n-1}{2}}s_{\frac{n-1}{2}}$ be different, for if any two were the same the determinant would vanish.

Therefore $r_1s_1\, r_2s_2 \dots r_{\frac{n-1}{2}}s_{\frac{n-1}{2}}$ must be $n-1$ of the numbers 1 to n.

There will be n cases to consider, viz. when they do not include 1, 2, ... n respectively.

When they do not include 1,

the coefficient of $\lambda_{23}\lambda_{45}\dots\lambda_{n-1\,n}$ is the determinant α_{11};

the coefficient of $\lambda_{r_1s_1}\lambda_{r_2s_2}\dots\lambda_{r_{\frac{n-1}{2}}s_{\frac{n-1}{2}}}$ is $\pm\alpha_{11}$, where $\pm$ is the sign appropriate to the permutation $r_1s_1\, r_2s_2 \dots r_{\frac{n-1}{2}}s_{\frac{n-1}{2}}$ considered as derived from $2345\dots n-1.n$.

Therefore ξ_1 contains the term

$$\alpha_{11}(\lambda_{23}\lambda_{45}\dots\lambda_{n-1\,n}+\dots\pm\lambda_{r_1s_1}\lambda_{r_2s_2}\dots\lambda_{r_{\frac{n-1}{2}}s_{\frac{n-1}{2}}}+\dots)=\alpha_{11}.\lambda_1,$$

where λ_1 is the square root of the coefficient of λ_{11} in the determinant Λ,

$$\begin{vmatrix} \lambda_{11}, & \lambda_{12} \dots\dots \lambda_{1n} \\ \lambda_{21}, & \lambda_{22} \dots\dots \lambda_{2n} \\ \dots & \dots\dots \\ \dots & \dots\dots \\ \lambda_{n1}, & \lambda_{n2} \dots\dots \lambda_{nn} \end{vmatrix}.$$

When $r_1s_1\, r_2s_2 \dots r_{\frac{n-1}{2}}s_{\frac{n-1}{2}}$ do not include 2, the coefficient of $\lambda_{34}\lambda_{56}\dots\lambda_{n1}$ is the determinant

$$\begin{vmatrix} \frac{dP_3}{dx_2}, & \frac{dP_3}{dx_3} \dots\dots & \frac{dP_3}{dx_n} \\ \frac{dP_4}{dx_2}, & \frac{dP_4}{dx_3} \dots\dots & \frac{dP_4}{dx_n} \\ \dots & \dots\dots & \dots \\ \dots & \dots\dots & \dots \\ \frac{dP_n}{dx_2}, & \frac{dP_n}{dx_3} \dots\dots & \frac{dP_n}{dx_n} \\ \frac{dP_1}{dx_2}, & \frac{dP_1}{dx_3} \dots\dots & \frac{dP_1}{dx_n} \end{vmatrix} = \alpha_{21}.$$

Digitized by Google

And in like manner it may be shown that ξ_1 contains the term $\lambda_2\alpha_{21}$, where λ_2 is the square root of the coefficient of λ_{22} in Λ, and so on.

Hence $$\xi_1 = \lambda_1\alpha_{11} + \lambda_2\alpha_{21} + \lambda_3\alpha_{31} + \ldots + \lambda_n\alpha_{n1},$$

and similarly, $$\xi_2 = \lambda_1\alpha_{12} + \lambda_2\alpha_{22} + \lambda_3\alpha_{32} + \ldots + \lambda_n\alpha_{n2},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\xi_n = \lambda_1\alpha_{1n} + \lambda_2\alpha_{2n} + \lambda_3\alpha_{3n} + \ldots + \lambda_n\alpha_{nn},$$

$\lambda_1, \lambda_2 \ldots \lambda_n$ being the square roots of the coefficients of $\lambda_{11}, \lambda_{22} \ldots \lambda_{nn}$ respectively in Λ.

Now observing that

$\xi_1, \xi_2, \ldots \xi_n$ are the square roots of the coefficients of $\xi_{11}, \xi_{22}, \ldots \xi_{nn}$ in the determinant $\begin{vmatrix} \xi_{11} & \xi_{12} & \ldots & \xi_{1n} \\ \xi_{21} & \xi_{22} & \ldots & \xi_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \xi_{n1} & \xi_{n2} & \ldots & \xi_{nn} \end{vmatrix}$ and that $\xi_{rs} = \dfrac{du_r}{dx_s} - \dfrac{du_s}{dx_r}$	$\lambda_1, \lambda_2, \ldots \lambda_n$ are the square roots of the coefficients of $\lambda_{11}, \lambda_{22}, \ldots \lambda_{nn}$ in the determinant $\begin{vmatrix} \lambda_{11} & \lambda_{12} & \ldots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \ldots & \lambda_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \lambda_{n1} & \lambda_{n2} & \ldots & \lambda_{nn} \end{vmatrix}$ and that $\lambda_{rs} = \dfrac{\delta f_r}{\delta P_s} - \dfrac{\delta f_s}{\delta P_r}$

it follows that the relations between $\xi_1, \xi_2 \ldots \xi_n$ and $x_1, x_2 \ldots x_n$ through the quantities $u_1, u_2 \ldots u_n$ are the same as those existing between $\lambda_1, \lambda_2 \ldots \lambda_n$ and $P_1, P_2 \ldots P_n$ through the quantities $f_1, f_2 \ldots f_n$.

And since in proving the equation $\dfrac{d\xi_1}{dx_1} + \ldots + \dfrac{d\xi_r}{dx_r} + \ldots + \dfrac{d\xi_n}{dx_n} = 0$ in Art. 3, the quantities $u_1, u_2 \ldots u_n$ might have been supposed perfectly arbitrary functions of $x_1, x_2 \ldots n_n$, it follows that

$$\frac{\delta\lambda_1}{\delta P_1} + \ldots + \frac{\delta\lambda_r}{\delta P_r} + \ldots + \frac{\delta\lambda_n}{\delta P_n} = 0.$$

Hence $\xi_1, \xi_2, \ldots \xi_n$ have been expressed in the form of the last article.

Now consider the case in which n is even.

$$\sqrt{\Xi} = \xi_{12}\xi_{34} \ldots \xi_{n-1,n} + \ldots \pm \xi_{t_1t_2}\xi_{t_3t_4} \ldots \xi_{t_{n-1}t_n} + \ldots\ldots$$

The coefficient of $\lambda_{r_1s_1}\lambda_{r_2s_2} \ldots \lambda_{r_{\frac{n}{2}}s_{\frac{n}{2}}}$ is

$P_{12}^{r_1s_1} P_{34}^{r_2s_2} \ldots P_{n-1,n}^{r_{\frac{n}{2}}s_{\frac{n}{2}}}$ + all terms which can be obtained from this by interchanging pairs of suffixes

+ .....

$\pm (P_{t_1t_2}^{r_1s_1} P_{t_3t_4}^{r_2s_2} \ldots P_{t_{n-1}t_n}^{r_{\frac{n}{2}}s_{\frac{n}{2}}}$ + all terms which can be obtained from this by interchanging pairs of suffixes)

+ .....

Digitized by Google

$$= \begin{vmatrix} \frac{dP_{r_1}}{dx_1} & \cdots\cdots & \frac{dP_{r_1}}{dx_n} \\ \frac{dP_{s_1}}{dx_1} & \cdots\cdots & \frac{dP_{s_1}}{dx_n} \\ \frac{dP_{r_2}}{dx_1} & \cdots\cdots & \frac{dP_{r_2}}{dx_n} \\ \frac{dP_{s_2}}{dx_1} & \cdots\cdots & \frac{dP_{s_2}}{dx_n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \frac{dP_{r_{\frac{n}{2}}}}{dx_1} & \cdots\cdots & \frac{dP_{r_{\frac{n}{2}}}}{dx_n} \\ \frac{dP_{s_{\frac{n}{2}}}}{dx_1} & \cdots\cdots & \frac{dP_{s_{\frac{n}{2}}}}{dx_n} \end{vmatrix}.$$

Therefore the coefficient of $\lambda_{r_1 s_1}\lambda_{r_2 s_2} \dots \lambda_{r_{\frac{n}{2}} s_{\frac{n}{2}}}$ will be $\pm A$, if $r_1 s_1 r_2 s_2 \dots r_{\frac{n}{2}} s_{\frac{n}{2}}$ be all different; the sign will be the sign appropriate to the permutation $r_1 s_1 r_2 s_2 \dots r_{\frac{n}{2}} s_{\frac{n}{2}}$ considered as derived from $1, 2, 3, 4 \dots n$. In all other cases the determinant will vanish;

$$\therefore \sqrt{\Xi} = A\,(\lambda_{12}\lambda_{34} \dots \lambda_{n-1.n} + \dots \pm \lambda_{r_1 s_1}\lambda_{r_2 s_2} \dots \lambda_{r_{\frac{n}{2}} s_{\frac{n}{2}}} + \dots\dots),$$

$$\therefore \sqrt{\Xi} = A\,.\sqrt{\Lambda}.$$

6. To prove that, using the language of ordinary fluid motion, the vortex lines $\frac{dx_1}{\xi_1} = \frac{dx_2}{\xi_2} = \dots = \frac{dx_n}{\xi_n}$ always contain the same particles of fluid. [Analytically, the sets of values of $x_1, x_2 \dots x_n$ which satisfy these equations always satisfy $(n-1)$ equations of the form $\Theta = \text{constant}$, where Θ is an integral of the Equation (III).]

If such a vortex line lie on the locus $\Theta(x_1, x_2 \dots x_n t) = 0$, then it is known that

$$\xi_1 \frac{d\Theta}{dx_1} + \xi_2 \frac{d\Theta}{dx_2} + \dots + \xi_n \frac{d\Theta}{dx_n} = 0.$$

Substituting for $\xi_1, \xi_2 \dots \xi_n$ their values from the last article, and expressing the differential coefficients of Θ in terms of the independent variables $P_1, P_2 \dots P_n t$

$$\begin{aligned} &(\lambda_1 \alpha_{11} + \lambda_2 \alpha_{21} + \dots + \lambda_n \alpha_{n1})\left(\frac{\delta\Theta}{\delta P_1}\frac{dP_1}{dx_1} + \frac{\delta\Theta}{\delta P_2}\frac{dP_2}{dx_1} + \dots + \frac{\delta\Theta}{\delta P_n}\frac{dP_n}{dx_1}\right) \\ &+ (\lambda_1 \alpha_{12} + \lambda_2 \alpha_{22} + \dots + \lambda_n \alpha_{n2})\left(\frac{\delta\Theta}{\delta P_1}\frac{dP_1}{dx_2} + \frac{\delta\Theta}{\delta P_2}\frac{dP_2}{dx_2} + \dots + \frac{\delta\Theta}{\delta P_n}\frac{dP_n}{dx_2}\right) \\ &+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &+ (\lambda_1 \alpha_{1n} + \lambda_2 \alpha_{2n} + \dots + \lambda_n \alpha_{nn})\left(\frac{\delta\Theta}{\delta P_1}\frac{dP_1}{dx_n} + \frac{\delta\Theta}{\delta P_2}\frac{dP_2}{dx_n} + \dots + \frac{\delta\Theta}{\delta P_n}\frac{dP_n}{dx_n}\right) = 0; \end{aligned}$$

$$\therefore A\left\{\lambda_1 \frac{\delta\Theta}{\delta P_1} + \lambda_2 \frac{\delta\Theta}{\delta P_2} + \dots + \lambda_n \frac{\delta\Theta}{\delta P_n}\right\} = 0.$$

Digitized by Google

Now this equation has $(n-1)$ independent integrals $\Theta_1, \Theta_2 \dots \Theta_{n-1}$ each of which is a function of $P_1, P_2 \dots P_n$ (not t), i.e. the vortex lines $\frac{dx_1}{\xi_1} = \frac{dx_2}{\xi_2} = \dots = \frac{dx_n}{\xi_n}$ are the intersections of $(n-1)$ loci which always contain the same particles.

Therefore the vortex lines always contain the same particles.

The same argument shows that the lines

$$\frac{dx_1}{\lambda_1 a_{11} + \lambda_2 a_{21} + \dots + \lambda_n a_{n1}} = \frac{dx_2}{\lambda_1 a_{12} + \lambda_2 a_{22} + \dots + \lambda_n a_{n2}} = \dots\dots = \frac{dx_n}{\lambda_1 a_{1n} + \lambda_2 a_{2n} + \dots + \lambda_n a_{nn}}$$

are always the intersections of $(n-1)$ loci whose equations satisfy the equation (III), whenever $\lambda_1 \lambda_2 \dots \lambda_n$ are any functions of $P_1 P_2 \dots P_n$.

7. It has been shown in Art. 2, that

$$u_1 dx_1 + \dots + u_n dx_n = dK + f_1 dP_1 + \dots + f_n dP_n.$$

But this form contains more quantities than Clebsch's form. It is required to reduce this to Clebsch's form in which n is supposed to be an odd number, and to investigate the meaning of the reduction.

If n be odd, it is known that $f_1 dP_1 + \dots + f_n dP_n$ is always reducible to the form

$$d\Phi + Q_{\frac{n+1}{2}} dQ_1 + Q_{\frac{n+3}{2}} dQ_2 + \dots + Q_{n-1} dQ_{\frac{n-1}{2}};$$

where Φ, $Q_1 \dots Q_{\frac{n-1}{2}}$, $Q_{\frac{n+1}{2}} \dots Q_{n-1}$ are functions of $f_1 \dots f_n P_1 \dots P_n$, and therefore of $P_1 \dots P_n$ only.

Therefore $\quad u_1 dx_1 + \dots + u_n dx_n = d(K + \Phi) + Q_{\frac{n+1}{2}} dQ_1 + Q_{\frac{n+3}{2}} dQ_2 + \dots + Q_{n-1} dQ_{\frac{n-1}{2}}.$

Now as Q_1, $Q_2 \dots Q_{\frac{n-1}{2}}$, $Q_{\frac{n+1}{2}} \dots Q_{n-1}$ are $(n-1)$ independent integrals of (III), they may together with one other, which may be called Q_n, be taken in the place of P_1, $P_2 \dots P_n$. Also since $\frac{\delta(K+\Phi)}{\delta t} = \frac{\delta K}{\delta t}$, $K + \Phi$ may be used in place of K. Hence reverting to the original notation $u_1 dx_1 + \dots + u_n dx_n$ may always be reduced to the form

$$dK + P_{\frac{n+1}{2}} dP_1 + P_{\frac{n+3}{2}} dP_2 + \dots + P_{n-1} dP_{\frac{n-1}{2}},$$

and this is the form which Clebsch has given for $u_1 dx_1 + u_2 dx_2 + \dots + u_n dx_n$.

To examine the meaning of it, it is necessary to calculate the values of ξ_1, $\xi_2 \dots \xi_n$ for this form; and as a preliminary step to this the determinant Λ.

Now this form is a particular case of that of Art. 2, viz. when

$$f_1 = P_{\frac{n+1}{2}}, f_2 = P_{\frac{n+3}{2}} \dots f_{\frac{n-1}{2}} = P_{n-1}, f_{\frac{n+1}{2}} = 0 \dots f_{n-1} = 0, f_n = 0.$$

Digitized by Google

The determinant corresponding to Λ, becomes

$$\left|\begin{array}{cccc|cccc|c}
0 & 0 & & 0 & 1 & 0 & & 0 & 0 \\
0 & 0 & & 0 & 0 & 1 & & 0 & 0 \\
 & & & & & & \ddots & & \\
0 & 0 & & 0 & 0 & 0 & & 1 & 0 \\
\hline
-1 & 0 & & 0 & 0 & 0 & & 0 & 0 \\
0 & -1 & & 0 & 0 & 0 & & 0 & 0 \\
 & & \ddots & & & & & & \\
0 & 0 & & -1 & 0 & 0 & & 0 & 0 \\
\hline
0 & 0 & & 0 & 0 & 0 & & 0 & 0
\end{array}\right|$$

The only one of the quantities $\lambda_1, \lambda_2 \ldots \lambda_n$ which does not vanish is the last, and that is 1.

Hence $$\xi_1 = \alpha_{n1}, \quad \xi_2 = \alpha_{n2}, \ldots \xi_n = \alpha_{nn}.$$

Now if Θ be a locus containing the lines

$$\frac{dx_1}{\xi_1} = \frac{dx_2}{\xi_2} = \ldots = \frac{dx_n}{\xi_n},$$

$$\xi_1 \frac{d\Theta}{dx_1} + \xi_2 \frac{d\Theta}{dx_2} + \ldots + \xi_n \frac{d\Theta}{dx_n} = 0,$$

therefore, $$\alpha_{n1} \frac{d\Theta}{dx_1} + \alpha_{n2} \frac{d\Theta}{dx_2} + \ldots + \alpha_{nn} \frac{d\Theta}{dx_n} = 0,$$

therefore, $$\left|\begin{array}{cccc}
\frac{dP_1}{dx_1} & \frac{dP_1}{dx_2} & \ldots\ldots & \frac{dP_1}{dx_n} \\
\frac{dP_2}{dx_1} & \frac{dP_2}{dx_2} & \ldots\ldots & \frac{dP_2}{dx_n} \\
\ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\frac{dP_{n-1}}{dx_1} & \frac{dP_{n-1}}{dx_2} & \ldots\ldots & \frac{dP_{n-1}}{dx_n} \\
\frac{d\Theta}{dx_1} & \frac{d\Theta}{dx_2} & \ldots\ldots & \frac{d\Theta}{dx_n}
\end{array}\right| = 0.$$

This is satisfied by $\Theta = P_1$, $\Theta = P_2 \ldots \Theta = P_{\frac{n-1}{2}}$, $\Theta = P_{\frac{n+1}{2}} \ldots \Theta = P_{n-1}$.

Therefore the loci

$$P_1 = \text{constant}, \ P_2 = \text{constant}, \ldots P_{\frac{n-1}{2}} = \text{constant}, \ P_{\frac{n+1}{2}} = \text{constant}, \ldots P_{n-1} = \text{constant},$$

intersect in the vortex lines, and this shows the difference between Clebsch's form for $u_1 dx_1 + \ldots + u_n dx_n$ and that which has been given in Art. 2, of which Clebsch's form is a particular case.

For a proof of the same theorem in the case $n = 3$, see an Article by the writer in *The Quarterly Journal of Pure and Applied Mathematics*, No. 65, page 11.

Digitized by Google

II. *On the measurement of Temperature by Water-vapour pressure.* By W. N. Shaw, M.A. (Plate I.)

[Read *November* 26, 1883.]

It is well known that the indications of a mercury thermometer are liable to errors that cannot be neglected when temperature measurements are carried to the degree of accuracy which in practice is frequently required. But at the same time, the mercury thermometer is so convenient an instrument in practical use that the plan generally adopted in measuring temperatures is to use a mercury thermometer, in spite of the errors to which it is liable, applying corrections, independently determined in the case of each thermometer, which enable us to refer the indications to a standard scale of temperature. The errors referred to arise from two causes and we may refer to them as separate errors, (1) the initial error arising from inequalities in the bore of the tube and the irregular expansion of the glass envelope between the limits of temperature for which the thermometer is used, and (2) the secular error arising from the gradual alteration of volume of the glass envelope which takes place after the thermometer has been graduated. This error increases continually although only slightly, with the lapse of time, but will be of considerable magnitude, amounting in some thermometers to as much as half a degree Centigrade, unless the instrument is allowed to remain for some six months or more after being filled before it is graduated.

The effect of this secular change is to produce an apparent elevation of the freezing-point as measured by the thermometer, and all other temperatures will be likewise increased, but the amount of apparent increase of higher temperatures diminishes as we rise on the scale, and would probably disappear if the thermometer were raised to the temperature at which it was filled, and the error would not then reappear on cooling until after the lapse of considerable time. In consequence of this the secular error will be altered if the thermometer is raised to any considerable temperature, and it is therefore necessary to use a particular mercury thermometer only for a small range of temperature within which the secular error may be regarded as constant, and we may then determine the secular error for any point within that range at the time the thermometer is used and correct for it by subtracting its value from any observed reading previously corrected for the initial error which remains always practically constant for the same thermometer.

In the present paper I propose only to deal with temperatures between 0°C. and 30°C., and within that range we may assume the secular error to be constant and equal

Digitized by Google

to its value at the freezing-point, so that it can always be determined by observing the reading of the thermometer when immersed in melting ice in the usual way.

Methods at present in use for obtaining comparable readings with mercury thermometers.

The secular error may be determined at any time, as already mentioned, by immersing the thermometer in melting ice; for determining the initial error we have at present a choice of three methods.

(1) By determining two fixed temperatures such as the freezing-point and boiling-point and calibrating the tube between those limits, we may determine the error for each degree due to the want of uniformity of the tube; and any two mercury thermometers thus corrected will be comparable, provided the expansion of the glass is uniform between the limits. The objection to this method is that the only fixed points easily obtainable are the freezing-point and boiling-point, and these two do not generally both appear on a sensitive thermometer in ordinary use. And moreover the raising of the thermometer to the boiling-point temporarily alters the freezing-point and thus introduces an uncertainty in the table of corrections.

(2) By comparison with an air-thermometer. It has been shewn by Regnault that any two air-thermometers give strictly comparable readings when their freezing and boiling-points have once been determined; the given thermometer may therefore be compared with an air-thermometer for different points on its scale by reading both at the same temperature.

This is a very practical method, but the air-thermometer requires very careful manipulation, and we have to face the difficulty of keeping the two thermometers at the same temperature during the observation.

(3) By comparison with Kew standard thermometers. The Kew Observatory undertakes to compare with their standards any thermometer sent to them by simultaneous readings of the thermometers when maintained at the same temperature by a special apparatus designed for the purpose. The Kew standard thermometers consist of instruments made of the same kind of glass, carefully annealed, and kept in stock for some years before being graduated. They are frequently referred to the fixed points, melting ice and boiling water, and have been compared with each other, so that a reliable standard scale of temperature has been established.

A table of corrections to 0·1 degree is issued by the Observatory for each thermometer compared with the Kew standard for every 10° F. or 5° C. This table gives the initial error together with the secular error at the time of comparison. At any other time the secular error can be determined by a reading of the freezing-point and the proper table of corrections formed for that time.

I have had occasion during the past few years to employ thermometers which required correction, and I have therefore had them compared at Kew, some of them more than once. Their secular errors have been determined as occasion required. I append a table of the results of these comparisons which will serve to shew what the magnitude

Digitized by Google

of thermometric errors may be expected to be, and also exhibit the variation of the secular error with lapse of time.

The first Kew comparison was the ordinary one; the comparison of January, 1882, was a special one giving corrections for every degree centigrade to a closer accuracy than 0°·1, kindly undertaken for me by Mr Whipple.

TABLES OF CORRECTIONS OF THERMOMETERS.

[The freezing points were not observed at Kew in January, 1882. The correction in brackets has been inferred from the corrections at the lowest temperatures available, viz. at 2°, 3° and 4° C.]

I. Thermometer by Casella, No. 12863, graduated to fifths of a degree Fahrenheit.

Temperature F.	Kew Corrections May, 1873.	Aug. 1881.	Kew Corrections Jan. 1882.	July, 1883.	Nov. 1883.
32°	−0·0	−0·2	(−0·12)	−0·20	−0·2
42°	−0·0		−0·17		
52°	−0·0		−0·15		
62°	−0·1		−0·10		
72°	−0·1		−0·10		
82°	−0·1		−0·10		

II. Thermometer by Negretti, No. 2344, graduated to half-degrees Fahrenheit.

Temperature F.	Kew Corrections Aug. 1879.	Aug. 1881.	Kew Corrections Jan. 1882.	July, 1883.	Nov. 1883.
32°	0·0	−0·5	(−0·32)	−0·45	−0·45
42°	0·0		−0·38		
52°	0·0		−0·45		
62°	0·0		−0·45		
72°	−0·1		−0·43		
82°	−0·2		−0·50		

III. Thermometer by Hicks, No. 79915, graduated to fifths of a degree Centigrade.

Temperature C.	Kew Corrections Dec. 1880.	Kew Corrections Jan. 1882.	July, 1883.	Nov. 1883.
0°	−0·1	(−0·20)	−0·22	−0·23
5°	−0·1	−0·19		
10°	−0·1	−0·22		
15°	−0·1	−0·17		
20°	−0·2	−0·25		
25°	−0·2	−0·26		

Digitized by Google

Fig. 1.

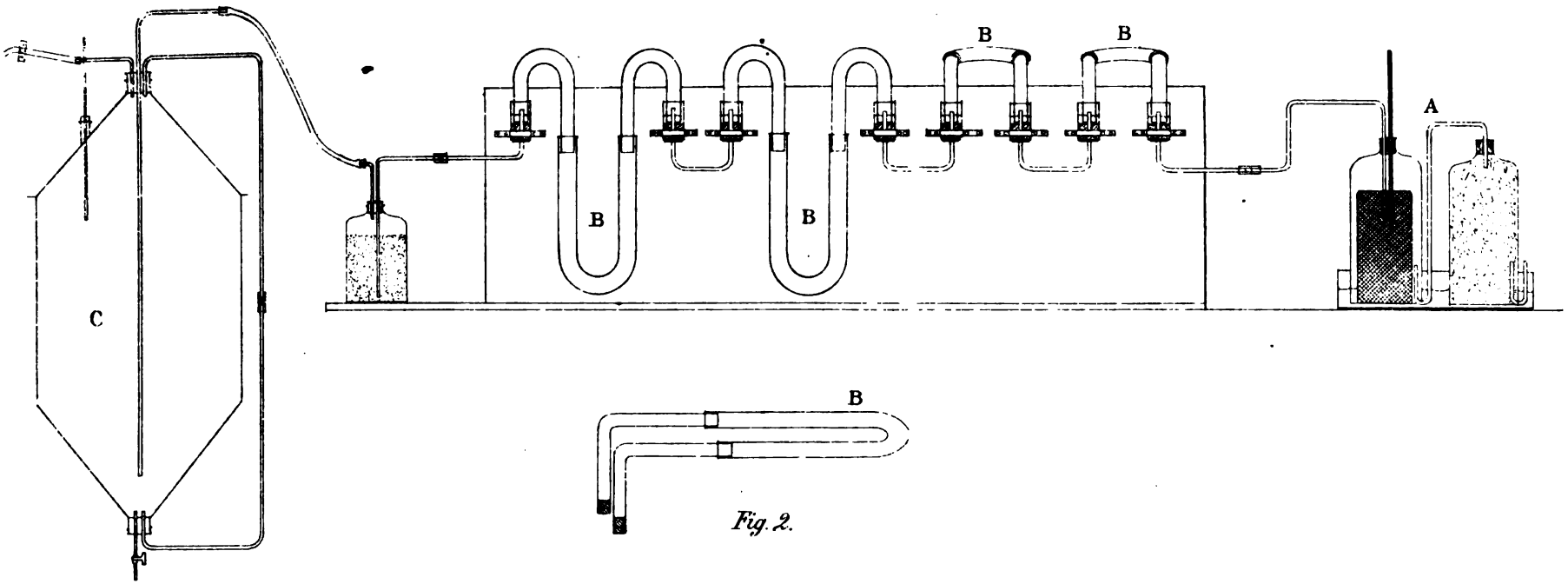


Fig. 2.

Digitized by Google

Digitized by Google

IV. Thermometer by Dr Geissler of Bonn (numbered C.L.C. 16), forming one of a collection purchased by the Cavendish Laboratory from the Loan Collection at South Kensington, graduated to tenths of a degree Centigrade.

Temperature C.	Kew Corrections Dec. 1880.	Aug. 1881.	Kew Corrections Jan. 1882.	July, 1883.	Nov. 1883.
0°	−0·2	−0·15	(−0·18)	−0·22	−·20
5°	−0·2		−0·19		
10°	−0·3		−0·26		
15°	−0·2		−0·30		
20°	−0·3		−0·34		
25°	−0·2		−0·26		

A glance at these tables of corrections shews that the various thermometers differ very considerably in regard to the variation of the secular error, the first example for instance shewing an increase of secular error of only 0°·2 in ten years, while the second shews an increase of 0°·45 in four years. The tables may be also taken to indicate that the method of taking a freezing-point and deducting any additional secular error thus discovered from all the readings does not give a strictly accurate table of corrections, the Geissler thermometer being the only instance in which the differences between the Kew corrections in January, 1882, and December, 1880, are approximately the same at different temperatures throughout the range.

Correction of thermometers by means of water-vapour pressure.

It is the purpose of the present paper to suggest a method of referring the readings of any thermometer to a standard scale without the necessity either of direct comparison with a standard thermometer or of subjecting the instrument to any considerable alteration of temperature.

It is generally accepted by physicists that the saturation pressure of the vapour of water in vacuo depends upon the temperature and upon nothing else. The question has been the subject of many experiments, but has been regarded as settled since Regnault (*Relation des Expériences. Mémoires de l'Académie*, T. XXI.) published his table of the pressures of aqueous vapour at different temperatures. This table expresses the relation between the pressures of aqueous vapour and the corresponding temperatures as referred to the standard scale of temperature adopted by Regnault. This relation being perfectly definite we might, by repeating Regnault's experiments with a given thermometer, use Regnault's table of tensions in order to refer the readings of the thermometer to Regnault's scale of temperatures. This would be practically using an instrument adapted to shew the

Digitized by Google

pressure of water-vapour at different temperatures as a continuous intrinsic thermoscope. Such an instrument, which may be called a water-steam thermometer, is figured by Sir W. Thomson in his article on Heat in the *Ency. Brit.* Vol. XI. p. 568, and its advantages for the purpose are there pointed out. They are mainly that the indications of such an instrument would depend on the temperature only, and not on the state of the glass envelope or other varying quantity at the time of the observation, and the different specimens of the substances used in the manufacture of such a thermometer are for the purpose perfectly identical in their properties, and in consequence two such thermometers, made quite independently, would give at once the same indication for the same temperature without any previous comparison.

The practical comparison of an ordinary thermometer with such an instrument would be in many respects identical with the comparison of a mercury and an air thermometer.

Such a water-steam thermometer is, however, a difficult instrument to construct and to manage. We proceed therefore to describe how the water-vapour pressure may be measured without employing such an instrument.

It was first enunciated by Dalton that the pressure of water-vapour in a closed vessel, in presence of water at the same temperature, is the same, no matter whether there be air present in the vessel or not. Experiments with a view to the verification of this law of Dalton's were undertaken by Regnault, *Ann. de Chimie*, 3rd Series, Vol. XIV. 1845. By a slightly modified arrangement of his apparatus for determining the tension in vacuo he determined the tension in air and in nitrogen gas. The pressure in air was observed for thirty-four temperatures lying between the limits 0°C. and 38°C. The results obtained shew a tension in air less than that derived from the table for vacuum by amounts varying between ·10 mm. and ·74 mm., the mean of the differences between calculation and observation being ·44 mm. The results obtained with nitrogen gas are very similar, the mean error for the first set of observations with nitrogen being ·56 mm. These differences, though very irregular in amount are considerable, and always in the same direction, and might be held to shew that Dalton's law is only approximately true, but Regnault, at the time he published the observations, himself suggested that they might be due to some constant error which he could not then discover, and in a subsequent paper on the tensions of ether and other vapours (*Mémoires de l'Académie*, T. XXVI.) he adduces reasons in favour of that suggestion, attributing in fact the diminished pressure of vapour to the molecular action of the glass side of the vessel upon the saturated vapour contained in the air producing a condensation upon the glass, the slowness of diffusion preventing the pressure reaching its maximum value by consequent evaporation. This method of accounting for the discrepancies between calculation and experiment in reference to Dalton's Law has been subsequently confirmed by experiments of Herwig (*Pogg. Ann.* CXXXVII.) upon compression of vapours, where it was found that the pressure of the vapour could be increased beyond the point at which a deposit was first formed on the sides of the vessel, and the vacuum saturation tension was the increased pressure and not the pressure at which the deposit is first formed. We may therefore conclude that Dalton's law is strictly true provided the air is saturated in such a manner as to avoid the molecular action of the sides of the containing vessel.

Digitized by Google

By Dalton's law the pressure of saturated steam is the same whether air be present in the vessel containing the water and steam or not. If then

e be the pressure of steam in air in millimetres,

Δ the density of dry air at 0°C. and 760 mm. pressure,

t the temperature of the moist air,

α the coefficient of expansion of air per degree centigrade,

d the specific gravity of steam referred to dry air at the same pressure and temperature,

f the density of the steam at the given temperature and pressure, i.e. the mass of one cubic centimetre,

then
$$\frac{e}{760} = \frac{(1 + \alpha t)}{\Delta} \cdot \frac{f}{d}.$$

Of these quantities, as t only enters as a factor of a small correction it may be read on any thermometer; f may be observed by causing a known volume of the air to pass over a substance which will absorb the whole of the moisture (and nothing else) and determining the increase of weight so caused; α and Δ are known constants, and we can therefore use the equation given to determine e if d be known.

The specific gravity of steam, however, referred to air at the same temperature and pressure, is a quantity which has not yet been completely determined for all temperatures and pressures. Theoretically the specific gravity of steam referred to hydrogen may be determined from its molecular weight, and the specific gravity referred to air may then be calculated from the known specific gravity of hydrogen; this theoretical value would of course be *constant for all temperatures and pressures*, and equal to 0·622, and Regnault (*Ann. de Chimie*, 3rd Series, XIV.) undertook several series of experiments to ascertain if such were the case. The first two series of experiments were made on water-vapour in vacuo, and shewed that the number quoted ·622 was, within the limits of error of experiment, the true value of the specific gravity provided the fraction of saturation of the vapour experimented on did not exceed 0·8. The third series of experiments was made with air artificially saturated with moisture at a known temperature. A volume of this air was made to pass through two drying tubes of sulphuric acid and pumice by means of an aspirator whose capacity was accurately determined, the gain in weight of the drying tubes gave the quantity of water-vapour contained in unit volume, and this could also be calculated from the known pressure of saturation at the temperature of the saturating vessel assuming the theoretical value ·622 for the specific gravity of the vapour. A comparison of the results obtained serves to shew whether this assumption of ·622 as the value of the specific gravity is justifiable or not. The series comprises sixty-eight experiments on air saturated at temperatures varying between 0°C. and 27°C.

The following table (V) shews the mean percentage difference between the observed

Digitized by Google

and calculated values of the mass of moisture per unit of volume at the different temperatures.

TABLE V.

Temperature of saturated air.	Number of experiments.	Percentage difference.	Corresponding error of Thermometer reading.
0°	9	·44	0°·065
14°	9	·77	0°·12
7°	7	·86	0°·12
20°·5	21	·45	0°·07
24°·5	22	·90	0°·15

The differences thus tabulated lie all in the same direction, the observed weight of moisture being in each case too small. They may be accounted for in four different ways, (1) the pressure of aqueous vapour in the saturated air may be slightly less than that given by the table of tensions in vacuo, (2) the density of vapour in saturated air may be less by about 1 per cent. than the theoretical density, (3) the temperature readings of the saturated space may be slightly inaccurate, (4) the differences may be due to the incidental errors of the experiments. The irregularity in the numbers for different temperatures tends rather to shew that either of the first two suggestions is insufficient completely to account for the observations although part of each difference may be due to one or other of these causes. The amount of thermometric error required in each case to account for the differences is tabulated in the fourth column of the table. Regnault gives no details as to the method of correcting the thermometer he employed in the experiments, but his known familiarity with thermometers of every kind and their errors makes it highly improbable that he could have overlooked these in the case before us. In reference to the possible experimental errors it should be mentioned that Regnault made a number of experiments upon the dessicating power of the sulphuric acid and pumice used, with the result of being completely assured as to its efficiency. The precautions adopted in order to secure the observations against the possible sources of error that are suggested by the arrangement of the apparatus are not definitely stated.

The construction of the apparatus for a repetition of Regnault's experiments is so simple that it could be arranged in any laboratory, and the observations themselves require only the ordinary amount of care in manipulation. We may use them to determine the temperature of the saturated space and compare the results so obtained with the temperature given by a thermometer enclosed in the space, and thus correct the thermometer for that temperature. We have only to adopt such a corrected value of the specific gravity d for that temperature as is given by experiments with previously corrected thermometers, provided that the value of the specific gravity as given by such experiments is always the same for the same temperature.

I have had occasion in the course of some experiments undertaken for the Meteorological Office to repeat the experiments of Regnault in order to assure myself of the efficiency of the absorbing substances which I proposed to use on employing the chemical

Digitized by Google

method for determining the pressure of aqueous vapour in the air. The investigation involved the question as to how far the method is liable to unavoidable error, and the results will therefore serve to shew to what extent the method suggested above may be relied upon for giving the value of a thermometer correction at the temperature of the saturated space. I purpose giving the details of these experiments for the information of those who may have to use a similar method, as I have been unable to find any satisfactory account of the sources of error and the methods of avoiding them.

The apparatus, which shews the general arrangement, consists of three distinct parts:

(1) The aspirator (C) for causing the passage of a known volume of air over the dessicating substances in the drying tubes.

(2) The weighed drying tubes (B, B) for determining by their increase of weight the quantity of moisture in a known volume.

(3) A saturater (A) for supplying saturated air at the temperature of the room.

Between the drying tubes and the aspirator is placed an additional drying tube or a bottle filled with chloride of calcium, to prevent moisture reaching the weighed tubes from the aspirator.

(1) *The aspirator* is of the ordinary form, a copper cylinder with conical ends; a tap is fixed at the bottom, and through a cork at the top pass three glass tubes, the first for the delivery of the air from the drying tubes passes nearly to the bottom of the aspirator to ensure a *uniform* flow of air, the second connected with a similar one entering at the bottom serves as a gauge, the third passes to the lower surface of the cork and is used to fill the aspirator by being connected with an aspirating pump. The aspirator is also provided with a thermometer passing through the side of the upper cone.

The volume of the aspirator was determined by completely filling it with water as for an experiment, and running the water out gradually into a litre flask of known weight when empty and weighing the flask each time when full, the flask being carefully dried between successive fillings.

Two aspirators were employed numbered A and B, and the quantity of water in each was determined twice with the following result:

Aspirator A,	1st observation	16365·9	grammes,	temperature 18°.
	2nd „	16371·2	„	„
Aspirator B,	1st observation	16384·0	„	„
	2nd „	16383·8	„	„

We have therefore, allowing for the density of the water used and the temperature,

Volume of A at 0°	16384 cc.
„ B at 0°	16400 cc.

The volumes at any temperature t will then be

Aspirator A, $16384\ (1 + \cdot 000052\, t)$ cubic centimetres.

„ B, $16400\ (1 + \cdot 000052\, t)$ „

One of these will then be the volume of water run out in any experiment. The water will be replaced partly by air which has come through the drying tubes and

Digitized by Google

partly by the vapour of the water formed in the aspirator. We may *assume that the air in the aspirator at the end of the experiment is saturated with moisture.*

Let V be the volume of the aspirator at its final temperature T,

E the saturation pressure of aqueous vapour at that temperature,

B the height of the barometer at the time.

Then the air from the drying tubes remaining in the aspirator at the end of the experiment is under a pressure equal to $B-E$, and if e was the pressure of the vapour in the air before it entered the drying tubes and t its temperature, the dry air pressure was $B-e$. Its volume was therefore $\frac{B-E}{B-e} \cdot \frac{1+\alpha t}{1+\alpha T} \cdot V$. α being the coefficient of expansion of air.

This then was the volume of moist air which entered the drying tubes, which is the element the aspirator is intended to determine.

(2) *The drying tubes.* A slightly modified form was used. Instead of being closed with corks perforated by glass tubes, glass connections of wider bore were used which were thickened and ground into the U-tubes. These latter were of the ordinary size, about 6 inches long and half-inch internal diameter. The long tubular stoppers were bent over in the case of sulphuric acid and pumice tubes through two right angles, and in the case of phosphoric acid tubes (shewn in fig. 2) through one right angle. The wide ends of these tubes could then pass over narrower tubes coming vertically through the bottoms of small mercury cups, and thus forming the connections between the drying tubes and the other parts of the apparatus. The connections are thus made by means of mercury joints. These joints were tested and found to be quite tight for differences of external and internal pressure many times greater than those occurring in the experiment. The arrangement is very convenient as the tubes can be simply lifted from their places and as easily replaced; they require carefully brushing to remove the adhering mercury and the ends are closed for weighing with small india-rubber stoppers. The liability to error in consequence of moisture on the surface of the mercury is probably not so great as that to which the tubes would be exposed by using india-rubber connections. The drying tubes were filled either with phosphoric anhydride or with rather coarse fragments of pumice saturated with the strongest sulphuric acid (Sp. Gr. 1·84). Experiments will be detailed below to shew that either of these substances is perfectly efficient for the purpose of withdrawing all the moisture from the air passed over it. A number of experiments have shewn me that drying tubes filled with recently fused chloride of calcium although in many ways convenient are not capable of extracting all the moisture from air[1].

[1] The experiments were of two kinds.

(1) Two chloride of calcium tubes were arranged in front of two sulphuric acid tubes and an aspirator full of saturated air passed through all four. The gain of weight in each of the four for three observations is given below.

1st tube. ($CaCl_2$)	2nd tube. ($CaCl_2$)	3rd tube. (H_2SO_4)	4th tube. (H_2SO_4)
·2025 gm.	+·0010 gm.	+·0208 gm.	
·2220 ,,	·0000 ,,	+·0245 ,,	+·0013
·2226 ,,	·0000 ,,	+·0190 ,,	+·0000

(2) The two aspirators were used to determine by two independent simultaneous observations the amount of moisture in the air of the room, first by pumice tubes, and secondly by chloride of calcium tubes. The amounts are given below.

By pumice tubes.	By chloride of calcium tubes.	Difference.
·1550 gm.	·1435 gm.	·0115
·1498 ,,	·1352 ,,	·0146

Digitized by Google

Correction of the weight of the drying tubes for weighing in air. The main part of the weight of the tubes is the weight of the glass, and the specific gravity of the pumice and of the strong sulphuric acid does not differ much from that of glass; we may therefore calculate the correction for weighing in air on the assumption that the specific gravity of the whole tube is that of flint glass, which we may take to be 3·5. This would make the correction to weight in vacuo for a tube of 150 grammes equal to 30 milligrammes. The effect of a barometric variation of 1 c.m. upon such a tube would therefore be to alter its apparent weight by ·39 mgm., and a variation of 1° C. of temperature would produce an alteration of ·11 mgm. in the apparent weight. The changes in barometric pressure and temperature between two successive weighings may therefore be such as to cause the apparent weight of the tube to alter by a considerable fraction of a milligramme. Now the amount of moisture is determined by the difference of weight of the tube at the two weighings, and accordingly any error in the weighing due to neglecting to correct for weighing in air, will be of the same *absolute* magnitude, and of very much greater relative importance, in the weight of moisture absorbed by the tube. The variations of pressure and temperature, however, between the successive weighings of the tubes during the observations were not sufficient to produce any appreciable effects upon the results.

(3) *The saturater.* This part of the apparatus is similar in principle to that used by Regnault. Two long bell jars stand in a shallow dish of distilled water, the one jar is filled with well washed sponge, and the second contains a cylinder covered with muslin which with it stands in the water at the bottom of the dish. The air supplied to the aspirator is drawn by means of a glass tube passing through the cork in the top of the jar from the middle of the muslin cage, and close to the opening of the tube is the bulb of the thermometer which also passes through the cork, the place of the air thus removed is supplied by air passing from the outside through the sponge vessel and delivered into the second vessel outside the muslin cage. During an observation, which lasted generally about two hours, the thermometer was read every quarter of an hour by means of a telescope placed at some distance in order to avoid any error of parallax, and the mean of the readings taken as the temperature of the saturated air. The muslin cage served to protect the thermometer from external radiation as well as to ensure the complete saturation of the air.

A series of observations was first made with a view of testing the drying tubes and other parts of the apparatus. There were two points to be determined, first whether the dessicating substances used could be regarded as completely drying the air passed over them, and secondly whether one drying tube was sufficient for the purpose or two or more were necessary. For this purpose four drying tubes were mounted, two being filled with phosphoric anhydride in the ordinary form of white powder, the other two were filled with pumice moistened with carboy sulphuric acid (Sp. Gr. 1·84). The pumice was broken into coarse fragments and before being used was saturated with sulphuric acid and heated to redness.

The saturated air was then sent through all four tubes and the gain in weight of

Digitized by Google

each tube determined. From the *gain in weight of the first tube alone,* the tension of vapour in the saturater was calculated by the formula

$$e = \frac{760\,(1 + \alpha t)}{\Delta d} \cdot \frac{w}{V} \cdot \frac{B - e}{B - E}\,\frac{1 + \alpha T}{1 + \alpha t}.$$

And this was compared with the tension as given by Regnault's table for the temperature indicated by the thermometer in the saturater corrected by a table of Kew corrections and a determination of the freezing-point during the course of the series of observations. The subjoined tables give the results obtained. Table VI. is a specimen of the observations as they were taken and Table VII. gives the collected results.

TABLE VI.

Experiment 10. Aspirator A.

Time.	Temperature of Saturater.	Temperature of Aspirator.	Barometer.	Weight of Pumice Tube I. II.	Weight of Pumice Tube III. IV.	Weight of Phosphoric Tube III. IV.
3·55	19·5			176·4613	177·5045	146·9780
4·10	19·5					
4·30	19·5					
4·44	19·45					
5·00	19·45					
5·30	19·45					
5·50	19·40	19·4	29·750	176·7315	177·5061	146·9798
Means and Differences.	19·46			·2702	·0016	·0018
Corrections.	− ·12	− ·3				

Tension of Vapour calculated from Tube I. II. 16·66 mm.

Digitized by Google

TABLE VII.

Number of Experiment.	Date.	Aspirator.	Gain in weight Tube 1, in grammes.	Gain in weight Tube 2.	Gain in weight Tube 3.	Gain in weight Tube 4.	Mean corr^d. temp. of saturater (t).	Corr^d. temp. of aspirator (T).	Tension in mm. cal^d. from Tube 1.	Tension in mm. cal^d. from temp. t of saturater.	Difference.	Duration of Experiment.
			Phosphoric Acid		Sulphuric Acid							h. m.
1	July 4	B	·2915	− ·0050	+ ·0048	+ ·0028	21·05	22·7	18·23	18·55	− ·32	1 20
2	„	A	·2928	+ ·0008	+ ·0117	+ ·0018	20·93	22·2	18·29	18·41	− ·12	2 15
3	„ 6	„	·2916	+ ·0023	+ ·0080	+ ·0040	21·05	21·8	18·17	18·55	− ·38	2 23
4	„ 7	„	·2944	+ ·0005	+ ·0011	− ·0008	21·06	21·7	18·34	18·56	− ·22	2 3
5	„ „	„	·3074	+ ·0023	+ ·0033	+ ·0017	21·59	21·5	19·12	19·18	− ·06	2 7
6	„ 9	„	·2931	+ ·0007	+ ·0022	+ ·0006	20·91	21·45	18·25	18·39	− ·14	2 45
7	„ „	„	·3020	− ·0001	− ·0002	− ·0009	21·31	21·0	18·77	18·85	− ·08	1 15
			Sulphuric Acid		Phosphoric Acid							
8	„ 10	„	·2870	+ ·0027	+ ·0019	—	20·64	21·2	17·85	18·09	− ·24	3 2
9	„ 11	B	·2785	− ·0005	+ ·0018	—	20·00	21·4	17·35	17·39	− ·04	2 0
10	„ 13	„	·2702	+ ·0016	+ ·0018	—	19·34	19·1	16·66	16·70	− ·04	1 55
11	„ 14	„	·2557	+ ·0013	+ ·0010	—	18·55	18·7	15·74	15·80	− ·06	2 0
12	„ 12	„	·2682	+ ·0023	+ ·0013	—	19·08	?	16·53	16·53	− ·00	2 0

Digitized by Google

In the first seven experiments the two phosphoric acid tubes were placed first, and in the last three the sulphuric acid tubes were in that position. The columns giving the increase of weight in the different tubes shew that these nearly always gained a small amount, but that amount is very irregular and is about the same whether the phosphoric acid tubes or the sulphuric acid are placed first, and the calculated tension is in nearly every case within 1 p.c. of the tabulated tension. This being about the same error that occurs in Regnault's observations we may take it that the first tube was sufficient to completely dry the air passed through it and that the increase of weight in the other tubes was due to some other cause.

The different connections were made partly by glass and partly by india-rubber tubing, and this suggested itself as a possible source of the observed differences. A number of observations were therefore taken with a view of determining how far this might be the case.

I. A glass tube about 5 feet long was mounted as a connection between two mercury cups; and air, first dried by passing through a phosphoric acid tube, was passed through the long glass tube and then through a sulphuric acid tube and the weight of the latter determined before and after the passage of the air. There was accordingly nothing but the glass tube between the two drying tubes. The results were as follows for four experiments.

TABLE VIII.

Date.	Increase of weight of the sulphuric acid tube in grammes.
July 18	+·0029
„ 19	+·0002
„ 20	−·0004
„ 21	−·0002

It appears therefore that after the tube had been first dried no further increase of weight of the sulphuric acid tube occurred, and that glass tubes may be used as connections without any fear of error.

II. The glass tube was replaced by an india-rubber tube 6 feet long, and similar observations taken, fifteen experiments being made. The sulphuric acid tube always gained in weight although every precaution was taken to keep the india-rubber tube dry between the experiments. The least amount of moisture was obtained when a second observation was taken, immediately after the completion of a first. The increase of weight of the sulphuric acid tube generally amounted to about 15 milligrammes. An india-rubber tube cannot therefore be used with any security for connecting two drying tubes.

We may therefore conclude that the increase in weight of the drying tubes after the first in the table of results given was due for the most part to moisture derived from the india-rubber connections. For the last three observations in the table, p. 12, these connections were made as short as possible so that the amount of india-rubber surface

Digitized by Google

exposed to the dry air might be small. With the apparatus in that form the drying tubes gained very much less in weight than before, and we may give the results obtained from these three observations as an instance of the accuracy which may be expected in correcting a thermometer by observations of this kind.

The first column of the subjoined table gives the temperature of the saturater as determined by a thermometer corrected by a table of Kew corrections. The second column gives the temperature as taken from Regnault's table of pressures, the pressure being calculated from the weight observations.

Thermometer 87400 corrected by Kew Table.	From weight observations.	Difference.
19°·34	19°·30	– ·04
18°·55	18°·49	– ·06
19°·08	19°·08	– ·00

I have since arranged another apparatus which is the same in principle but which is slightly modified in some details, the air after leaving the saturater is passed through two glass globes and its temperature is again read immediately before entering the drying tubes as well as in the saturater.

The following table gives the results obtained.

Thermometer (C. L. C. 21) in saturater corrected by Kew Table.	Temperature calculated from weighing.	Difference.
14°·34	14°·21	– ·13
15°·74	15°·45	
12°·29	12°·29	·00

With regard to the second observation it should be mentioned that the air in that case had fallen in temperature to 15°·27 or 0°·47 below the temperature of the saturater before entering the drying tubes, so that it would appear that the air was then more than saturated or the calculated result is really 0°·12 too high instead of being, as apparent at first sight, too low. It is possible therefore that in this observation the air contained a small amount of moisture in excess of that required to saturate it at the temperature at which it passed into the tubes. This excess of moisture may have been held in the form of a very light cloud or the air may have been supersaturated.

It appears then that from this method of correcting thermometers we might expect results accurate within one-tenth of a degree, and that is probably not wider than the limit of accuracy of the thermometer as corrected by the Kew tables. The errors are at present all in the same direction and part is probably constant, and therefore a more extended series of observations would enable us to determine the value of the constant

Digitized by Google

error at each temperature which is inseparable from the method and give the thermometer corrections within less than the tenth of a degree. If that be so, the fact that the apparatus is of simple construction would make it a thoroughly practical method of determining the error at any temperature. It also has the additional advantage as compared with the Kew comparison method that the experiments can be undertaken in the place where the thermometer is used.

CAVENDISH LABORATORY,
Nov. 26, 1883.

Digitized by Google

III. *On the pulsations of spheres in an elastic medium.* By A. H. Leahy, M.A.

1. The motion due to the pulsations of spheres of the same period of pulsation in an incompressible fluid has been investigated by Professor Bjerknes of Christiania*, by whom the following results have been obtained. If the pulsations of two spheres are in the same phase of vibration, there will be an apparent force on each of the bodies, which varies according to the law of the inverse square of the distance, and tends to make the spheres approach one another; but, if the pulsations are in phases differing by half a complete period, there will be a force tending to repel the spheres from one another, and varying according to the same law These results have been experimentally verified, and similar effects have been shewn by some experiments, described in the *Journal of Telegraph Engineers* for 1882, to hold in air. An apparatus shewing these attractive and repulsive effects, together with several other "inverse analogies," to use Dr Bjerknes' phrase, between electro magnetic effects and pulsations under water of spheres and cylinders was exhibited at the Paris exhibition†.

2. These phenomena, together with several others of a kindred character, may be explained by the following general considerations. Suppose a periodic force of the nature of surface tensions or pressures to be acting on a sphere, whose centre is fixed in space, and which is itself pulsating with a simple harmonic motion. Then, since the magnitude of the force which acts upon the body varies as the superficial area, it is clear that the effect of the force will be greatest, when the surface of the body is greatest. If therefore the force is a simple harmonic function of the time, and has the same period as that of the pulsations of the body, it is clear that its effect during one complete oscillation will be to urge the body in the direction in which the force acted when the area of the sphere was a maximum. For, considering any two instants the time between which is half a complete period, it is clear that the force at each of these instants will be the same in magnitude and opposite in direction; so that the resultant effect will be to urge the body in the direction which the force had when the superficial area of the sphere was the greater. Thus we have only to consider the effect of the force during that half period when the sphere is greater than its mean value; i.e. than its value at a time midway between the instants of greatest contraction and expansion. Let τ be the time when the sphere is greatest. Then, if $2p$ is the complete period, we shall only have to

* See the Reports of the *Proceedings of the Scientific Society of Christiania*, 1875, and the *Repertorium der Mathematik von Königsberger und Zeuner*, 1876, p. 268.

† See *La Lumière Electrique*, 5th Oct. and 9th Nov. 1881, and *Engineering*, 1882.

Digitized by Google

consider the force between the instants $\tau+\frac{p}{2}$ and $\tau-\frac{p}{2}$. But if τ' is the time when the force on the sphere is zero, where τ' lies between $\tau+\frac{p}{2}$ and $\tau-\frac{p}{2}$, it is clear that the force at the instants $\tau'+\alpha$ and $\tau'-\alpha$ are the same in magnitude and opposite in direction. Also, if $\tau'-\tau$ is positive, the effect of the force at the time $\tau'+\alpha$ will be less than at the time $\tau'-\alpha$; since the area of the body which is acted on is less at the former instant than at the latter. Thus the action on the sphere during the period between the instants $\tau-\frac{p}{2}$ and τ' will exceed the action in the opposite direction during the period between τ' and $\tau+\frac{p}{2}$, and the resultant action during a complete oscillation will be the same in direction as at the time τ, when the area of the sphere was a maximum. A similar result will follow if $\tau'-\tau$ is negative.

We have therefore, in order to determine the direction in which a periodic force of the character described urges a pulsating sphere, merely to determine the direction which the force has when the area of the sphere is a maximum. Now, in the case of two spheres A and B pulsating with their centres fixed in an incompressible fluid; it can easily be seen that the change of pressure due to the pulsations of A increases with the time differential of the velocity along the radius vector from the centre of A. The action on B due to this change of pressure is of course greater on that side which faces A than on the opposite side, and the force will therefore be a repulsion when the velocity due to the pulsations of A is increasing, and an attraction when the velocity is diminishing. Now when the volume of A is greater than its mean value the velocity is diminishing; hence, if the pulsations of A and B are in the same phase of vibration, there will be an attractive force on B when its volume is greatest, and the general effect of the changes of pressure due to A's pulsation will be an attraction towards A. Similarly, if the pulsations are in opposite phases, the effect will be a repulsion.

But if these changes of volume are executed in a medium having properties similar to those of the ether, in which the vibrations producing the sensation of light are supposed to be propagated, the results which have been given above will not continue to hold. For, in this case there will be no flux at the surface of B, if the displacements are not large; and the force will not depend upon the velocity, or upon the changes of velocity, but upon the absolute deformations. If the waves of displacement are long, compared with the distance between A and B, the medium will be compressed as A expands; and the effect at the surface of B will be a repulsion if the volume of A is greater than its mean value. Thus the effects produced in an incompressible fluid will be reversed if the oscillations are performed in an elastic medium, and like phases of pulsation will give rise to a repulsion and unlike phases to an attraction on the pulsating bodies. This way of looking at the problem appears to indicate, that, if spheres are pulsating in an elastic medium, the period of pulsation being such as to give rise to waves which are long, compared with the distances between the spheres, the results obtained by Professor Bjerknes will be reversed. If the distance of B from A exceed a quarter wave length it is evident that this result will not

Digitized by Google

be true. Supposing for example, that the medium under consideration is the same as that in which light-vibrations are performed, we shall have, taking the approximate velocity of propagation to be 200,000 miles a second, a wave length of 200 miles corresponding to 1000 vibrations in a second. Thus for all distances of A and B at which any sensible effect can be observed we can take the phase of the vibration to be the same; but if the distance exceeds 50 miles our argument does not apply.

3. These considerations are founded on a principle which seems to underlie many cases of differential action; namely, that if a body be acted on by a force F, where F is a simple harmonic function of the time; and, if the action of the force on the body due to variations in the position, magnitude, or shape of the body be expressed by FF', where F' is also a simple harmonic function of the time of the same period as F; then the effect of the force on the body will be to urge it in that direction which F had when F' was a maximum. This principle can also be extended to the case where F' is any periodic function of the same period as F, provided that F' has only one maximum value during the complete period $2p$, and also satisfies the condition $F'(\tau+\alpha)=F'(\tau-\alpha)$, where τ is the time when F' is a maximum. The truth of this principle can be established by the same considerations as those employed in § 2; since we shall have $F'(t)$ diminishing, as the numerical value of $t-\tau$ increases from zero to p; the complete period being $2p$; and the whole of the argument at the beginning of § 2 will apply. As an example of differential action which can be treated by the principle just enunciated may be mentioned that of a body placed in a field of force, where the force at any point has for components $L\sin\frac{\pi t}{p}$, $M\sin\frac{\pi t}{p}$, $N\sin\frac{\pi t}{p}$, where L, M, N are functions of the co-ordinates of the point. Let the body be constrained by some independent cause to move, so that at the time t its position is such that the force acting on it has for components $L\sin\frac{\pi t}{p}$, $M\sin\frac{\pi t}{p}$, $N\sin\frac{\pi t}{p}$, where L, M, N are periodic functions of the time, of period equal to $2p$, which have only one maximum value during that period, and satisfy the functional equation $f(\tau+\alpha)=f(\tau-\alpha)$, where τ is the time when f is a maximum. It will then be found that the action of any component $L\sin\frac{\pi t}{p}$ will be to urge the body in that direction in which the component acted when L was a maximum.

4. These considerations do not however give the law of the action of the force, either in this case, or in the case of a pulsating body which was mentioned in § 2. In order to completely investigate the mutual action of two pulsating bodies in an elastic medium it will be necessary to find the displacement at any point due to their joint effect, and it will be found that the law of attraction in the case of unlike phases, and of repulsion in the case of like phases will be that of the inverse square of the distance to the first order of approximation. The term of next order of importance will always be a repulsion and will vary according to the law of the inverse cube. In the following work I propose to establish these results.

Digitized by Google

General expressions for a periodic and steady displacement symmetrical about an axis.

The equations of motion of an elastic medium* are

$$\left.\begin{aligned} r^2 \sin\theta \frac{d^2u}{dt^2} &= \frac{\lambda+2\mu}{\rho} r^2 \sin\theta \frac{d\epsilon}{dr} + \frac{\mu}{\rho}\left(\frac{d\gamma}{d\theta} - \frac{d\beta}{d\phi}\right) \\ r \sin\theta \frac{d^2v}{dt^2} &= \frac{\lambda+2\mu}{\rho} \sin\theta \frac{d\epsilon}{d\theta} + \frac{\mu}{\rho}\left(\frac{d\alpha}{d\phi} - \frac{d\gamma}{dr}\right) \\ r \frac{d^2w}{dt^2} &= \frac{\lambda+2\mu}{\rho} \frac{1}{\sin\theta} \frac{d\epsilon}{d\phi} + \frac{\mu}{\rho}\left(\frac{d\beta}{dr} - \frac{d\alpha}{d\theta}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(1),$$

where u is the displacement along radius vector, v is the displacement along the tangent to the meridian tending from the pole, and w is the displacement along a parallel of longitude tending from the fixed meridian; r, θ, ϕ being the co-ordinates of a particle in its undisturbed position; λ and μ being the coefficients of elasticity of the medium, ρ its density, and ϵ, α, β, γ being defined by the equations

$$\left.\begin{aligned} \epsilon &= \frac{1}{r^2}\frac{d}{dr}(r^2u) + \frac{1}{r\sin\theta}\frac{d}{d\theta}(v\sin\theta) + \frac{1}{r\sin\theta}\frac{dw}{d\phi} \\ \alpha &= \frac{1}{r^2\sin\theta}\left\{\frac{d}{d\phi}(rv) - \frac{d}{d\theta}(rw\sin\theta)\right\} \\ \beta &= \frac{1}{\sin\theta}\left\{\frac{d}{dr}(rw\sin\theta) - \frac{du}{d\phi}\right\} \\ \gamma &= \sin\theta\left\{\frac{du}{d\theta} - \frac{d}{dr}(rv)\right\} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(2).$$

The particular values of u, v, w depending upon $\frac{\lambda+2\mu}{\rho}$, which are propagated with the velocity $\sqrt{\frac{\lambda+2\mu}{\rho}}$, must make the terms vanish which depend upon $\frac{\mu}{\rho}$.

Thus we must have

$$\frac{d}{d\phi}(rv) - \frac{d}{d\theta}(rw\sin\theta) = r^2\sin\theta\frac{d\psi}{dr},$$

$$\frac{d}{dr}(rw\sin\theta) - \frac{du}{d\phi} = \sin\theta\frac{d\psi}{d\theta},$$

$$\frac{du}{d\theta} - \frac{d}{dr}(rv) = \frac{1}{\sin\theta}\frac{d\psi}{d\phi}.$$

Substituting in equations (1) we get $\frac{d^2}{dt^2}\left(\frac{d\psi}{dr}\right) = 0$, etc. Thus since ψ is essentially periodic, we must have $\frac{d\psi}{dr} = 0$, $\frac{d\psi}{d\theta} = 0$, $\frac{d\psi}{d\phi} = 0$, or $\alpha = \beta = \gamma = 0$.

* Lamé's *Elasticity*, Art. 84.

Digitized by Google

These conditions give

$$u = \frac{dF}{dr}, \qquad rv = \frac{dF}{d\theta}, \qquad rw \sin\theta = \frac{dF}{d\phi} \quad \text{.....(3)},$$

where F is a function of the same period as u, v, w. Hence $\epsilon = \nabla^2 F$, where

$$\nabla^2 = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr}\right) + \frac{1}{r^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d}{d\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{d^2}{d\phi^2},$$

and equations (1) reduce to

$$\frac{d^2 F}{dt^2} = \frac{\lambda + 2\mu}{\rho}\nabla^2 F \quad \text{.....(4)}.$$

Next taking the values of u, v, w which depend upon $\frac{\mu}{\rho}$, and which travel with velocity $\sqrt{\frac{\mu}{\rho}}$, we have $\epsilon = 0$, or

$$\frac{d}{dr}(r^2 u \sin\theta) + \frac{d}{d\theta}(rv \sin\theta) + \frac{d}{d\phi}(rw) = 0.$$

Hence we must have

$$\left.\begin{aligned} r^2 \sin\theta . u &= \frac{dM}{d\phi} - \frac{dN}{d\theta} \\ r \sin\theta . v &= \frac{dN}{dr} - \frac{dL}{d\phi} \\ r . w &= \frac{dL}{d\theta} - \frac{dM}{dr} \end{aligned}\right\} \quad \text{.....(5)}.$$

Substituting in equations (1), we get

$$\frac{d}{d\theta}\left[\frac{d^2 L}{dt^2} - \frac{\mu}{\rho}\left\{\nabla_1^2 L - \frac{2\sin\theta}{r^3}\frac{d}{d\theta}\left(\frac{M}{\sin\theta}\right) - \frac{2\cot\theta}{r^2}\frac{dM}{dr} - \frac{2}{r^2 \sin^2\theta}\frac{dN}{d\phi}\right\}\right]$$
$$= \frac{d}{dr}\left\{\frac{d^2 M}{dt^2} - \frac{M}{\rho}\left(\nabla_2^2 M - \frac{2\cot\theta}{r^2 \sin^2\theta}\frac{dN}{d\phi}\right)\right\},$$

$$\frac{d}{d\phi}\left[\frac{d^2 L}{dt^2} - \frac{\mu}{\rho}\left\{\nabla_1^2 L - \frac{2\sin\theta}{r^3}\frac{d}{d\theta}\left(\frac{M}{\sin\theta}\right) - \frac{2\cot\theta}{r^2}\frac{dM}{dr} - \frac{2}{r^2 \sin^2\theta}\frac{dN}{d\phi}\right\}\right] = \frac{d}{dr}\left\{\frac{d^2 N}{dt^2} - \frac{\mu}{\rho}\nabla_2^2 N\right\},$$

$$\frac{d}{d\phi}\left\{\frac{d^2 M}{dt^2} - \frac{\mu}{\rho}\left(\nabla_2^2 M - \frac{2\cot\theta}{r^2 \sin^2\theta}\frac{dN}{d\phi}\right)\right\} = \frac{d}{d\theta}\left\{\frac{d^2 N}{dt^2} - \frac{\mu}{\rho}\nabla_2^2 N\right\};$$

where

$$\left.\begin{aligned} \nabla_1^2 &= \frac{d^2}{dr^2} + \frac{1}{r^2}\frac{d^2}{d\theta^2} + \frac{\cot\theta}{r^2}\frac{d}{d\theta} + \frac{1}{r^2 \sin^2\theta}\frac{d^2}{d\phi^2} \\ \nabla_2^2 &= \frac{d^2}{dr^2} + \frac{1}{r^2}\frac{d^2}{d\theta^2} - \frac{\cot\theta}{r^2}\frac{d}{d\theta} + \frac{1}{r^2 \sin^2\theta}\frac{d^2}{d\phi^2} \end{aligned}\right\} \quad \text{.....(6)}.$$

These results give

$$\left.\begin{aligned} \frac{d^2 L}{dt^2} &= \frac{\mu}{\rho}\left\{\nabla_1^2 L - \frac{2\sin\theta}{r^3}\frac{d}{d\theta}\left(\frac{M}{\sin\theta}\right) - \frac{2\cot\theta}{r^2}\frac{dM}{dr} - \frac{2}{r^2 \sin^2\theta}\frac{dN}{d\phi}\right\} \\ \frac{d^2 M}{dt^2} &= \frac{\mu}{\rho}\left\{\nabla_2^2 M - \frac{2\cot\theta}{r^2 \sin^2\theta}\frac{dN}{d\phi}\right\} \\ \frac{d^2 N}{dt^2} &= \frac{\mu}{\rho}\nabla_2^2 N \end{aligned}\right\} \quad \text{.....(7)}.$$

Digitized by Google

5. In order to get a displacement symmetrical about an axis, we put $w=0$ and $\frac{d}{d\phi}=0$.

This will give

$$\left.\begin{aligned} u &= \frac{dF}{dr} - \frac{1}{r^2 \sin\theta}\frac{dN}{d\theta} \\ v &= \frac{1}{r}\frac{dF}{d\theta} + \frac{1}{r\sin\theta}\frac{dN}{dr} \end{aligned}\right\} \quad \text{.....(1)},$$

where F satisfies the equation

$$\frac{d^2F}{dt^2} = \frac{\lambda+2\mu}{\rho}\left\{\frac{d^2F}{dr^2} + \frac{2}{r}\frac{dF}{dr} + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dF}{d\theta}\right)\right\} \quad \text{.....(2)},$$

and N satisfies the equation

$$\frac{d^2N}{dt^2} = \frac{\mu}{\rho}\left\{\frac{d^2N}{dr^2} + \frac{\sin\theta}{r^2}\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{dN}{d\theta}\right)\right\} \quad \text{.....(3)}.$$

Now, whatever be the forms of F and N, they can be expanded in series of Legendre's coefficients $A_n P_n(\mu)$, where $\mu=\cos\theta$ and A_n is a function of r and t only. Also, since $\sin\theta\frac{dP_n}{d\theta}$ can be expressed in terms of Legendre's coefficients, F and N can also be expressed in series of form $B_n \sin\theta\frac{dP_n}{d\theta}$, where B_n is a function of r and t only.

Put $F=\Sigma A_n P_n(\mu)$ and $N=\Sigma B_n \sin\theta\frac{dP_n(\mu)}{d\theta}$ and the equations (2) and (3) reduce to

$$\frac{d^2A_n}{dt^2} = \frac{\lambda+2\mu}{\rho}\left\{\frac{d^2A_n}{dr^2} + \frac{2}{r}\frac{dA_n}{dr} - \frac{n(n+1)}{r^2}A_n\right\},$$

$$\frac{d^2B_n}{dt^2} = \frac{\mu}{\rho}\left\{\frac{d^2B_n}{dr^2} - \frac{n(n+1)}{r^2}B_n\right\}.$$

Let us now suppose the displacement to be oscillatory, and steady, of period equal to $\frac{2\pi}{p}$. We shall get, on this supposition, if $k^2=\frac{\rho p^2}{\lambda+2\mu}$, $h^2=\frac{\rho p^2}{\mu}$,

$$A_n = f_n(ikr)\,e^{ipt},$$

$$B_n = F_n(ihr)\,e^{ipt},$$

where f_n, F_n are the solutions of the equations

$$\frac{d^2f_n}{d(kr)^2} + \frac{2}{kr}\frac{df_n}{d(kr)} + \left\{1 - \frac{n(n+1)}{(kr)^2}\right\}f_n = 0 \quad \text{and} \quad \frac{d^2F_n}{d(hr)^2} + \left\{1 - \frac{n(n+1)}{(hr)^2}\right\}F_n = 0.$$

The solutions of these equations are

$$\left.\begin{aligned} f_n(ikr) &= A_n r^n\left(\frac{1}{r}\frac{d}{dr}\right)^{n+1} e^{ikr} + A_n' r^n\left(\frac{1}{r}\frac{d}{dr}\right)^{n+1} e^{-ikr} \\ F_n(ihr) &= B_n r^{n+1}\left(\frac{1}{r}\frac{d}{dr}\right)^{n+1} e^{ihr} + B_n' r^{n+1}\left(\frac{1}{r}\frac{d}{dr}\right)^{n+1} e^{-ihr} \end{aligned}\right\} \quad \text{.....(4)}.$$

Digitized by Google

Hence, if f_n and F_n have the meanings given above;

$$\left.\begin{aligned} u &= \Sigma P_n(\mu)\left\{\frac{d.f_n(ikr)}{dr}+\frac{n(n+1)}{r^2}F_n(ihr)\right\}e^{ipt} \\ v &= \Sigma\frac{1}{r}\frac{dP_n(\mu)}{d\theta}\left\{f_n(ikr)+\frac{d.F_n(ihr)}{dr}\right\}e^{ipt} \end{aligned}\right\}\text{.....(5)},$$

with two similar terms obtained by changing the sign of i, will give the complete value of a periodic and steady vibration, which is symmetrical about an axis.

6. The series $f_n(ikr)$ and $F_n(ihr)$ can be expanded in powers of kr and hr respectively.

For putting $r^2=z$, we have $\qquad \frac{1}{r}\frac{d}{dr}=2\frac{d}{dz}$,

therefore

$$\left.\begin{aligned} f_n(ikr) &= A_n z^{\frac{n}{2}}.2^{n+1}\left(\frac{d}{dz}\right)^{n+1}e^{ik\sqrt{z}} \\ F_n(ihr) &= B_n z^{\frac{n+1}{2}}.2^{n+1}\left(\frac{d}{dz}\right)^{n+1}e^{ih\sqrt{z}} \end{aligned}\right\}\text{.....(1)},$$

with other terms obtained by changing the sign of i.

Hence, if when $n-s$ is negative, $\frac{2n-2s!}{n-s!}$ means $(-1)^{n-s}.\frac{s-n!}{2s-2n!}$,

$$\left.\begin{aligned} f_n(ikr) &= (-1)^n A_n ikr^{-(n+1)}\left\{\frac{2n!}{2^n.n!}\ldots\ldots+(-1)^s\frac{2n-2s!}{2^n.s!.n-s!}(ikr)^{2s}+\frac{2^{n+1}.s+1!}{2s+2!\,s-n!}(ikr)^{2s+1}+\ldots\right\} \\ F_n(ihr) &= (-1)^n B_n.ihr^{-n}\left\{\frac{2n!}{2^n.n!}\ldots\ldots+(-1)^s.\frac{2n-2s!}{2^n.s!\,n-s!}(ihr)^{2s}+\frac{2^{n+1}.s+1!}{2s+2!\,s-n!}(ihr)^{2s+1}\ldots\ldots\right\} \end{aligned}\right\}(2),$$

the highest odd powers of ikr and of ihr in the series being $(ikr)^{2n+1}$ and $(ihr)^{2n+1}$.

Hence we get

$$\left.\begin{aligned} u_n &= (-1)^{n+1}A_nP_n(\mu).r^{-(n+2)}.ik\left\{\frac{2n!\,(n+1)}{2^n.n!}\ldots+(-1)^{s+1}.\frac{2n-2s!\,(2s-n-1)}{2^n.s!\,n-s!}(ikr)^{2s}-\frac{2^{n+1}.s+1!\,(2s-n)}{2s+2!\,s-n!}(ikr)^{2s+1}\ldots\right\} \\ &\quad+(-1)^nB_nP_n(\mu)r^{-(n+2)}.ih\left\{\frac{2n!\,n(n+1)}{2^n.n!}\ldots+(-1)^s.\frac{2n-2s!\,n(n+1)}{2^n.s!\,n-s!}(ihr)^{2s}+\frac{2^{n+1}.s+1!\,n(n+1)}{2s+2!\,s-n!}(ihr)^{2s+1}\ldots\right\} \\ v_n &= (-1)^nA_n\frac{dP_n(\mu)}{d\theta}.r^{-(n+2)}.ik\left\{\frac{2n!}{2^n.n!}\ldots+(-1)^s.\frac{2n-2s!}{2^n.s!\,n-s!}(ikr)^{2s}+\frac{2^{n+1}.s+1!}{2s+2!\,s-n!}(ikr)^{2s+1}\ldots\right\} \\ &\quad+(-1)^{n+1}.B_n\frac{dP_n(\mu)}{d\theta}r^{-(n+2)}.ih\left\{\frac{2n!\,n}{2^n.n!}\ldots+(-1)^{s+1}.\frac{2n-2s!\,(2s-n)}{2^n.s!\,n-s!}(ihr)^{2s}-\frac{2^{n+1}.s+1!\,(2s-n-1)}{2s+2!\,s-n!}(ihr)^{2s+1}\ldots\right\} \end{aligned}\right\}(3),$$

7. Let us now suppose that the medium extends without limit. When r is very great $r^{n+1}\left(\frac{1}{r}\frac{d}{dr}\right)e^{ikr}=(ik)^{n+1}e^{ikr}$. Hence at a very great distance from the origin the coefficients of $P_n(\mu)$ and $\frac{dP_n(\mu)}{d\theta}$ in equation (5), § 5 become

Digitized by Google

$$u_n = \frac{ik}{r}\{C_n e^{i(pt+kr)} - C_n' e^{i(pt-kr)}\} + \frac{n(n+1)}{r^2}\{D_n e^{i(pt+hr)} + D_n' e^{i(pt-hr)}\},$$

$$v_n = \frac{1}{r^2}\{C_n e^{i(pt+kr)} + C_n' e^{i(pt-kr)}\} + \frac{ih}{r}\{D_n e^{i(pt+hr)} - D_n' e^{i(pt-hr)}\}.$$

The terms $e^{i(pt+hr)}$ and $e^{i(pt+kr)}$ represent disturbances travelling inwards. Hence, if the disturbance is to be zero at infinity, we have $C_n = 0$, $D_n = 0$; and the amount of the displacement at any point in a medium extending to infinity and bounded internally by a surface vibrating in any assigned manner is given by two functions u and v; of the form given in equation (5), § 5, where

$$f_n(ikr) = A_n r^n \left(\frac{1}{r}\frac{d}{dr}\right) e^{-ikr},$$

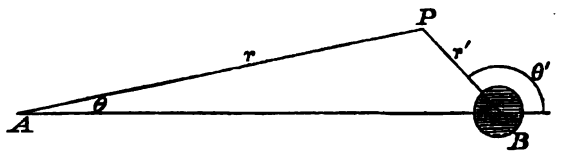
$$F_n(ihr) = B_n r^{n+1} \left(\frac{1}{r}\frac{d}{dr}\right) e^{-ihr}.$$

To investigate the disturbance produced by the presence of a small fixed sphere on the axis of symmetry, if there is no slipping at the surface of the sphere.

8. We shall now investigate the disturbance produced by the presence of a small fixed sphere on the axis of symmetry, where the disturbance if the sphere were removed would be given by the expressions

$$u = \text{displacement along } AP = \Sigma P_n(\mu)\left\{\frac{df_n}{dr} + \frac{n(n+1)}{r^2}F_n\right\} e^{ipt},$$

$$v = \text{displacement perpendicular to } AP = \Sigma \frac{1}{r}\frac{dP_n(\mu)}{d\theta}\left\{f_n + \frac{dF_n}{dr}\right\} e^{ipt}.$$



where

$$f_n = A_n r^n \left(\frac{1}{r}\frac{d}{dr}\right)^{n+1} e^{-ikr},$$

$$F_n = A_n' r^{n+1} \left(\frac{1}{r}\frac{d}{dr}\right)^{n+1} e^{-ihr},$$

the waves being supposed to be long compared with the distance between the origin A and the centre of sphere B. Let radius of B be b, and let distance between A and centre of B be c. In order to express the conditions that the displacement should be zero at the surface of B, we must find the resolved parts along and perpendicular to BP of the displacement given above by u and v.

Digitized by Google

Taking the coefficient of $P_n(\mu)$,

$$\left.\begin{aligned} u_n' &= \text{displacement along } BP = u_n \cdot \frac{r' + c\cos\theta'}{r} + v_n \frac{c\sin\theta'}{r} \\ v_n' &= \text{displacement perpendicular to } BP = -u_n \frac{c\sin\theta'}{r} + v_n \frac{r' + c\cos\theta'}{r} \end{aligned}\right\} \quad \text{..... (1).}$$

Taking the leading terms only of expressions for u_n, v_n given in equations (3) of paragraph 6; and omitting the time factor e^{ipt}, we get

$$\left.\begin{aligned} u_n' &= (-1)^n . ic . \frac{2n!}{2^n . n!} . \frac{A_n k - A_n' h}{r^{n+3}} \left\{ \sin\theta' \frac{dP_n(\mu)}{d\theta} - (n+1)\left(\cos\theta' + \frac{r'}{c}\right) P_n(\mu) \right\} \\ v_n' &= (-1)^n . ic . \frac{2n!}{2^n . n!} . \frac{A_n k - A_n' h}{r^{n+3}} \left\{ \left(\cos\theta' + \frac{r'}{c}\right) \frac{dP_n(\mu)}{d\theta} + (n+1) P_n(\mu) . \sin\theta' \right\} \end{aligned}\right\} \quad \text{..... (2).}$$

We shall have to express u_n' and v_n' in series involving Legendre's functions of μ', where $\mu' = \cos\theta'$ and their differential coefficients with respect to θ'.

This operation can be facilitated by the following transformations.

By a known formula*,

$$P_n(\mu) = (-1)^n . \frac{r^{n+1}}{n!} \frac{d^n}{dc^n} . \frac{1}{r},$$

therefore, differentiating $\frac{1}{r}$ n times with respect to c,

$$\frac{P_n(\mu)}{r^{n+1}} = \frac{1}{c^{n+1}} \left\{ 1 - (n+1) P_1(\mu') \frac{r'}{c} + \frac{(n+1)(n+2)}{2!} P_2(\mu') . \frac{r'^2}{c^2} \ldots \right\} \quad \text{.....(3).}$$

Differentiating this result (first) with respect to θ' keeping r' constant, and then with respect to r', keeping θ' constant, we get

$$\left.\begin{aligned} &\frac{1}{r^{n+3}} \left\{ \sin\theta' \frac{dP_n(\mu)}{d\theta} - \left(\cos\theta' + \frac{r'}{c}\right)(n+1) P_n(\mu) \right\} \\ &\quad = -\frac{1}{c^{n+3}} \left\{ (n+1) P_1(\mu') - \frac{(n+1)(n+2)}{1!} P_2(\mu') \frac{r'}{c} + \frac{(n+1)(n+2)(n+3)}{2!} P_3(\mu') \frac{r'^2}{c^2} \ldots \right\} \\ &\frac{1}{r^{n+3}} \left\{ \left(\cos\theta' + \frac{r'}{c}\right) \frac{dP_n(\mu)}{d\theta} + (n+1) \sin\theta' P_n(\mu) \right\} \\ &= -\frac{1}{c^{n+3}} \left\{ (n+1) \frac{dP_1(\mu')}{d\theta'} - \frac{(n+1)(n+2)}{2!} \frac{dP_2(\mu')}{d\theta'} \frac{r'}{c} + \frac{(n+1)(n+2)(n+3)}{3!} \frac{dP_3(\mu')}{d\theta'} \frac{r'^2}{c^2} \ldots \right\} \end{aligned}\right\} \text{..(4).}$$

Reducing the values of u_n', v_n' given in result (2) above by means of relations (4) we get

$$u_n' = (-1)^n . ic . \frac{2n!}{2^n . n!} \left\{ (n+1) P_1(\mu') - \frac{(n+1)(n+2)}{1!} P_2(\mu') \frac{r'}{c} + \ldots \right\} \frac{A_n k - nA_n' h}{c^{n+3}},$$

$$v_n' = (-1)^n . ic . \frac{2n!}{2^n . n!} \left\{ (n+1) \frac{dP_1(\mu')}{d\theta'} - \frac{(n+1)(n+2)}{2!} \frac{dP_2(\mu')}{d\theta'} \frac{r'}{c} + \ldots \right\} \frac{A_n k - nA_n' h}{c^{n+3}};$$

* See Maxwell's *Electricity*, Vol. I. art. 132, equation (28).

Digitized by Google

or
$$\left.\begin{aligned} u_n' &= (-1)^{n+1}.\,ic.\frac{2n!}{2^n.(n!)^2}\sum_{m=0}^{m=\infty}(-1)^m.\frac{m+n!}{m-1!}P_m(\mu').\frac{r'^{m-1}}{c^{m-1}}.\frac{A_nk-nA_n'h}{c^{n+2}} \\ v_n' &= (-1)^{n+1}.\,ic.\frac{2n!}{2^n.(n!)^2}\sum_{m=0}^{m=\infty}(-1)^m.\frac{m+n!}{m!}\frac{dP_m(\mu')}{d\theta'}.\frac{r'^{m-1}}{c^{m-1}}.\frac{A_nk-nA_n'h}{c^{n+2}} \end{aligned}\right\}\text{.....(5).}$$

In order that the surface conditions may be satisfied, we shall have to take $u_n'+(u_n')$ for the whole displacement along BP; and $v_n'+(v_n')$ for the whole displacement perpendicular to BP; where (u_n') and (v_n') are to be of such magnitude as to cause the whole displacement due to u_n, v_n to be zero at the surface of the sphere B. These terms (u_n') (v_n') must be of the form given in equations (3), § 6, namely,

$$\left.\begin{aligned} (u_n') &= \sum_{m=0}^{m=\infty}(-1)^{m+1}P_m(\mu')B_m.(ik)r'^{-(m+2)}\left\{\frac{2m!\,(m+1)}{2^m.\,m!}-\frac{2m-2!\,(m-1)}{2^m.1!\,m-1!}(ikr')^2+\frac{2m-4!\,(m-3)}{2^m.2!\,m-2!}(ikr')^4\ldots\right\} \\ &+\sum_{m=0}^{m=\infty}(-1)^mP_m(\mu').B_m'(ih)r'^{-(m+2)}\left\{\frac{2m!\,m(m+1)}{2^m.\,m!}-\frac{2m-2!\,m(m+1)}{2^m.1!\,m-1!}(ihr')^2+\frac{2m-4!\,m(m+1)}{2^m.2!\,m-2!}(ihr')^4\ldots\right\} \\ (v_n') &= \sum_{m=1}^{m=\infty}(-1)^m.\frac{dP_m(\mu')}{d\theta'}.B_m.(ik)r'^{-(m+2)}\left\{\frac{2m!}{2^m.\,m!}-\frac{2m-2!}{2^m.1!\,m-1!}(ikr')^2+\frac{2m-4!}{2^m.2!\,m-2!}(ikr')^4\ldots\right\} \\ &+\sum_{m=1}^{m=\infty}(-1)^{m+1}\frac{dP_m(\mu')}{d\theta'}.B_m'.(ih)r'^{-(m+2)}\left\{\frac{2m!\,m}{2^m.\,m!}-\frac{2m-2!\,(m-2)}{2^m.1!\,m-1!}(ihr')^2+\frac{2m-4!\,(m-4)}{2^m.2!\,m-2!}(ihr')^4\ldots\right\} \end{aligned}\right\}(6),$$

where the constants B_m, B_m' have to be chosen so as to satisfy the conditions

$$\left.\begin{aligned} u_n'+(u_n')&=0 \\ v_n+(v_n')&=0 \end{aligned}\right\},\ \text{when } r'=b.$$

These equations of condition will give

$$B_n.ik=(-1)^n.\,ic\,\frac{2n!(A_nk-nA_n'h)}{2^n.(n!)^2c^{n+2}}.\frac{2^{m-1}.\,m+n!}{2m-2!\,\{mk^2+(m+1)h^2\}}.\frac{b^{2m-1}}{c^{m-1}}$$
$$\left[2m+1-\frac{m(2m+1)k^4+4(m+1)h^4+(m+1)(2m-3)h^2k^2}{2(2m-3)\{mk^2+(m+1)h^2\}}b^2\right],$$

$$B_n'.ih=(-1)^n.\,ic\,\frac{2n!\,(A_nk-nA_n'h)}{2^n.(n!)^2c^{n+2}}.\frac{2^{m-1}.\,m+n!}{m.\,2m-2!\,\{mk^2+(m+1)h^2\}}.\frac{b^{2m-1}}{c^{m-1}}$$
$$\left[2m+1-\frac{(m+1)(2m+1)h^4+4mk^4+m(2m-3)h^2k^2}{2(2m-3)\{mk^2+(m+1)h^2\}}b^2\right],$$

to the second approximation. It is necessary to determine B_m and B_m' to this order of approximation, for, if we neglect the squares, it will be found on substitution that we get $(u_n')=0$, $(v_n')=0$.

Substituting in equations (6) and writing C_n for $(-1)^n ik.A_n$, C_n' for $(-1)^n.ih.nA_n'$, we get

$$\left.\begin{aligned} (u_n') &= \frac{2n!\,(C_n-C_n')}{2^n(n!)^2c^{n+2}}\sum_{m=0}^{m=\infty}(-1)^{m+1}P_m(\mu')\frac{m+n!\,[(2m+1)\{(m-1)k^2-(m+1)h^2\}r'^2-(m+1)(2m-1)(k^2-h^2)b^2]}{2.\,m!\,\{mk^2+(m+1)h^2\}r'^{m+2}}\frac{b^{2m-1}}{c^{2m-1}}.c^{4m} \\ (v_n') &= \frac{2n!\,(C_n-C_n')}{2^n(n!)^2c^{n+2}}\sum_{m=1}^{m=\infty}(-1)^m\frac{dP_m(\mu')}{d\theta'}.\frac{m+n!\,[(2m+1)\{mk^2-(m-2)h^2\}r'^2-m(2m-1)(k^2-h^2)b^2]}{2.\,m!\,\{mk^2+(m+1)h^2\}r'^{m+2}}\frac{b^{2m-1}}{c^{2m-1}}c^{4m} \end{aligned}\right\}(7).$$

Digitized by Google

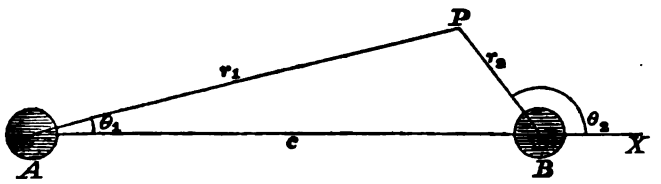
The displacement, whose components are given in equations (7), will be that produced by the presence of B in the field of vibration, if high powers of kr and hr are neglected.

9. Let us now consider the displacement produced by the simultaneous pulsations of two small spheres in an elastic medium, the waves in which are long compared with the distance between the spheres; the centres of the spheres being supposed fixed in space, and the displacements such that no slipping takes place at the surfaces of the spheres.

Let the radii of the spheres at the time t be given by the equations

$$r_a = a(1 + u_a \sin pt),$$
$$r_b = b(1 + u_b \sin pt),$$

where u_a, u_b are small quantities.



The displacement at any point due to the pulsations of A alone, if we neglect the disturbance due to the presence of B in the field of vibration, will be compounded of

$$u_1 = \text{displacement along } AP = \frac{d}{dr_1}\{f_1(kr_1 - pt)\},$$

$$v_1 = \text{displacement perpendicular to } AP = 0,$$

where
$$f_1(kr_1 - pt) = A\,\frac{\cos(pt - kr_1 + \alpha)}{r_1};$$

A and α being determined by the boundary conditions.

Similarly the displacement due to the pulsations of B alone will be compounded of

$$u_2 = \text{displacement along } BP = \frac{d}{dr_2}\{f_2(kr_2 - pt)\},$$

$$v_2 = \text{displacement perpendicular to } BP = 0,$$

where
$$f_2(kr_2 - pt) = \frac{B\cos(pt - kr_2 + \beta)}{r_2};$$

B and β being determined by the boundary conditions.

Putting f_1 and f_2 into the exponential form, we have

$$\left.\begin{aligned} f_1 &= \frac{iA\cos\alpha - A\sin\alpha}{2k}\left(\frac{1}{r_1}\frac{d}{dr_1}\right)e^{-i(kr_1 - pt)} - \frac{iA\cos\alpha + A\sin\alpha}{2k}\left(\frac{1}{r_1}\frac{d}{dr_1}\right)e^{i(kr_1 - pt)} \\ f_2 &= \frac{iB\cos\beta - B\sin\beta}{2k}\left(\frac{1}{r_2}\frac{d}{dr_2}\right)e^{-i(kr_2 - pt)} - \frac{iB\cos\beta + B\sin\beta}{2k}\left(\frac{1}{r_2}\frac{d}{dr_2}\right)e^{i(kr_2 - pt)} \end{aligned}\right\}\ldots(1).$$

Digitized by Google

The equations to find the constants are obtained by expressing that u_1 must be $au_a \sin pt$ at the surface of A and u_2 $bu_b \sin pt$ at the surface of B for all values of t.

These conditions give

$$\left.\begin{aligned} A\{ka\cos(ka-\alpha)-\sin(ka-\alpha)\} &= a^3u_a \\ ka\sin(ka-\alpha)+\cos(ka-\alpha) &= 0 \\ B\{kb\cos(kb-\beta)-\sin(kb-\beta)\} &= b^3u_b \\ kb\sin(kb-\beta)+\cos(kb-\beta) &= 0 \end{aligned}\right\}\text{.....(2)},$$

whence the constants can be determined.

The functions u_1 and u_2 will give the whole displacement to a high degree of approximation at considerable distances from the spheres, the radii of which are supposed to be small. But it is evident that, in the immediate neighbourhood of the pulsating bodies, these values for the displacement cannot safely be taken; for, the surface conditions $u = au_a \sin pt$ and $v = 0$ when $r_1 = a$, and the corresponding conditions when $r_2 = b$ will not be satisfied. We shall therefore have to investigate the disturbance at the surfaces of the spheres A and B, in order to find the terms that have to be added to complete the solution. This investigation would be very difficult in the general case; but, since in the case under consideration we take the waves to be long, so that the lowest powers only of kr need to be retained in the neighbourhood of the vibrating bodies, we can by successive applications of equations (7), § 8, obtain a solution to any order of approximation that may be required.

Taking account only of the lower powers of kc, ka, kb; we shall add terms u_2', v_2' of the forms given in the equations just alluded to, so that at those points, whose original distance from the centre of B was b, the displacement may be compounded of $au_b \sin pt$ along the radius vector and zero perpendicular to it. The conditions at the surface of A will not be satisfied by these values, but, if we transform to the centre of A as pole, we can add terms which will satisfy the conditions at the surface of that body. This series of operations will in general have to be continued indefinitely; but, if an approximate solution only is required, and if it should appear that the surface conditions are satisfied for both bodies if a certain order of small quantities is rejected, we shall have obtained a complete solution to that order of approximation.

Neglecting all the powers of ka and kb except the lowest, we have, by (2),

$$A = a^3u_a, \quad B = b^3u_b, \quad \sin\alpha = 1, \quad \sin\beta = 1,$$

therefore equations (1) become

$$\left.\begin{aligned} f_1(kr_1-pt) &= -\frac{a^3u_a}{2k}\left(\frac{1}{r_1}\frac{d}{dr_1}\right)e^{i(kr_1-pt)} - \frac{a^3u_a}{2k}\left(\frac{1}{r_1}\frac{d}{dr_1}\right)e^{-i(kr_1-pt)} \\ f_2(kr_2-pt) &= -\frac{b^3u_b}{2k}\left(\frac{1}{r_2}\frac{d}{dr_2}\right)e^{i(kr_2-pt)} - \frac{b^3u_b}{2k}\left(\frac{1}{r_2}\frac{d}{dr_2}\right)e^{-i(kr_2-pt)} \end{aligned}\right\}\text{.....(3)},$$

Digitized by Google

whence we obtain to a first approximation

$$u_1 = \frac{a^3 u_a}{2r_1^2} ie^{-ipt}, \qquad u_2 = \frac{b^3 u_b}{2r_2^2} ie^{-ipt} \quad \text{.....(4)},$$

with two other terms obtained by changing the sign of i.

Introducing the condition that, at all points which were when undisturbed on the surface of the sphere B, the displacement is to be $bu_b ie^{-ipt}$ along the radius vector and zero perpendicular to it; we shall find that, in order to find the whole disturbance, we shall have to introduce terms u_2', v_2' of the same form as those given in equation (7), § 8, so that

$$\left.\begin{aligned} u_2' &= ic\frac{a^3 u_a}{2c^3} e^{-ipt}\Sigma(-1)^{n+1}P_n(\mu_2).\frac{n\left[(2n+1)\{(n-1)k^2-(n+1)h^2\}r_2^2-(n+1)(2n-1)(k^2-h^2)b^2\right]}{2\{nk^2+(n+1)h^2\}r_2^{n+2}}\frac{b^{2n-1}}{c^{n-1}} \\ v_2' &= ic\frac{a^3 u_a}{2c^3} e^{-ipt}\Sigma(-1)^n\frac{dP_n(\mu_2)}{d\theta_2}.\frac{(2n+1)\{nk^2-(n-2)h^2\}r_2^2-n(2n-1)(k^2-h^2)b^2}{2\{nk^2+(n+1)h^2\}r_2^{n+2}}\frac{b^{2n-1}}{c^{n-1}} \end{aligned}\right\} \quad \text{.....(5)},$$

if $\mu_2 = \cos\theta_2$; θ_2 being the angle which r_2 makes with BX.

Again, using the condition that at the surface of A, u should be equal to $au_a ie^{-ipt}$ and that v should be zero; we shall have to add terms u_1', v_1' in order to counteract the disturbance caused by the displacement u_2 along BP. These terms will, as before, be

$$\left.\begin{aligned} u_1' &= -ic\frac{b^3 u_b}{2c^3} e^{-ipt}\Sigma P_n(\mu_1).\frac{n\left[(2n+1)\{(n-1)k^2-(n+1)h^2\}r_1^2-(n+1)(2n-1)(k^2-h^2)a^2\right]}{2\{nk^2+(n+1)h^2\}r_1^{n+2}}\frac{a^{2n-1}}{c^{n-1}} \\ v_1' &= ic\frac{b^3 u_b}{2c^3} e^{-ipt}\Sigma\frac{dP_n(\mu_1)}{d\theta_1}.\frac{(2n+1)\{nk^2-(n-2)h^2\}r_1^2-n(2n-1)(k^2-h^2)a^2}{2\{nk^2+(n+1)h^2\}r_1^{n+2}}\frac{a^{2n-1}}{c^{n-1}} \end{aligned}\right\} \quad \text{.....(6)}.$$

The disturbance produced by u_2', v_2' at the surface of A and by u_1', v_1' at the surface of B will be found to be of the third order of the small quantities $\frac{a}{c}$ and $\frac{b}{c}$. If therefore we neglect the terms of the third order, we shall have a complete solution for the displacement at any point. This displacement is given by those parts of the terms u_1, u_1', v_1'; u_2, u_2', v_2' which do not involve small quantities of the third order.

At the surface of A the displacement will consist of

$$\left.\begin{aligned} u &= \text{displacement along } AP = \frac{a^3 u_a}{2r_1^2} ie^{-ipt} - \frac{b^3 u_b}{2c^3} P_1(\mu_1) ie^{-ipt} + \frac{ab^3 u_b}{2c^3}.\frac{3h^2 r_1^2+(k^2-h^2)a^2}{(k^2+2h^2)r_1^3} P_1(\mu_1) ie^{-ipt} \\ v &= \text{displacement perpendicular to } AP \\ &= -\frac{b^3 u_b}{2c^3}\frac{dP_1(\mu_1)}{d\theta_1} ie^{-ipt} + \frac{ab^3 u_b}{2c^3}.\frac{3(k^2+h^2)r_1^2-(k^2-h^2)a^2}{2(k^2+2h^2)r_1^3}\frac{dP_1(\mu_1)}{d\theta_1} ie^{-ipt} \end{aligned}\right\} \quad \text{.....(7)},$$

with similar terms obtained by changing the sign of i.

Similarly the displacement at the surface of B can be obtained.

Digitized by Google

10. *Investigation of the mutual action of two small spheres pulsating in an elastic medium, the waves being supposed to be long compared with the distance between the centres of the spheres.*

Let us suppose the centres to be fixed in space, and assume that no slipping takes place at the surfaces. Under these circumstances the displacement can be found as in § 9, and if small quantities of the third and higher orders be neglected we shall have for the components of the displacement at the surface of A,

$$u = \frac{a^3 u_a}{2r_1^2} ie^{-ipt} - \frac{b^3 u_b}{2c^2} P_1(\mu_1) ie^{-ipt} + \frac{ab^3 u_b}{2c^3} \cdot \frac{3h^2 r_1^2 + (k^2 - h^2) a^2}{(k^2 + 2h^2) r_1^3} P_1(\mu_1) ie^{-ipt},$$

$$v = -\frac{b^3 u_b}{2c^2} \frac{dP_1(\mu_1)}{d\theta_1} ie^{-ipt} + \frac{ab^3 u_b}{2c^3} \frac{3(k^2 + h^2) r_1^2 + (k^2 - h^2) a^2}{2(k^2 + 2h^2) r_1^3} \frac{dP_1(\mu_1)}{d\theta_1} ie^{-ipt},$$

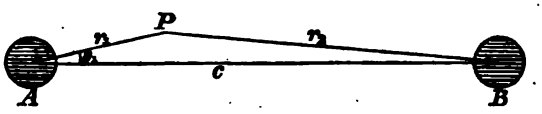
with two similar terms got by changing sign of i.

When $r_1 = a$ we shall have

$$u = \frac{au_a}{2} ie^{-ipt}, \quad v = 0, \quad \frac{du}{d\theta_1} = 0, \quad \frac{dv}{d\theta_1} = 0, \text{ and hence the dilatation } \epsilon = \frac{du}{dr_1}.$$

Also the force on the sphere A resolved along AB in the direction AB

$$= \int \left\{ \left(\lambda\epsilon + 2\mu \frac{du}{dr_1} \right) \cos\theta_1 - \mu \left(\frac{1}{r_1} \frac{du}{d\theta_1} + \frac{dv}{dr_1} - \frac{v}{r_1} \right) \sin\theta_1 \right\} d\sigma,$$



where $d\sigma$ is an element of the surface and the integration is performed over the whole of the sphere.

Since $\int P_n(\mu_1) \cos\theta_1 d\sigma = 0$ and $\int \frac{dP_n(\mu_1)}{d\theta_1} \sin\theta_1 d\sigma = 0$, unless $n = 1$, we need only attend to the terms in u and v involving $P_1(\mu_1)$ and $\frac{dP_1(\mu_1)}{d\theta_1}$.

Hence the force in the direction AB

$$= -\frac{3b^3 u_b}{2ac^2} \cdot ie^{-ipt} \int_0^\pi \frac{(\lambda + 2\mu) k^2 \cos^2\theta_1 + \mu h^2 \sin^2\theta_1}{k^2 + 2h^2} 2\pi r_a^2 \sin\theta d\theta.$$

Introducing the condition $(\lambda + 2\mu) k^2 = \mu h^2$ and taking account of the double sign of i, we get

$$\text{Force in direction } AB \text{ at the time } t = -\frac{12\pi\mu(\lambda + 2\mu)}{2\lambda + 5\mu} \frac{b^3}{ac^2} \cdot r_a^2 u_b \sin pt \ldots\ldots\ldots (1).$$

The resultant force on A can be found if we know the forms of u_a, u_b. If $u_a = \rho_1$, $u_b = \rho_2$, where ρ_1 and ρ_2 are constants, we shall have during a complete period $\frac{2\pi}{p}$

Digitized by Google

$$\text{Force on } A \text{ in direction } AB = -\frac{12\pi\mu(\lambda+2\mu)}{2\lambda+5\mu}\frac{b^3}{ac^3}\int_0^{\frac{2\pi}{p}} a^2\rho_2(1+\rho_1\sin pt)^2\sin pt\,dt$$

$$= -\frac{24\pi^2\mu(\lambda+2\mu)}{p(2\lambda+5\mu)}\rho_1\rho_2\frac{b^3a}{c^3} \quad \text{.....(2).}$$

This is a repulsion when ρ_1 and ρ_2 have the same sign and an attraction if they have opposite signs. In either case the force varies according to the inverse square of the distance between A and B.

11. *Second approximation.*

We shall now include the terms of the third order in the expression for the displacement. As in § 10, it is evident that the only terms which will enter into the expression for the force on A will be the coefficients of $P_1(\mu_1)$, and of $\frac{dP_1(\mu_1)}{d\theta_1}$. The coefficients of these terms in u_1, u_2, u_1', v_1' have already been considered, and the action arising from them has been given in equation (2), § 10. We have next to resolve the displacement given by u_2', v_2' in equations (5), § 9 along and perpendicular to AP. If the resolved components are (u_2') and (v_2') we get, if the terms of the fourth order are neglected,

$$\left.\begin{aligned}(u_2') &= \frac{iba^3u_0}{2c}e^{-ipt}\left\{\left(\cos\theta_1-\frac{r_1}{c}\right)P_1(\mu_2)\cdot\frac{3h^2r_2^2+(k^2-h^2)b^2}{(k^2+2h^2)r_2^4}+\sin\theta_1\frac{dP_1(\mu_2)}{d\theta_2}\cdot\frac{3(h^2+k^2)r_2^2-(k^2-h^2)b^2}{2(k^2+2h^2)r_2^4}\right\}\\ (v_2') &= -\frac{iba^3u_0}{2c}e^{-ipt}\left\{\sin\theta_1P_1(\mu_2)\frac{3h^2r_2^2+(k^2-h^2)b^2}{(k^2+2h^2)r_2^4}+\left(\cos\theta_1-\frac{r_1}{c}\right)\frac{dP_1(\mu_2)}{d\theta_2}\cdot\frac{3(h^2+k^2)r_2^2-(k^2-h^2)b^2}{2(k^2+2h^2)r_2^4}\right\}\end{aligned}\right\}\ (1).$$

These terms can be expanded in series involving Legendre's coefficients of μ_1 and their differentials with respect to θ_1. For by equation (3), § 8, we get

$$\frac{P_n(\mu_2)}{r_2^{n+1}} = (-1)^n\cdot\frac{1}{c^{n+1}}\left\{1+(n+1)P_1(\mu_1)\frac{r_1}{c}+\frac{(n+1)(n+2)}{2!}P_2(\mu_1)\frac{r_1^2}{c^2}+\ldots\right\} \quad \text{.....(2),}$$

therefore

$$\frac{\sin\theta_1\cdot P_n(\mu_2)}{r_2^{n+1}} = (-1)^{n+1}\cdot\frac{1}{c^{n+1}}\left[\frac{dP_1(\mu_1)}{d\theta_1}+\frac{n+1}{3}\frac{dP_2(\mu_1)}{d\theta_1}\frac{r_1}{c}\right.$$
$$\left.+\frac{(n+1)(n+2)}{5\,.\,2!}\left\{\frac{dP_3(\mu_1)}{d\theta_1}-\frac{dP_1(\mu_1)}{d\theta_1}\right\}\frac{r_1^2}{c^2}+\text{etc.}\right] \quad \text{.....(3),}$$

and
$$\frac{P_n(\mu_2)}{r_2^{n+1}}\left(\cos\theta_1-\frac{r_1}{c}\right) = (-1)^n\cdot\frac{1}{c^{n+1}}\left[P_1(\mu_1)+\frac{2(n+1)P_2(\mu_1)-(n-2)P_0(\mu_1)}{3\,.\,1!}\frac{r_1}{c}\right.$$
$$\left.+\frac{n+1}{5\,.\,2!}\{3(n+2)P_3(\mu_1)+2(n-3)P_1(\mu_1)\}\frac{r_1^2}{c^2}+\ldots\right] \quad \text{..... (4).}$$

Also differentiating equation (2) with respect to θ_1 keeping r_2 constant, we get

$$\frac{1}{r_2^{n+1}}\frac{dP_n(\mu_2)}{d\theta_2} = (-1)^n\cdot\frac{n(n+1)r_1}{c^{n+2}}\left\{\frac{1}{2!}\frac{dP_1(\mu_1)}{d\theta_1}+\frac{n+2}{3!}\frac{dP_2(\mu_1)}{d\theta_1}\frac{r_1}{c}\right.$$
$$\left.+\frac{(n+2)(n+3)}{4!}\frac{dP_3(\mu_1)}{d\theta_1}\frac{r_1^2}{c^2}+\ldots\right\} \quad \text{.....(5),}$$

Digitized by Google

therefore
$$\frac{\sin\theta_1}{r_2^{n+1}}\frac{dP_n(\mu_2)}{d\theta_2}=(-1)^n.\frac{n(n+1)r_1}{c^{n+2}}\left[\frac{1}{3.1!}\{P_2(\mu_1)-P_0(\mu_1)\}\right.$$
$$\left.+\frac{n+2}{5.2!}\{P_3(\mu_1)-P_1(\mu_1)\}\frac{r_1}{c}+\frac{(n+2)(n+3)}{7.3!}\{P_4(\mu_1)-P_2(\mu_1)\}\frac{r_1^2}{c^2}+\ldots\right]\ldots\ldots\ldots(6),$$

and
$$\frac{1}{r_2^{n+1}}\left(\cos\theta_1-\frac{r_1}{c}\right)\frac{dP_n(\mu_2)}{d\theta_2}=(-1)^n.\frac{n(n+1)r_1}{c^{n+2}}\left[\frac{1}{3.2!}\frac{dP_2(\mu_1)}{d\theta_1}\right.$$
$$\left.+\frac{1}{5.3!}\left\{2(n+2)\frac{dP_3(\mu_1)}{d\theta_1}+3(n-3)\frac{dP_1(\mu_1)}{d\theta_1}\right\}\frac{r_1}{c}+\frac{n+2}{7.4!}\left\{3(n+3)\frac{dP_4(\mu_1)}{d\theta_1}+4(n-4)\frac{dP_2(\mu_1)}{d\theta_1}\right\}\frac{r^2}{c^2}\ldots\right]$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

If all the terms of a higher order than the third be neglected, we get by the above equations

$$\left.\begin{aligned}(u_2')&=-\frac{iba^3u_a}{2c^3}e^{-ipt}\frac{3h^2}{k^2+2h^2}P_1(\mu_1)\\(v_2')&=-\frac{iba^3u_a}{2c^3}e^{-ipt}\frac{3h^2}{k^2+2h^2}\frac{dP_1(\mu_1)}{d\theta_1}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

Using equation (7), § 8, we find the terms which have to be added in order to make the whole displacement equal to $au_a ie^{-ipt}$ at the surface of A. They are

$$\left.\begin{aligned}u_1''&=\frac{iba^6u_a}{2c^3}e^{-ipt}.\frac{3h^2}{k^2+2h^2}\left\{\frac{3h^2}{(k^2+2h^2)a^2r_1}+\frac{k^2-h^2}{(k^2+2h^2)r_1^3}\right\}P_1(\mu_1)\\v_1''&=\frac{iba^6u_a}{2c^3}e^{-ipt}.\frac{3h^2}{k^2+3h^2}\left\{\frac{3(h^2+k^2)}{2(k^2+2h^2)a^2r_1}-\frac{k^2-h^2}{2(k^2+2h^2)r_1^3}\right\}\frac{dP_1(\mu_1)}{d\theta_1}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

Similarly, if we consider the whole displacement at the surface of B we can see that we shall have to add terms u_2'', v_2'' in order to satisfy the surface conditions to the third order of small quantities, where

$$\left.\begin{aligned}u_2''&=-\frac{iab^6u_b}{2c^3}e^{-ipt}.\frac{3h^2}{k^2+2h^2}\left\{\frac{3h^2}{(k^2+2h^2)b^2r_2}+\frac{k^2-h^2}{(k^2+2h^2)r_2^3}\right\}P_1(\mu_2)\\v_2''&=\frac{iab^6u_b}{2c^3}e^{-ipt}.\frac{3h^2}{k^2+2h^2}\left\{\frac{3(h^2+k^2)}{2(k^2+2h^2)b^2r_2}-\frac{k^2-h^2}{2(k^2+2h^2)r_2^3}\right\}\frac{dP_1(\mu_2)}{d\theta_2}\end{aligned}\right\}\ldots\ldots\ldots\ldots(10).$$

The effect of u_1'', v_1'' at the surface of B or of u_2'', v_2'' at the surface of A will be of the fourth order of small quantities, and can therefore be neglected to our approximation. Hence we have the complete expression for the coefficients of $P_1(\mu_1)$ and $\frac{dP_1(\mu_1)}{d\theta_1}$ to the third order of approximation.

The second term in the expression for the force on A may now be found as in § 10. It will be

$$-\frac{72\pi^2\mu(\lambda+2\mu)^2}{p(2\lambda+5\mu)^2}\rho_1^2\frac{ba^4}{c^3},$$

in the direction AB, if u_a and u_b are, as before, supposed to be constants. This gives a repulsive force, varying with the inverse cube of the distance between A and B.

Digitized by Google

12. If any explanations of the nature of gravitation, or of electrical forces, are to be based on suppositions of the transmission of motion, or stress through a medium intervening between the electrified bodies; it appears that such explanations must be based on the hypothesis that the medium behaves like an elastic solid, unless space is to be supposed to be filled with a second medium, whose properties differ from that in which light is supposed to be propagated. Thus it appears that hypotheses based on the theory of incompressible fluids must be erroneous; and it should be observed that even if two media are admitted, the second, if a fluid, can hardly be supposed to be absolutely incompressible. This appears to be a serious objection to a theory which has been adduced* to explain gravitation, based upon Bjerknes' investigation, in which it is supposed that all atoms are pulsating in phases not differing from one another by more than a quarter period, and that the intervening medium is an incompressible fluid. If this were the case, the law of universal attraction according to the law of the inverse square would follow; but unless the medium is supposed to be absolutely incompressible, in which case all pulsations would be instantaneously diffused throughout space, there would on this theory be repulsion between bodies at distances greater than a quarter wave length, and bodies would at certain distances repel one another, which is contrary to observation.

The electrical phenomena of attraction and repulsion are capable of explanation by the results proved in this paper. If we suppose two groups of particles to be pulsating, so that the phases of the particles in the first group lie between $\frac{p}{4}$ and $-\frac{p}{4}$, and that those of the other group lie between $\frac{3p}{4}$ and $-\frac{3p}{4}$; $2p$ being the complete period; we shall find, by an obvious extension of equation (2), § 10, that the particles of the same group will repel each other, the law of force being that of the inverse square of the distance, and will attract the particles belonging to the other group according to the same law of force. This is in accordance with observed electrical actions, and is based on the hypothesis that the medium which transmits electrical vibrations is the same as that by which the undulations of light are transmitted. The observed electrical effects of attraction and repulsion, conduction and induction, can be explained on this hypothesis, as in the ordinary two fluid theory. It should however be observed that the laws of attraction and repulsion are supposed to hold good only for distances which do not exceed a quarter wave length of the displacement corresponding to the period of the particles' pulsation. This is as yet unverified by experiment, but, if it be untrue, all hypotheses of electrical action arising from supposed periodic disturbances of the intervening medium owing to the electrical condition of the body will in like manner be vitiated. It should also be noticed that no hypothesis is made concerning the shape or density or other properties of the two groups of particles, which may be supposed to differ in any or all of these respects:—the only suppositions made concerning them are that they are pulsating, and that their phases satisfy the conditions above indicated. Several other electrical phenomena may be explained by a hypothesis similar to that given above. If we suppose the

* *Proceedings of the Cambridge Philosophical Society*, Vol. III. Part VII.

Digitized by Google

atoms composing matter to be also pulsating, then, by proper suppositions with regard to their phases, we can explain the cohesion of two or more atoms to form a molecule, the apparent comparative affinity for one or the other kind of electricity that bodies appear to possess, and several of the phenomena of electrolysis.

Digitized by Google

IV. *On the curves of constant intensity of homogeneous polarized light seen in a uniaxal crystal cut at right angles to the optic axis.* By C. SPURGE, B.A. (Plate II.)

[Read *January* 28, 1884.]

IN a paper published in the *Proceedings* of this Society I have demonstrated many important general properties of the curves given by the polar equation

$$\kappa = \pm \sin 2\theta \sin r^2$$

which are interesting from a physical point of view*.

The forms of the curves in the simple cases when $\kappa = 1$, $\kappa = 0$ when the curves are very nearly point curves and when r is very small were there shewn.

The object of the present communication is to shew what is the general form and relative position of the whole system of curves by giving an accurate tracing in which the positions of a number of points on the curves sufficiently close together are determined by actual numerical calculation (*vide* Plate II.).

The results of the calculation are given in the tables appended.

The method of conducting the calculation was as follows. The values of $\frac{\kappa}{\sin 2\theta}$ were tabulated for every two degrees of angle as regards θ and for values of κ increasing from ·1 to ·9. As these values can be easily computed from a table of natural cosecants it is unnecessary to give them. From the known values of $\frac{\kappa}{\sin 2\theta}$ (which is equal to $\sin r^2$) the least positive values of r^2 were obtained and reduced to circular measure. The values of r^2 thus obtained are given in Table I. The rest of the values of r^2 were found from the formula $n\pi \pm r^2$, where n is an integer and the square roots having been extracted the absolute values of r were determined to three places of decimals.

Thus the polar co-ordinates of a number of points were calculated.

To trace the curve lines were drawn through the origin at angular distances apart of 2°, and on these lines the values of r were set off according to a certain scale. The scale of the map which was adopted to get the curves within the limits of the sheet of curve paper was that a radius vector of absolute value 1 is represented by $\frac{1000}{8}$ scale divisions.

* See *Proc. Camb. Phil. Soc.* Vol. v. Pt. II.

Digitized by Google

The values of r in scale divisions are exhibited in Tables II. *a*, *b*, *c*, *d*, *e*. Thus the curves were accurately traced except near the points of contact of the tangents from the origin to the curves where r varies very considerably for a small increment in θ. Consequently the form of the curves near the points of contact was traced by describing a small arc of the circle of curvature at such a point. The values of ρ, the radius of curvature obtained for this purpose, will be found in the central column of the Tables II.

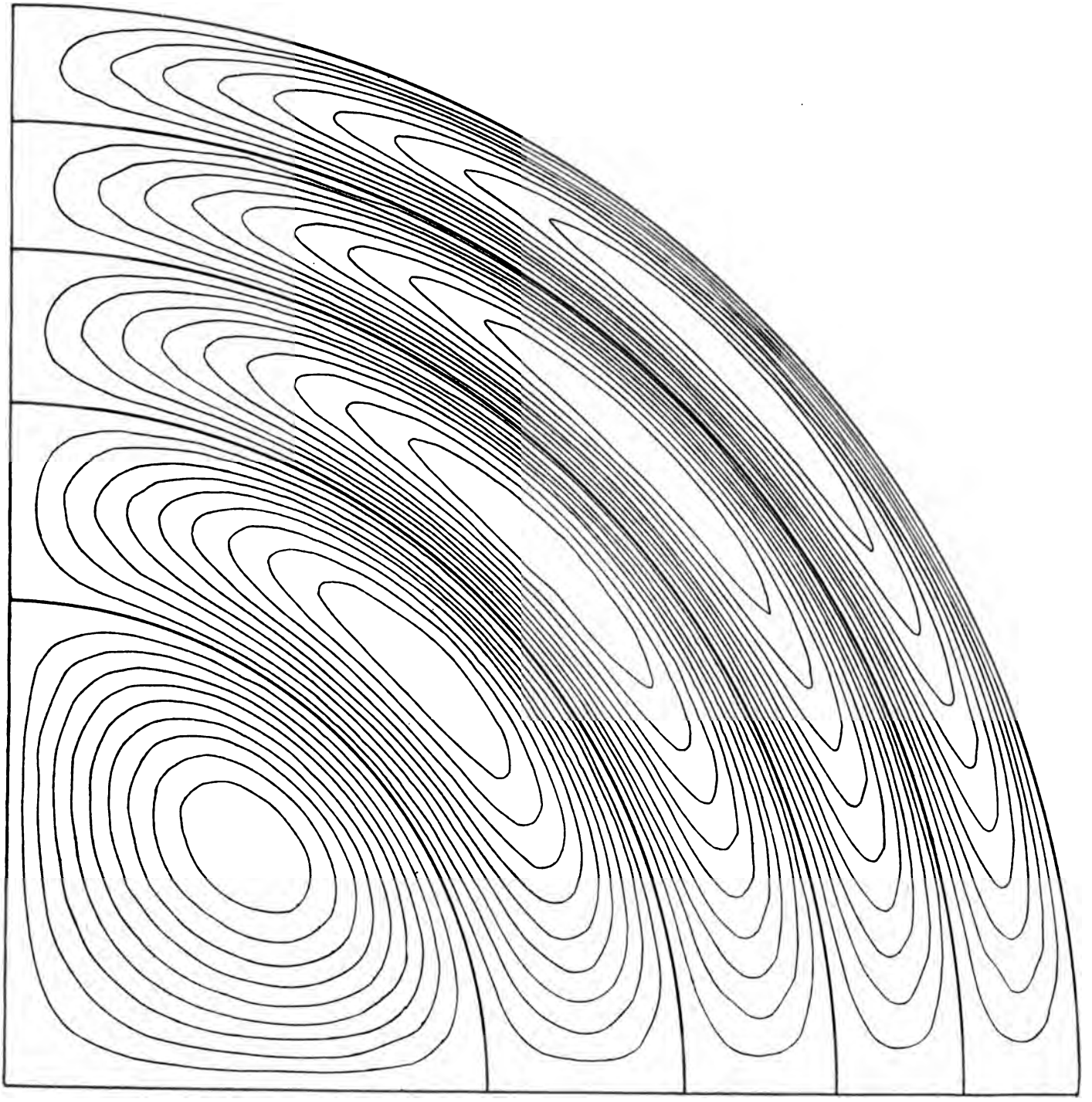
The tracing represents that part of the curve which lies in the first quadrant. It is unnecessary to give the rest as the curve is symmetrical in the four quadrants.

In conclusion I desire to express my best thanks to Mr R. T. Glazebrook for valuable suggestions at some stages of the work.

I. LEAST POSITIVE VALUES OF r^2.

$\kappa =$	·1	·2	·3	·4	·5	·6	·7	·8	·9
$\theta = 2°$									
4°	·8016846								
6°	·5017646	1·2940404							
8°	·3712760	·8118925							
10°	·2967151	·6245851	1·0698771						
12°	·2484059	·5141109	·8294764	1·3885397					
14°	·2146478	·4400743	·6932103	1·0198187					
16°	·1898462	·3870041	·6017966	·8553947	1·2331574				
18°	·1709617	·3471953	·5356386	·7484725	1·0172207				
20°	·1562069	·3163974	·4855752	·6716642	·8911563	1·2037727			
22°	·1444575	·2920450	·4465615	·6136097	·8034824	1·0426320			
24°	·1349727	·2724860	·4155464	·5683631	·7380093	·9396977	1·2283712		
26°	·1272435	·2566079	·3905549	·5323991	·6873646	·8654746	1·0936380		
28°	·1209162	·2436471	·3702681	·5032981	·6473930	·8091933	1·0053557	1·3053458	
30°	·1157280	·2330437	·3537414	·4801136	·6154795	·7653923	·9412419	1·1777850	
32°	·1114910	·2243963	·3403113	·4612198	·5899277	·7309284	·8927845	1·0975240	
34°	·1080631	·2174156	·3294897	·4460591	·5695664	·7038013	·8556141	1·0408361	1·3280466
36°	·1053409	·2118740	·3209187	·4340898	·5535743	·6826833	·8270294	·9994424	1·2416414
38°	·1032446	·2076108	·3143347	·4249170	·5413652	·6666641	·8058638	·9693010	1·1878071
40°	·1017171	·2046282	·3095476	·4182593	·5325280	·6551219	·7906323	·9481548	1·1527489
42°	·1007210	·2024824	·3064253	·4139219	·5267818	·6476385	·7808107	·9346756	1·1312789
44°	·1002286	·2014821	·3048841	·4117826	·5239504	·6439579	·7760110	·9281077	1·1210286

Digitized by Google



Digitized by Google

Digitized by Google

II (a). FIRST OVALS.

Tables of r in scale divisions.

(Should be multiplied by $\frac{6}{1000} = \cdot 006$ to reduce back to absolute values.)

Values for tangents are marked +	$\kappa =$ 0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1	·9	·8	·7	·6	·5	·4	·3	·2	·1	0
$\theta =$ 0°	0 to 209										Radius of curvature at points of contact.									209 to 295	
2°	0																				295
+ 2° 52′		209	—	—	—	—	—	—	—	—	$\rho = 421$	—	—	—	—	—	—	—	—	209	
4°	0	149																		255	295
+ 5° 46′			209	—	—	—	—	—	—	—	$\rho = 207$	—	—	—	—	—	—	—	209		
6°	0	118	189																226	271	295
8°	0	101	150																254	277	295
+ 8° 43′				209	—	—	—	—	—	—	$\rho = 134$	—	—	—	—	—	—	209			
10°	0	91	132	172														240	264	281	295
+ 11° 47′					209	—	—	—	—	—	$\rho = 97$	—	—	—	—	—	209				
12°	0	83	119	152	196												221	253	270	283	295
14°	0	77	110	139	168												243	261	274	285	295
+ 15°						209	—	—	—	—	$\rho = 73$	—	—	—	—	209					
16°	0	72	104	129	154	185										230	252	265	276	286	295
18°	0	69	98	122	144	168										243	258	269	278	287	295
+ 18° 26′							209	—	—	—	$\rho = 54$	—	—	—	209						
20°	0	66	94	116	136	157	183								232	250	262	271	280	288	295
22°	0	63	90	111	130	149	170								241	255	265	273	281	288	295
+ 22° 12′								209	—	—	$\rho = 43$	—	—	209							
24°	0	61	87	107	125	143	161	185						230	247	258	267	275	282	289	295
26°	0	59	84	104	121	138	155	174						238	251	261	269	276	283	289	295
+ 26° 33′									209	—	$\rho = 32$	—	209								
28°	0	58	82	101	118	134	150	167	190				226	243	254	263	271	277	284	290	295
30°	0	57	80	99	115	131	145	162	181				233	247	257	265	272	278	284	290	295
32°	0	55	79	97	113	128	142	157	175				238	250	259	266	273	279	284	290	295
+ 32° 4′										209	— $\rho = 20$ —	209									
34°	0	55	78	96	111	126	140	154	170	192		224	241	252	260	267	274	279	285	290	295
36°	0	54	77	94	110	124	138	151	167	186		230	244	253	261	268	274	280	285	290	295
38°	0	53	76	93	109	123	136	149	164	181		233	245	255	262	268	275	280	285	290	295
40°	0	53	75	93	108	122	135	148	162	179		235	247	255	262	269	275	280	285	290	295
42°	0	53	75	92	107	121	134	148	161	177		236	247	255	263	269	27	280	285	290	295
44°	0	53	75	92	107	120	134	147	160	176		237	248	256	263	269	275	281	286	291	295
45°	0	53	75	92	107	120	134	147	160	176	209	237	248	256	263	269	275	281	286	291	295

Digitized by Google

II (b). SECOND OVALS.

TABLES OF r IN SCALE DIVISIONS.

(Should be multiplied by $\frac{6}{1000} = \cdot 006$ to reduce back to absolute values.)

Values for tangents are marked +	$\kappa =$ 0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1	·9	·8	·7	·6	·5	·4	·3	·2	·1	0
$\theta = 0°$	295 to 362																			362 to 418	
2°	295										Radius of curvature at points of contact.										418
+ 2° 52′		362	—	—	—	—	—	—	—	—	$\rho = 81$	—	—	—	—	—	—	—	—	362	
4°	295	331																		390	418
+ 5° 46′			362	—	—	—	—	—	—	—	$\rho = 40$	—	—	—	—	—	—	—	362		
6°	295	318	351																372	401	418
8°	295	312	331																390	405	418
+ 8° 43′				362	—	—	—	—	—	—	$\rho = 26$	—	—	—	—	—	—	362			
10°	205	309	323	342														380	396	408	418
+ 11° 47′					362	—	—	—	—	—	$\rho = 19$	—	—	—	—	—	362				
12°	295	307	319	332	354												369	389	400	409	418
14°	295	305	315	326	340												382	394	403	410	418
+ 15°						362	—	—	—	—	$\rho = 14$	—	—	—	—	362					
16°	295	304	313	322	333	348										374	388	397	405	411	418
18°	295	303	311	319	329	340										382	392	399	406	412	418
+ 18° 26′							362	—	—	—	$\rho = 11$	—	—	—	362						
20°	295	303	310	317	325	335	347								375	387	395	401	407	412	418
22°	295	302	309	316	323	331	341								381	390	397	402	408	412	418
+ 22° 12′								362	—	—	$\rho = 8$	—	—	362							
24°	295	302	308	314	321	328	336	348						375	385	392	398	403	408	413	418
26°	295	302	307	313	319	326	333	343						380	388	394	400	404	409	413	418
+ 26° 33′									362	—	$\rho = 6$	—	362								
28°	295	301	306	312	318	324	331	339	351				372	383	390	395	401	405	409	413	418
30°	295	301	306	311	317	323	329	336	346				376	385	391	396	401	406	410	414	418
32°	295	300	306	311	317	322	328	334	343				379	387	392	397	401	406	410	414	418
32° 4′										362	— $\rho = 4$ —	362									
34°	295	300	305	311	316	321	327	333	341	352		371	381	388	393	398	402	406	410	414	418
36°	295	300	305	310	316	320	326	332	339	349		374	383	389	394	398	402	407	410	414	418
38°	295	300	305	310	315	320	325	331	338	347		376	384	389	395	399	403	407	410	414	418
40°	295	300	305	310	315	319	325	330	337	345		377	385	390	395	399	403	407	410	414	418
42°	295	300	305	310	315	319	324	330	337	344		378	385	390	396	399	403	407	410	414	418
44°	295	300	305	309	314	319	324	330	336	344		379	386	391	396	400	404	407	411	414	418
45°	295	300	305	309	314	319	324	330	336	344	362	379	386	391	396	400	404	407	411	414	418

Digitized by Google

II (c). THIRD OVALS.

Tables of r in scale divisions.

(Should be multiplied by $\frac{6}{1000} = \cdot 006$ to reduce back to absolute values.)

Values for tangents are marked +	$\kappa =$ 0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1	·9	·8	·7	·6	·5	·4	·3	·2	·1	0
$\theta = 0°$	418 to 467																			467 to 512	
2°	418																				512
+ 2° 52′		467	—	—	—	—	—	—	—	—	$\rho = 38$	—	—	—	—	—	—	—	—	467	
4°	418	444																		489	512
+ 5° 46′			467	—	—	—	—	—	—	—	$\rho = 18$	—	—	—	—	—	—	—	467		
6°	418	434	459																475	498	512
8°	418	430	444																489	501	512
+ 8° 43′				467	—	—	—	—	—	—	$\rho = 12$	—	—	—	—	—	—	467			
10°	418	427	437	452														482	494	503	512
+ 11° 47′					467	—	—	—	—	—	$\rho = 9$	—	—	—	—	—	467				
12°	418	426	434	445	461												472	488	497	505	512
14°	418	425	432	440	450												483	492	499	506	512
+ 15°						467	—	—	—	—	$\rho = 6$	—	—	—	—	467					
16°	418	424	430	437	445	457										477	488	495	501	506	512
18°	418	423	429	435	442	450										483	491	497	502	507	512
+ 18° 26′							467	—	—	—	$\rho = 5$	—	—	—	467						
20°	418	423	428	433	439	446	456								478	487	493	498	503	507	512
22°	418	423	427	432	437	444	451								482	489	495	499	504	507	512
+ 22° 12′								467	—	—	$\rho = 4$	—	—	467							
24°	418	422	427	431	436	442	448	457						477	485	491	496	500	504	508	512
26°	418	422	426	430	435	440	445	453						481	487	493	497	501	505	508	512
+ 26° 33′									467	—	$\rho = 3$	—	467								
28°	418	422	426	429	434	439	444	450	459				475	483	489	494	498	501	505	508	512
30°	418	422	425	429	433	438	442	448	455				478	485	490	495	498	502	505	509	512
32°	418	421	425	429	433	437	441	446	453				481	487	491	495	498	502	505	509	512
+ 32° 4′										467	— $\rho = 2$ —	467									
34°	418	421	425	429	432	436	440	445	451	460		474	482	488	492	495	499	502	506	509	512
36°	418	421	425	428	432	436	440	444	450	457		477	483	488	493	496	499	503	506	509	512
38°	418	421	425	428	432	435	439	444	449	455		478	484	489	493	496	499	503	506	509	512
40°	418	421	425	428	432	435	439	444	448	454		479	485	489	493	496	499	503	506	509	512
42°	418	421	425	428	432	435	439	444	448	454		479	485	489	493	496	499	503	506	509	512
44°	418	421	424	428	431	435	439	443	447	453		480	486	490	494	497	500	503	506	509	512
45°	418	421	424	428	431	435	439	443	447	453	467	480	486	490	494	497	500	503	506	509	512

Radius of curvature at points of contact.

Digitized by Google

II (d). FOURTH OVALS.

TABLES OF r IN SCALE DIVISIONS.

(Should be multiplied by $\frac{6}{1000} = \cdot 006$ to reduce back to absolute values.)

Values for tangents are marked +	$\kappa =$ 0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1	·9	·8	·7	·6	·5	·4	·3	·2	·1	0
$\theta = 0°$	512 to 553																			553 to 591	
2°	512										Radius of curvature at points of contact.										591
+ 2° 52′		553									$\rho = 23$									553	
4°	512	533																		572	591
+ 5° 46′			553								$\rho = 11$								553		
6°	512	525	546																559	579	591
8°	512	521	533																571	582	591
+ 8° 43′				553							$\rho = 7$							553			
10°	512	519	528	540														565	576	584	591
− 11° 47′					553						$\rho = 5$						553				
12°	512	518	525	534	548												557	571	578	585	591
14°	512	517	523	530	538												566	574	580	586	591
+ 15°						553					$\rho = 4$					553					
16°	512	517	522	528	534	544										561	570	576	582	586	591
18°	512	516	521	526	531	538										566	573	578	582	587	591
+ 18° 26′							553				$\rho = 3$				553						
20°	512	516	520	525	529	535	543								562	569	575	579	583	587	591
22°	512	516	519	524	528	533	539								566	571	576	580	584	587	591
+ 22° 12′								553			$\rho = 2 \cdot 3$			553							
24°	512	515	519	523	527	531	536	544						561	568	573	577	581	584	587	591
26°	512	515	518	522	526	530	534	540						564	570	574	578	581	585	587	591
+ 26° 33′									553		$\rho = 1 \cdot 7$		553								
28°	512	515	518	521	525	529	533	538	546				559	567	571	575	579	582	585	587	591
30°	512	515	518	521	525	528	532	536	543				562	568	572	576	579	582	585	588	591
32°	512	515	518	521	524	527	531	535	540				564	569	573	577	579	582	585	588	591
+ 32° 4′										553	$\rho = 1 \cdot 1$	553									
34°	512	515	517	521	524	527	531	534	539	546		559	566	570	574	577	580	582	586	588	591
36°	512	515	517	520	523	526	530	534	538	544		561	567	570	574	577	580	583	586	588	591
38°	512	515	517	520	523	526	530	533	537	543		562	567	571	574	578	580	583	586	588	591
40°	512	515	517	520	523	526	530	533	537	542		563	567	571	574	578	580	583	586	588	591
42°	512	515	517	520	523	526	530	533	536	542		563	567	571	574	578	580	583	586	588	591
44°	512	514	517	520	523	526	529	532	536	541		564	568	572	575	578	581	583	586	588	591
45°	512	514	517	520	523	526	529	532	536	541	553	564	568	572	575	578	581	583	586	588	591

Digitized by Google

II (e). FIFTH OVALS.

TABLES OF r IN SCALE DIVISIONS.

(Should be multiplied by $\frac{6}{1000} = \cdot 006$ to reduce back to absolute values.)

Values for tangents are marked +	$\kappa =$ 0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1	·9	·8	·7	·6	·5	·4	·3	·2	·1	0
$\theta = 0°$	591 to 627																			627 to 660	
2°	591										Radius of curvature at points of contact.										660
+ 2° 52′		627	—	—	—	—	—	—	—	—	$\rho = 15$	—	—	—	—	—	—	—	—	627	
4°	591	609																		643	660
+ 5° 46′			627	—	—	—	—	—	—	—	$\rho = 8$	—	—	—	—	—	—	—	627		
6°	591	603	621																633	650	660
8°	591	600	610																643	653	660
+ 8° 43′				627	—	—	—	—	—	—	$\rho = 5$	—	—	—	—	—	—	627			
10°	591	598	606	615														638	647	654	660
+ 11° 47′					627	—	—	—	—	—	$\rho = 4$	—	—	—	—	—	627				
12°	591	597	603	610	623												631	643	650	655	660
14°	591	596	601	607	614												639	646	651	656	660
+ 15°						627	—	—	—	—	$\rho = 3$	—	—	—	—	627					
16°	591	596	600	604	611	619										634	642	648	652	657	660
18°	591	595	599	603	608	614										639	645	649	653	657	660
+ 18° 26′							627	—	—	—	$\rho = 2$	—	—	—	627						
20°	591	595	598	602	606	611	618								635	642	646	650	654	657	660
22°	591	595	598	601	605	609	615								638	644	647	651	654	657	660
+ 22° 12′								627	—	—	$\rho = 1 \cdot 6$	—	—	627							
24°	591	594	597	601	604	608	613	619						634	640	645	648	652	654	658	660
26°	591	594	597	600	603	607	611	616						637	642	646	649	652	655	658	660
+ 26° 33′									627	—	$\rho = 1 \cdot 3$	—	627								
28°	591	594	597	600	603	606	610	614	621				632	639	643	647	649	653	655	658	660
30°	591	594	597	599	602	605	609	612	618				635	640	644	647	650	653	655	658	660
32°	591	593	597	599	602	605	608	611	616				637	641	645	648	650	653	655	658	660
+ 32° 4′										627	— $\rho = \cdot 75$	—	627								
34°	591	593	596	599	601	604	607	610	615	621		632	638	642	646	648	651	653	655	658	660
36°	591	593	596	598	601	604	607	610	614	619		634	639	642	646	648	651	654	655	658	660
38°	591	593	596	598	601	604	607	609	613	618		635	640	643	646	648	651	654	655	658	660
40°	591	593	596	598	601	604	607	609	613	617		636	640	643	646	648	651	654	655	658	660
42°	531	593	596	598	600	604	607	609	612	617		636	640	643	646	648	652	654	655	658	660
44°	591	593	596	598	600	603	606	609	612	617		637	641	644	647	649	652	654	656	658	660
45°	591	593	595	598	600	603	606	609	612	616	627	637	641	644	647	649	652	654	656	658	660

Digitized by Google

Digitized by Google

V. *On a certain Atomic Hypothesis.* By Professor KARL PEARSON, M.A. Communicated by H. T. STEARN, M.A.

[Read *February* 2, 1885.]

On a certain Atomic Hypothesis.

1. THAT view of the physical universe which regards it as an absolutely continuous medium seems to be rapidly replacing the old molecular hypotheses. In some manner or other we suppose certain portions of this medium to be differentiated off from the rest, these portions are indestructible, are possessed of an internal energy and an external energy (i.e. one of translation), and by reason of this energy give rise to that group of phenomena which we are accustomed to class as attributes of 'matter.' One extremely beautiful hypothesis would make these differentiated portions, which for the purposes of this paper I shall term *atoms*, consist in vortices. It might perhaps be sufficient to suppose that these atoms were portions of the universal medium in motion, that the portions themselves were unchangeable and their motion indestructible, in character. It would then be the aim of science to explain by means of these atoms, that is, by the very fact of their motion, vibratory and translatory, within this continuous medium all the usual phenomena of matter, gravitation, electricity, heat, light, magnetism, etc. Let us mark the difference, and it is a fundamental one, between the old and the new view of the universe, the old view endowed its atoms with certain inherent forces, and having done so, more or less completely ignored the existence of the medium; the new view endows its atoms with no inherent forces, but with motion—it looks to the action of this motion on the medium to explain the action of one atom on another. The old view saw everywhere in the universe force, the new view finds everywhere motion—that is a gross way of putting the difference. Wherever we now find force, we must seek for atomic vibrations, oscillations or translations, which will produce the requisite stream lines in the continuous medium, and so the apparent force. The whole problem of physics is reduced to one of hydrodynamics, and a new meaning seems to be thrown upon the recurrence of certain typical equations in nearly all branches of physics. The strongest light hitherto cast upon this view of the universe has been from the researches of Prof. Bjerknes of Christiania on the phenomena which he has classed under the term Hydromagnetism. Two spherical bodies oscillating in a fluid medium are found to act with similar forces to two magnetic molecules placed at their centres with axes in the directions of the oscillations. If we suppose our ultimate atoms to be spherical, this result throws considerable light on the nature of that atomic motion in the all pervading medium, which may give rise to magnetic phenomena.

Digitized by Google

2. The object of this paper will be to seek a corresponding analogy between the results of some motion of the atoms and phenomena which are usually classed as gravitation, cohesion, chemical combination, etc. It will be noted that this theory does not deny the truth, for instance, of such a hypothesis as the Kinetic Theory of Gases, rather it throws new light upon it, by suggesting the nature of the "forces" which act between two free molecules of a gas. The motion of an atom will be of two kinds, vibratory and translatory (the motion of rotation is for our *present* purpose neglected); now if we suppose the vibratory motion to be of the same kind as the vibrations which we term light, the velocity of translation of an atom due to its motion with the planetary system through space, will be an indefinitely small quantity compared with the rate at which a point on the surface of the atom is capable of moving. For example, let the atoms be spherical, and let q_1, q_2 be the translatory velocities of the atoms in directions h_1, h_2; let a_1, a_2 be the radii of the spheres and γ the distance between their centres; d_1, d_2 the outward velocity of the extremity of any normal, supposed the same all over the same atom; ρ the 'density' of the medium; then, it has been shewn (*Quarterly Journal of Math.* Vol. XX. p. 77) that there are terms in the total energy of the two-atom system, which, so far as terms of the order $\left\{\frac{\text{radius}}{\text{central distance}}\right\}^3$ are concerned, may be written

$$\frac{4\pi d_1 d_2 a_1^3 a_2^3 \rho}{\gamma} - \pi q_1 q_2 a_1^3 a_2^3 \rho \frac{d^2}{dh_1 dh_2}\left(\frac{1}{\gamma}\right) + 2\pi d_2 q_1 a_1^3 a_2^3 \rho \frac{d}{dh_1}\left(\frac{1}{\gamma}\right) + 2\pi d_1 q_2 a_1^3 a_2^3 \rho \frac{d}{dh_2}\left(\frac{1}{\gamma}\right).$$

Now if q_1, q_2 be merely velocities of the atoms through space of a kind which we may term planetary motion and d_1, d_2 velocities corresponding to light vibrations, the first term is here all important and the others may be neglected. If however q_1, q_2 were velocities due to oscillations of the spherical atoms capable of producing phenomena analogous to the magnetic, then the last three terms might be as important as the first. We might then divide the terms of the above energy-expression as follows:

1st term. Involves only vibrations of the atoms *perhaps* analogous to those which produce light, heat, etc.

2nd term. Involves only oscillations of the atoms *perhaps* analogous to those which produce magnetic phenomena.

3rd and 4th terms. Involve both vibrations and oscillations of the atoms and *perhaps* give rise to phenomena analogous to those in which light and magnetism are intimately connected.

It will be noted that these are only analogies and analogies too of the vaguest description, yet it cannot be denied that they suggest possible directions of research, which might at least throw light, if only of a negative kind, on the constitution of the "continuous medium."

3. For the object we have at present in view we shall suppose no "hydromagnetic" oscillations of the atoms to exist, we shall also neglect all interstellar motion of the atoms, and suppose them to be "endowed" only with vibratory motions. To bring the analysis within comparatively narrow limits, we shall suppose a very simple character for our

Digitized by Google

atoms and their vibrations, namely, that they are spherical portions differentiated from the continuous medium (henceforth merely for brevity to be termed *the ether*) and that their vibrations are uniform over their entire surfaces, or as it were pulsations of their entire surfaces. These pulsations may of course consist of a number of terms of different periods, and when an atom is "free," as in the case of gas or vapour, at very small pressure, it will be pulsating with a certain number of "beats" of fixed period characteristic of that particular class of atom, these may be termed its *natural pulsations*. When however an atom is constrained by its proximity to other atoms, it is probable that a certain number of forced pulsations take the place of the natural pulsations; it is probable that these will not be uniform over the surface of the sphere, but have a *polar* character; for the purpose however of our present investigation we shall ultimately suppose all vibrations of the atoms whether natural or forced to be uniform or non-polar, we shall denote by pulsation such a non-polar vibration. The *natural* pulsations in the case of a *free* atom will be analogous to those peculiar vibrations in the case of a molecule which in the kinetic theory of gases are supposed to give the bright lines of the gas's spectrum*. Now, so far as the light spectrum is concerned, we have reason to believe that at small pressure and not too high temperature we get a single bright line *only* from any gas or vapour. Hence we might from analogy assume that there was only one natural pulsation in any given class of atoms; as however such assumption would not alter our analysis it seems unnecessary at present to adopt it. As our atoms become more and more constrained by increase, for example, of pressure, a series of forced pulsations is imposed on the free pulsations, till at last in the solid state the latter are absolutely indistinguishable. Let us follow the analogy with the spectrum: as the pressure of the gas increases, the number of bright lines increases, the spectrum becomes more and more complex and after passing through various stages of semi-continuity, at last in the case of a solid becomes absolutely continuous; in other words, the molecules in the case of a solid (or liquid) must within certain limits be giving out all conceivable vibrations. What is the analogy in the case of our pulsating atoms? namely that when they are in a state of great constraint as in the case when they form a solid or liquid body, the enforced pulsations give a continuous range of "beats"; even as a piece of iron or a piece of platinum give the same continuous spectrum, so the atoms of any two solids though they may have different *natural* pulsations, yet owing to the *enforced* pulsations have all the same continuous range of "beats." This is an analogy which will be found of the highest suggestiveness, when we endeavour to explain the phenomenon of gravitation. Furthermore, we may remark that it is not difficult to understand how a molecule may be composed of a group of like or different spherical atoms; such a molecule may itself have *free* vibrations, while the atoms which compose it have, owing to their mutual action, constrained pulsations; the free vibrations of this molecule may have no apparent relation to the natural pulsations of the atoms which compose it, even as the spectrum of a compound bears no obvious relation to those of its elements. Or again a substance composed of like atoms may exist in various molecular states each with its own spectrum. All our hypothesis has

* Lockyer's *Studies in Spectrum Analysis*, p. 131.

Digitized by Google

hitherto forced us to assume is this: that when any substance is completely disassociated, i.e. its molecules broken up into their ultimate atoms and these atoms sufficiently far from each other to be vibrating freely, then these atoms are spherical differentiated portions of the ether with certain definite natural pulsations only.

4. We proceed to consider the hydrodynamical theory of such pulsating spherical atoms. We suppose that a velocity potential exists and therefore that there is no *rotational* motion of the ether, at least outside the differentiated spherical portions. As surface condition at such portion or atom we make the normal velocity of the fluid equal to that of the surface. It will be sufficient to consider two atoms, the results will be with certain modifications capable of extension to any number.

Let a_1, a_2 be the radii of the two atoms; r_1, r_2 the distances of any point in the ether from their centres, and γ the distance between their centres. Then $\frac{d}{d\gamma}$ will denote a differentiation in the direction of γ; in order to mark the direction of this differentiation we shall use $\frac{d}{d\gamma_{12}}$ when we are considering γ as measured from the centre of first to the centre of second atom; $\frac{d}{d\gamma_{21}}$ to denote the opposite direction, so that $d\gamma_{12} = -d\gamma_{21}$. Let d_1, d_2 be the *outward* velocities of the extremity of any normal of either sphere respectively at time t. The normals ν_1, ν_2 will be measured outward into the fluid. The conditions which the velocity potential ψ must satisfy are the following: at an infinite distance it must be constant or zero, throughout space not occupied by the atoms we must have

$$\nabla^2\psi = 0,$$

and finally at the surfaces of the two atoms we must have

$$\frac{d\psi}{d\nu_1} = d_1, \quad \frac{d\psi}{d\nu_2} = d_2 \text{ respectively.}$$

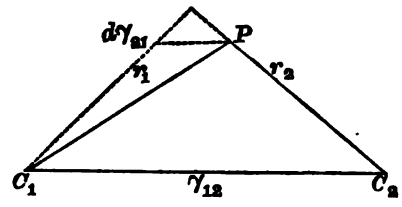
A value of ψ which satisfies the first two conditions is given by

$$\psi = \Sigma_0^\infty \left\{ A_p \left(\frac{d}{d\gamma_{12}}\right)^p \frac{1}{r_1} + B_p \left(\frac{d}{d\gamma_{21}}\right)^p \frac{1}{r_2} \right\} \text{.....(i).}$$

Can we by a proper choice of the constants A_p, B_p make it satisfy the third set of conditions? If we can, ψ will then be the required solution.

Let P be any point whose distances from the centres C_1 and C_2 are r_1 and r_2. Let the $\angle C_2C_1P = \theta$ and $\cos\theta = \mu$. Then, if γ be a constant, r_2 is a function of r_1 and μ,

$$r_2^2 = \gamma^2 + r_1^2 - 2\mu r_1\gamma.$$



And hence, if γ be greater than ν_1,

$$\frac{1}{r_2} = \frac{1}{\gamma} \cdot \Sigma_0^\infty \left(\frac{r_1}{\gamma}\right)^n \cdot P_n(\mu) \text{ .....(ii),}$$

where $P_p(\mu)$ denotes the pth zonal harmonic of μ.

Digitized by Google

Again
$$\frac{d}{d\gamma_{21}} = \frac{dr_1}{d\gamma_{21}} \cdot \frac{d}{dr_1} + \frac{d\mu}{d\gamma_{21}} \cdot \frac{d}{d\mu},$$
the subject to be operated upon being a function of r_2.

Now
$$d\gamma_{21} \cdot \mu = -dr_1,$$
$$d\gamma_{21} \sin\theta = r_1 d\theta;$$
and therefore
$$d\gamma_{21} \sin^2\theta = -r_1 d\mu;$$
therefore
$$\frac{d}{d\gamma_{21}} = -\left\{\frac{1-\mu^2}{r_1}\frac{d}{d\mu} + \mu\frac{d}{dr_1}\right\}.$$

Hence
$$\frac{d}{d\gamma_{21}}\{r_1^n \cdot P_n(\mu)\} = -r_1^{n-1}\left\{(1-\mu^2)\frac{dP_n}{d\mu} + n\mu P_n\right\} = -nr_1^{n-1}P_{n-1}(\mu);$$
therefore
$$\frac{d^2}{d\gamma_{21}^2}\{r_1^n P_n(\mu)\} = n(n-1)r_1^{n-2}P_{n-2}(\mu).$$

And generally,
$$\frac{d^p}{d\gamma_{21}^p}\{r_1^n P_n(\mu)\} = (-1)^p n(n-1)\ldots(n-p+1)\, r_1^{n-p}P_{n-p}(\mu).$$

We have then
$$\frac{d^p}{d\gamma_{21}^p}\left(\frac{1}{r_2}\right) = \frac{(-1)^p}{\gamma^{p+1}}\Sigma_p^\infty\left(\frac{r_1}{\gamma}\right)^{n-p} n(n-1)\ldots(n-p+1)\, P_{n-p}(\mu) \quad\text{.....(iii).}$$

Further,
$$\frac{d^p}{d\gamma_{12}^p}\left(\frac{1}{r_1}\right) = \frac{(-1)^p\,\underline{|p}}{r_1^{p+1}}P_p(\mu) \quad\text{.....(iv).}$$

Differentiating these with regard to r_1, we have
$$\left.\begin{aligned}\frac{d^{p+1}}{dr_1 d\gamma_{21}^p}\left(\frac{1}{r_2}\right) &= \frac{(-1)^p}{\gamma^{p+2}}\Sigma_{p+1}^\infty\left(\frac{r_1}{\gamma}\right)^{n-p-1} n.\overline{n-1}\ldots\overline{n-p}\,P_{n-p}(\mu)\\ \frac{d^{p+1}}{dr_1 d\gamma_{12}^p}\left(\frac{1}{r_1}\right) &= \frac{(-1)^{p+1}\,\underline{|p+1}}{r_1^{p+2}}P_p(\mu)\end{aligned}\right\}\quad\text{.....(v).}$$

To find $\frac{d\psi}{d\nu_1}$ we must substitute a_1 for r_1 in (v) and then take (v) in conjunction with (i); we obtain
$$\frac{d\psi}{d\nu_1} = d_1 = \Sigma_{p=0}^{p=\infty}\left[\frac{A_p(-1)^{p+1}\,\underline{|p+1}}{a_1^{p+2}}P_p(\mu) + \frac{(-1)^p B_p}{\gamma^{p+2}}\Sigma_{n=p+1}^{n=\infty}\left(\frac{a_1}{\gamma}\right)^{n-p-1} n\,\overline{n-1}\ldots\overline{n-p}\,P_{n-p}(\mu)\right]$$
.....(vi).

5. It remains to equate to zero the coefficients of the various zonal harmonics. We may note however that equation (vi) gives us a means of solving a still more general problem; namely, if instead of a simple pulsation as given by d_1, we had a vibration of the atom symmetrical about a pole coinciding with the central distance, i.e. if d_1 was a function of the form
$$\phi_0 P_0(\mu) + \phi_1 P_1(\mu) + \phi_2 P_2(\mu) + \text{etc.}$$

Digitized by Google

A vibration however about a pole *not peculiar* to the atom such as this would not be one of its *free* vibrations; it would however be the pole about which the free pulsations of the second atom would create *forced* vibrations in the first. In order to see more clearly the nature of these forced vibrations, we shall for the present follow the more general problem, and assume

$$d_1 = \Sigma_0^\infty \dot{\phi}_p . P_p(\mu), \qquad d_2 = \Sigma_0^\infty \dot{\phi}'_p . P_p(\mu').$$

It will always be possible to return to our original problem by putting all the ϕ's $= 0$, except $\dot{\phi}_0$ and $\dot{\phi}_0'$.

Equate the coefficients of $P_0(\mu)$ in (vi),

$$\dot{\phi}_0 = -\frac{A_0}{a_1^2}, \qquad A_0 = -\dot{\phi}_0 . a_1^2.$$

Hence, $$\dot{\phi}_0' = -\frac{B_0}{a_2^2}, \qquad B_0 = -\dot{\phi}_0' . a_2^2, \text{ by symmetry.}$$

Equate the coefficients of $P_p(\mu)$,

$$\dot{\phi}_p = \frac{(-1)^{p+1}\,\underline{|p+1}}{a_1^{p+2}} A_p + \frac{p}{\gamma^2}\left(\frac{a_1}{\gamma}\right)^{p-1}\left\{B_0 - (p+1)\frac{B_1}{\gamma} + (p+2)(p+1)\frac{B_2}{\gamma^2} - (p+3)(p+2)(p+1)\frac{B_3}{\gamma^3} + \text{etc.}\right\} \text{.....(vii).}$$

By symmetry,

$$\dot{\phi}_p' = \frac{(-1)^{p+1}\,\underline{|p+1}}{a_2^{p+2}} B_p + \frac{p}{\gamma^2}\left(\frac{a_2}{\gamma}\right)^{p-1}\left\{A_0 - (p+1)\frac{A_1}{\gamma} + (p+2)(p+1)\frac{A_2}{\gamma^2} - (p+3)(p+2)(p+3)\frac{A_3}{\gamma^3} + \text{etc.}\right\} \text{.....(viii).}$$

We have here types of the two sets of equations which enable us by continual approximation to determine the value of the A's and B's in terms of the ϕ's and ϕ''s.

We see generally that

$$A_p = \frac{(-1)^{p+1} a_1^{p+2}}{\underline{|p+1}} \dot{\phi}_p + \text{terms of the order } \left(\frac{a_1}{\gamma}\right)^{p+1},$$

$$B_p = \frac{(-1)^{p+1} a_2^{p+2}}{\underline{|p+1}} \dot{\phi}_p' + \text{terms of the order } \left(\frac{a_2}{\gamma}\right)^{p+1}.$$

Now γ is greater than $a_1 + a_2$, and hence, the atomic radii being supposed nearly equal, $\frac{a_1}{\gamma}$ and $\frac{a_2}{\gamma}$ will be less than $\frac{1}{2}$. Therefore for any considerable value of p we should be justified in putting

$$A_p = \frac{(-1)^{p+1} a_1^{p+2}}{\underline{|p+1}} \dot{\phi}_p, \qquad B_p = \frac{(-1)^{p+1} a_2^{p+2}}{\underline{|p+1}} \dot{\phi}_p'.$$

For example, in the case of A_{10} and B_{10} the terms omitted would be, roughly speaking, $\frac{1}{2000}$th of those retained.

Digitized by Google

6. Before proceeding to calculate the values of the coefficients, let us determine the form of the total kinetic energy of the fluid. Supposing the density of the fluid unity, we require the value of

$$-\frac{1}{2}\int \psi \frac{d\psi}{d\nu}\, d\sigma$$

taken over the surface of the two atoms.

Now at the surface of the first atom

$$\psi_1 = \Sigma_{p=0}^{p=\infty}\left[A_p \frac{(-1)^p \lfloor p}{a_1^{p+1}} P_p(\mu) + \frac{B_p (-1)^p}{\gamma^{p+1}} \Sigma_{n=p}^{n=\infty} \left(\frac{a_1}{\gamma}\right)^{n-p} n(n-1)\ldots(n-p+1)\, P_{n-p}(\mu)\right],$$

and
$$\frac{d\psi}{d\nu_1} = \Sigma_0^\infty \dot{\phi}_p P_p(\mu), \qquad \int P_p^2(\mu)\, d\sigma_1 = \frac{4\pi a_1^2}{2p+1};$$

therefore
$$-\frac{1}{2}\int \psi_1 \frac{d\psi}{d\nu_1}\, d\sigma_1 = -\Sigma_0^\infty \frac{2\pi A_p \dot{\phi}_p (-1) p \lfloor p}{(2p+1)\, a_1^{p-1}}$$
$$-\Sigma_0^\infty \frac{\dot{\phi}_p \,.\, 2\pi a_1}{2p+1}\left(\frac{a_1}{\gamma}\right)^{p+1}\left\{B_0 - (p+1)\frac{B_1}{\gamma} + (p+2)(p+1)\frac{B_2}{\gamma^2}\right.$$
$$\left. - (p+3)(p+2)(p+1)\frac{B_3}{\gamma^3} + \text{etc.}\right\}.$$

Now for the series in B's under the second summation we can substitute from equation (vii).

We find for all values of $p > 0$,

$$\frac{a_1^2}{p}\left(\dot{\phi}_p - \frac{(-1)^{p+1}\lfloor p+1}{a_1^{p+2}} A_p\right) = \left(\frac{a_1}{\gamma}\right)^{p+1}\left\{B_0 - \frac{B_1}{\gamma}(p+1) + \frac{B_2}{\gamma^2}(p+2)(p+1) - \text{etc.}\right\}.$$

Hence
$$-\frac{1}{2}\int \psi_1 \frac{d\psi}{d\nu_1}\, d\sigma = -\Sigma_0^\infty \frac{2\pi A_p \dot{\phi}_p (-1)^p \lfloor p}{(2p+1)\, a_1^{p-1}}$$
$$-\frac{2\pi a_1^2 \dot{\phi}_0}{\gamma}\left\{B_0 - \frac{B_1}{\gamma} + \lfloor 2 \frac{B_2}{\gamma^2} - \lfloor 3 \frac{B_3}{\gamma^3} + \text{etc.}\right\}$$
$$-\Sigma_1^\infty \frac{\dot{\phi}_p^2 \,.\, 2\pi a_1^2}{p(2p+1)} + \Sigma_1^\infty \frac{2\pi A_p \dot{\phi}_p (-1)^{p+1} \lfloor p+1}{p(2p+1)\,.\, a_1^{p-1}}$$
$$= -2\pi a_1 A_0 \dot{\phi}_0 - \frac{2\pi a_1^2 \dot{\phi}_0}{\gamma}\left\{B_0 - \frac{B_1}{\gamma} + \lfloor 2 \frac{B_2}{\gamma^2} - \text{etc.}\right\}$$
$$-\Sigma_1^\infty \frac{\dot{\phi}_p^2 \,.\, 2\pi a_1^2}{p(2p+1)} + \Sigma_1^\infty \frac{(-1)^{p-1}\lfloor p-1\, 2\pi A_p \dot{\phi}_p}{a_1^{p-1}}.$$

We may therefore write the total kinetic energy of the fluid as follows, remembering the values of A_0 and B_0,

Digitized by Google

$$
\begin{aligned}
K_1 = {} & 2\pi a_1^3 \dot{\phi}_0^2 + 2\pi a_2^3 \dot{\phi}_0'^2 \\
& - \frac{2\pi a_1^3 \dot{\phi}_0}{\gamma} \left\{ B_0 - \frac{B_1}{\gamma} + \lfloor 2 \cdot \frac{B_2}{\gamma^2} - \text{etc.} \right\} \\
& - \frac{2\pi a_2^3 \dot{\phi}_0'}{\gamma} \left\{ A_0 - \frac{A_1}{\gamma} + \lfloor 2 \cdot \frac{A_2}{\gamma^2} - \text{etc.} \right\} \\
& - \Sigma_1^\infty \frac{\dot{\phi}_p^2 \cdot 2\pi a_1^3}{p(2p+1)} + \Sigma_1^\infty \frac{(-1)^{p-1} \lfloor p-1 \, 2\pi A_p \dot{\phi}_p}{a_1^{p-1}} \\
& - \Sigma_1^\infty \frac{\dot{\phi}_p'^2 \cdot 2\pi a_2^3}{p(2p+1)} + \Sigma_1^\infty \frac{(-1)^{p-1} \lfloor p-1 \, 2\pi B_p \dot{\phi}_p'}{a_2^{p-1}} \quad \text{.....(ix).}
\end{aligned}
$$

Now we have supposed the velocity of the extremity of any radius of the first atom to be given by $d_1 = \Sigma \dot{\phi}_p P_p(\mu)$, hence we may represent the increase in the length of the radius as given by $\Sigma \phi_p . P_p(\mu)$, and ϕ_p may be termed a generalized co-ordinate of the atom.

7. If we wish to calculate exactly the kinetic and potential energies of an atom we must adopt some hypothesis as to its nature. Suppose, for example, it were an uniformly elastic spherical solid, then we might deduce the internal displacements at any point corresponding to a normal surface displacement $= \Sigma \phi_p P_p(\mu)$, from these displacements obtain the forces of elasticity, and so, by Clapeyron's Theorem, the internal work. [The displacements are found from equations (xii) and (ix) and the corresponding forces from equation (x) of a paper on the "Distortion of a Solid Elastic Sphere," *Quarterly Journal of Mathematics*, Vol. XVI. p. 377.] On calculating such internal work, it will be seen that it contains not only squares but products of the generalized co-ordinates; every co-ordinate ϕ_p occurring in two products, namely, $\phi_p \phi_{p-1}$ and $\phi_p \phi_{p+1}$. Remembering this special case, we shall not assume that the ϕ's are the *normal* co-ordinates of our atom, but suppose the potential energy V to be of the form:

$$V = \tfrac{1}{2} \Sigma \left(\tau_p \phi_p^2 + 2\rho_{pq} \phi_p \phi_q \right).$$

Similarly the kinetic energy of the atom will be supposed in like manner to be given by

$$K_2 = \tfrac{1}{2} \Sigma \left\{ \lambda_p \dot{\phi}_p^2 + 2\mu_{pq} \dot{\phi}_p \dot{\phi}_q \right\}.$$

The like letters dotted will be used to denote similar quantities for the second atom. Hence, if L be the difference of the kinetic and potential energies for the whole system,

$$L = K_1 + K_2 + K_2' - V - V',$$

and the equations of motion will be of the type

$$\frac{d}{dt}\left(\frac{dL}{d\dot{\phi}}\right) - \frac{dL}{d\phi} = 0 \quad \text{.....(x).}$$

8. We may here stop to note a very curious possibility; since K_1 is a quadratic function of the $\dot{\phi}$'s

$$= \Sigma_0 \frac{2\pi a_1^3 \dot{\phi}_p^2}{(p+1)(2p+1)} + \Sigma_0^\infty \frac{2\pi a_2^3 \dot{\phi}_p'^2}{(p+1)(2p+1)},$$

Digitized by Google

so far as the terms independent of γ are concerned; it follows that the vibrations in the fluid *appear* to increase λ_p by $\frac{2\pi a_1^3}{(p+1)(2p+1)}$. Now in the expression for the kinetic energy of the atom, the λ's are what express the *mass* of the atom. But if all the λ's were zero, they would still have an apparent value, i.e. $\frac{2\pi a_1^3}{(p+1)(2p+1)}$, owing to the fact that the vibrations were taking place in the fluid. Hence we are not *compelled* to suppose our atoms to have any mass at all, they will still have an *apparent* mass owing to the vibrations taking place in the fluid. It is a strange possibility that atoms, the supposed foundation of all matter, may be the one thing in the universe immaterial; and that their supposed mass may arise purely from the fact of their existing in the ether! A like train of argument applies to *linear* motion of atoms in the ether, they too have an apparent mass, which may be their only mass. It must be noted however that in order to obtain the vibrations it is necessary to suppose them or *their surfaces* possessed of a potential energy similar to that given by the V's above. The notion of an atom as a perfect vacuum (even as to ether) in the boundary of which in some manner potential energy is resident seems at first sight startling, but it does not appear altogether inconceivable.

9. Suppose now γ very great, so that the atoms have no mutual action, then:

$$\frac{dL}{d\dot{\phi}_p} = \frac{4\pi a_1^3 \dot{\phi}_p}{(p+1)(2p+1)} + \lambda_p \dot{\phi}_p + \mu_{p0}\dot{\phi}_0 + \mu_{p1}\dot{\phi}_1 + \mu_{p2}\dot{\phi}_2 + \text{etc.},$$

$$-\frac{dL}{d\phi_p} = \tau_p \phi_p + \rho_{p0}\phi_0 + \rho_{p1}\phi_1 + \rho_{p2}\phi_2 + \text{etc.}$$

Hence the equation of motion gives:—

$$\ddot{\phi}_p \left\{\lambda_p + \frac{4\pi a_1^3}{(p+1)(2p+1)}\right\} + \Sigma_{q=0}^{q=\infty} \mu_{pq}\ddot{\phi}_q + \Sigma_{q=0}^{q=\infty} \rho_{pq}\phi_q + \tau_p\phi_p = 0.$$

We may write down as many other equations as there are ϕ's. Assume a vibration of period $\frac{2\pi}{n}$ and we obtain the following determinant from which to calculate n^2.

$$\begin{vmatrix} \tau_0 - n^2(\lambda_0 + 4\pi a_1^3), & \rho_{01} - n^2\mu_{01}, & \rho_{02} - n^2\mu_{02}, & \cdots & \cdots \\ \rho_{10} - n^2\mu_{10}, & \tau_1 - n^2\left(\lambda_1 + \frac{4\pi a_1^3}{6}\right), & \rho_{12} - n^2\mu_{12}, & \cdots & \cdots \\ \rho_{20} - n^2\mu_{20}, & \rho_{21} - n^2\mu_{21}, & \tau_2 - n^2\left(\lambda_2 + \frac{4\pi a_1^3}{15}\right), & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0.$$

This determines the *free* vibrations produced by a "single pole" disturbance, if the atom is disturbed and then left to vibrate freely in the ether. In the case where the

Digitized by Google

free vibration is to be a pulsation, all the ϕ's except ϕ_0 must vanish, and we find

$$n_1 = \sqrt{\frac{\tau_0}{\lambda_0 + 4\pi a_1^3}};$$

that is what we have considered as the peculiar vibration of an atom, that which finds its analogy in the single bright line of the spectrum in the case of a gas or vapour at low pressure and not too high temperature.

10. Let us suppose for a moment that the natural pulsation of our atom *does* coincide with the single bright line vibration. Then the period of the light wave will be given by $2\pi\sqrt{\frac{\lambda_0 + 4\pi a_1^3\rho}{\tau_0}}$, where ρ, the ether density, has been introduced to preserve the order of quantities. Now λ_0 (if it be not zero) will vary as the volume of the atom combined with its density, and τ_0 will of course be of the third dimensions in space but *may* only vary as a_1^2; for example, if the potential energy is resident in the surface, it will also contain some factor representing the mechanical structure of the atom—for example its elasticity. Hence we see that the period will very probably be a function of the density or radius or both of the atom. It is a noteworthy fact that "there is a connection between the atomic weight of the metalloids and the region of the spectrum in which their lines appear under similar conditions" (Lockyer, p. 143). A careful examination of the position of the 'single bright line' and the atomic weights of various substances might possibly throw some light on the value of λ_0 and τ_0, and so on the mechanical nature of atoms. Does the spectrum order of the single bright lines in the cases of the elementary gases in any way follow the order of their atomic weights?, or of atomic diameters as calculated in the Kinetic Theory of Gases? (See Oskar Meyer, *Die kinetische Theorie der Gase*, p. 206).

11. We may here note why it possibly is, that the natural pulsation is of more importance than any of the other free vibrations of the atom. These latter may occur about any pole or any number of poles in the same atom, while in another atom they may be taking place about an entirely different pole or system of poles. Suppose two atoms to collide and then separate and pass beyond their sphere of mutual action; each will vibrate freely about the point of colliding contact, but the position of the corresponding poles in each will have no relation whatever to the poles of the vibrations produced by the contact of another pair of atoms. The free pulsations in all cases will be the same and from any small volume will have a common direction; the free vibrations will however have no common direction, and so produce nothing like the effect of the free pulsation upon the ether.

12. Let us now suppose the atoms to be so close as to enforce vibrations on each other. We shall look at this problem in the following manner. When no such influence existed we calculated the free vibrations of a single atom, we now ask what are the *free* vibrations of a system of two atoms acting upon each other? Such free vibrations are when considered with reference to one atom alone *forced* vibrations, they occur only owing to the mutual position of the two. What are really the free vibrations of a system of atoms

Digitized by Google

mutually reacting? Here we shall find *a new set of periods differing from the old, in that they are functions of the distance between any two atoms.*

Now the expression for K_1 is a quadratic function of $\dot{\phi}$ and ϕ', which may be expressed as follows:—

$$
\begin{aligned}
K_1 = {} & \phi_0^2 \cdot \left\{2\pi a_1^2 + \frac{{}_1\beta_0}{\gamma} + \frac{{}_2\beta_0}{\gamma^2} + \right\} \\
& + \phi_0'^2 \cdot \left\{2\pi a_2^2 + \frac{{}_1\beta_0'}{\gamma} + \frac{{}_2\beta_0'}{\gamma^2} + \ldots\right\} \\
& + \Sigma_{p=1}^{p=\infty} \phi_0 \phi_p \cdot \left\{\frac{\epsilon_p}{\gamma^{p+4}} + \text{higher terms in } \frac{1}{\gamma}\right\} \\
& + \Sigma_{p=1}^{p=\infty} \phi_0' \phi_p' \cdot \left\{\frac{\epsilon'_p}{\gamma^{p+4}} + \ldots\right\} \\
& + \Sigma_{p=1}^{p=\infty} \Sigma_{s=1}^{s=\infty} \phi_p \phi_s \left\{\frac{\eta_{ps}}{\gamma^{p+s+4}} + \ldots\right\} \\
& + \Sigma_{p=1}^{p=\infty} \Sigma_{s=1}^{s=\infty} \phi_p' \phi_s' \cdot \left\{\frac{\eta'_{ps}}{\gamma^{p+s+4}} + \ldots\right\} \\
& + \Sigma_{p=1}^{p=\infty} \Sigma_{s=1}^{s=\infty} \phi_p \phi_s' \left\{\frac{\zeta_{ps}}{\gamma^{p+s+1}} + \ldots\right\} \\
& + \Sigma_{p=0}^{p=\infty} \phi_0 \phi_p' \left\{\frac{\xi_p}{\gamma^{p+1}} + \ldots\right\} \\
& + \Sigma_{p=0}^{p=\infty} \phi_0' \phi_p \left\{\frac{\xi_p'}{\gamma^{p+1}} + \ldots\right\} \\
& + \Sigma_{p=1}^{p=\infty} \phi_p^2 \left\{\frac{2\pi a_1^3}{(p+1)(2p+1)} + \frac{\chi_p}{\gamma^{2p+4}} + \ldots\right\} \\
& + \Sigma_{p=1}^{p=\infty} \phi_p'^2 \left\{\frac{2\pi a_2^3}{(p+1)(2p+1)} + \frac{\chi_p'}{\gamma^{2p+4}} + \ldots\right\}.
\end{aligned}
$$

We note at once from this that the terms due to the mutual influence—

(*a*) Bring in products of the ϕ's, but that these products become of small importance when the order of the ϕ's is considerable or the distance between the atoms sensible.

(*b*) Affect also the coefficients of the squares of the generalised velocity-components, but when the order of the component is considerable to a very small extent. In other words the mutual action affects most the co-ordinates of the lowest orders, in particular the *natural pulsation.*

13. Proceeding as before to form the equations of motions, we obtain a series of linear relations between the ϕ's, the coefficients of every term being functions of $\frac{1}{\gamma}$. Assuming values of the type $C \cos nt$ for the ϕ's, we obtain linear equations between the C's and n^2. Eliminating the C's by a determinant, we arrive at an equation to determine n^2. The

Digitized by Google

coefficients of the powers of n in this equation are functions of $\frac{1}{\gamma}$, and therefore its roots may be expressed in the form

$$n_s = S_0 + \frac{S_1}{\gamma} + \frac{S_2}{\gamma^2} + \text{etc.}$$

If γ be great, $n_s = S_0$, and $\frac{2\pi}{S_0}$ will be one of the vibrations of one or other atom considered as vibrating freely. If γ decrease, we shall have to consider the next term $\frac{S_1}{\gamma}$; if this however be still small, the period will differ but little from that given by S_0. Still from one pair of atoms to another γ will vary, and thus we shall have a number of periods grouped, as it were, round the natural period. If these vibrations correspond to light vibrations, we should expect to find the spectrum lines spreading out into bands as the atoms of a gas are brought nearer together. Finally, let the atoms be brought very close together, then we shall have to consider a long series of terms

$$\frac{S_1}{\gamma} + \frac{S_2}{\gamma^2} + \frac{S_3}{\gamma^3} + \text{etc.},$$

altering n_s very considerably from S_0. In other words the vibrations given by any two atoms will differ very considerably from the free vibrations of either. Furthermore the vibrations given by different pairs will depend on their distances, and so vary from pair to pair. Hence there will be a *continuous series* of periods between certain limits, built up out of the vibrations of systems of atoms at different distances. If these vibrations correspond to light vibrations, we should, when the atoms are close together, expect a *continuous spectrum.*

Supposing for an instant that a chemical combination consisted of *two* atoms in a very close union, we see that the free vibrations of the two would differ very materially from the free vibrations of either alone, hence we should not expect the spectrum of such a combination to be identical with the spectra of its two components superposed.

14. Hitherto we have been dealing only with *two* atoms, we may now ask what difference is introduced into the above results if three (or more) atoms are all within their mutual range of action?

The calculation of the value of K_1 for three atoms in the most general case of polar vibrations presents considerable difficulties; we shall content ourselves with examining the nature of the *pulsations* of any number of atoms mutually reacting. It is not difficult to shew that in the case of three atoms:—

$$K_1 = 2\pi a_1^2 \dot{\phi}_0^2 + 2\pi a_2^2 \dot{\phi}_0'^2 + 2\pi a_3^2 \dot{\phi}_0''^2 + \frac{4\pi a_2^2 a_3^2}{\gamma_{23}} \dot{\phi}_0' \dot{\phi}_0'' + \frac{4\pi a_1^2 a_3^2}{\gamma_{31}} \dot{\phi}_0' \dot{\phi}_0'' + \frac{4\pi a_1^2 a_2^2}{\gamma_{12}} \dot{\phi}_0 \dot{\phi}_0'$$

$$+ \text{terms with higher powers of } \frac{1}{\gamma}.$$

Digitized by Google

Forming as before the equations of motion, we find

$$(4\pi a_1^2 + \lambda_0)\,\ddot{\phi}_0 + \tau_0\phi_0 + \frac{4\pi a_1^2 a_3^2}{\gamma_{31}}\,\ddot{\phi}_0'' + \frac{4\pi a_1^2 a_2^2}{\gamma_{12}}\,\ddot{\phi}_0' = 0,$$

with two similar equations.

Now assume $\phi_0 = C_1 \cos \overline{nt}$, $\phi_0' = C_2 \cos \overline{nt}$, $\phi_0'' = C_3 \cos \overline{nt}$, where $\overline{nt}$ is written shortly for $nt + \alpha$. Then

$$C_1(\tau_0 - n^2\{4\pi a_1^2 + \lambda_0\}) - C_2\frac{4\pi a_1^2 a_2^2}{\gamma_{12}}n^2 - C_3\frac{4\pi a_1^2 a_3^2}{\gamma_{31}}n^2 = 0.$$

Let
$$\begin{cases} \nu_1^2 = \dfrac{\tau_0}{4\pi a_1^2 + \lambda_0}, \\ Q_1 = \dfrac{4\pi a_2^2 a_3^2}{\gamma_{23}}, \qquad Q_2 = \dfrac{4\pi a_1^2 a_3^2}{\gamma_{31}}, \qquad Q_3 = \dfrac{4\pi a_1^2 a_2^2}{\gamma_{12}}. \end{cases}$$

Then
$$C_1\tau_0\left(\frac{1}{\nu_1^2} - \frac{1}{n^2}\right) + C_2 Q_3 + C_3 Q_2 = 0,$$

similarly
$$C_1 Q_3 + C_2\tau_0'\left(\frac{1}{\nu_2^2} - \frac{1}{n^2}\right) + C_3 Q_1 = 0,$$

$$C_1 Q_2 + C_2 Q_1 + C_3\tau_0''\left(\frac{1}{\nu_3^2} - \frac{1}{n^2}\right) = 0.$$

Eliminate C_1, C_2, C_3, and we have the following well-known characteristic equation to determine $\frac{1}{n^2}$:—

$$\tau_0\tau_0'\tau_0''\left(\frac{1}{\nu_1^2} - \frac{1}{n^2}\right)\left(\frac{1}{\nu_2^2} - \frac{1}{n^2}\right)\left(\frac{1}{\nu_3^2} - \frac{1}{n^2}\right) - Q_1^2\tau_0\left(\frac{1}{\nu_1^2} - \frac{1}{n^2}\right)$$
$$- Q_2^2\tau_0'\left(\frac{1}{\nu_2^2} - \frac{1}{n^2}\right) - Q_3^2\tau_0''\left(\frac{1}{\nu_3^2} - \frac{1}{n^2}\right) + 2Q_1Q_2Q_3 = 0.$$

Therefore the three values of n are the lengths of the semi-axes of the quadric

$$\frac{x^2}{\nu_1^2} + \frac{y^2}{\nu_2^2} + \frac{z^2}{\nu_3^2} + 2\frac{Q_1}{\sqrt{\tau_0'\tau_0''}}yz + 2\frac{Q_2}{\sqrt{\tau_0\tau_0''}}xz + 2\frac{Q_3}{\sqrt{\tau_0\tau_0'}}yx = 1,$$

and $\frac{1}{n_1^2}$, $\frac{1}{n_2^2}$, $\frac{1}{n_3^2}$ are separated by the quantities

$$\frac{1}{\nu_1^2} - \frac{Q_2Q_3}{\tau_0 Q_1}, \qquad \frac{1}{\nu_2^2} - \frac{Q_1Q_3}{\tau_0' Q_2}, \qquad \frac{1}{\nu_3^2} - \frac{Q_1Q_2}{\tau_0'' Q_3}.$$

15. Let us consider some particular cases.

a. Suppose the free pulsations of the three atoms not to be nearly equal, and let ν_1, ν_2, ν_3 be in order of magnitude, so that following the analogy of the spectrum the bright lines of the three atoms would be grouped from the violet to the red in order.

Digitized by Google

We find approximately,

$$\frac{1}{n_1^2}=\frac{1}{\nu_1^2}-\frac{Q_2^2}{\tau_0\tau_0''}\frac{\nu_1^2\nu_3^2}{\nu_1^2-\nu_3^2}-\frac{Q_3^2}{\tau_0\tau_0'}\frac{\nu_1^2\nu_2^2}{\nu_1^2-\nu_2^2},$$

$$\frac{1}{n_2^2}=\frac{1}{\nu_2^2}-\frac{Q_3^2}{\tau_0'\tau_0}\frac{\nu_2^2\nu_1^2}{\nu_2^2-\nu_1^2}-\frac{Q_1^2}{\tau_0'\tau_0''}\frac{\nu_2^2\nu_3^2}{\nu_2^2-\nu_3^2},$$

$$\frac{1}{n_3^2}=\frac{1}{\nu_3^2}-\frac{Q_1^2}{\tau_0''\tau_0'}\frac{\nu_3^2\nu_2^2}{\nu_3^2-\nu_2^2}-\frac{Q_2^2}{\tau_0''\tau_0}\frac{\nu_3^2\nu_1^2}{\nu_3^2-\nu_1^2};$$

and it follows that $\frac{1}{n_1^2}$ is less than $\frac{1}{\nu_1^2}$,

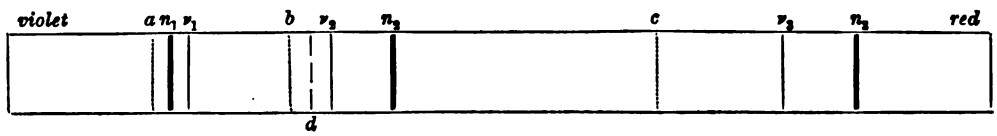
$\frac{1}{n_3^2}$ is greater than $\frac{1}{\nu_3^2}$;

and therefore greater than $\frac{1}{\nu_3^2}-\frac{Q_1Q_2}{\tau_0''Q_3}$.

Hence the following quantities must be in order of magnitude,

$$\frac{1}{n_3^2},\ \frac{1}{\nu_3^2},\ \frac{1}{\nu_3^2}-\frac{Q_1Q_2}{\tau_0''Q_3},\ \left\{\frac{1}{\nu_2^2}\text{ and }\frac{1}{n_2^2}\right\},\ \frac{1}{\nu_2^2}-\frac{Q_1Q_3}{\tau_0'Q_2},\ \frac{1}{\nu_1^2},\ \frac{1}{n_1^2},\ \frac{1}{\nu_1^2}-\frac{Q_2Q_3}{\tau_0Q_1},$$

where $\frac{1}{\nu_2^2}$ and $\frac{1}{n_2^2}$ lie between the limits given, but which is greater is not determined. If we draw a figure corresponding to the spectrum, we have the following result.



Here ν_1, ν_2, ν_3 represent the bright lines corresponding to the free pulsations of the three atoms, a, b, c denote the position of the lines corresponding to the quantities which separate the roots of the cubic for $\frac{1}{n^2}$.

The thick lines n_1, n_2, n_3 denote the spectrum given by the system of three atoms, according however to the mutual distances between the atoms, n_2 may be right or left of ν_2, i.e. may take up a position marked by the line d.

We note that the position of the lines n_1, n_2, n_3 within the limits thus determined will vary according to the distances of the atoms from each other.

Suppose now the molecule of a certain substance to be composed of three different atoms, then when that substance is in such a condition (as a gas or vapour at low pressure and not too high temperature) that its molecules are vibrating freely, its spectrum will be of the kind here described, and will in part consist of two bright lines, the one nearer to the violet and the other nearer to the red than any bright line due to its component atom when they are pulsating freely.

Digitized by Google

We may express the surface displacements of the three atoms as follows:

$$\phi_0 = C_1 \cos \overline{n_1 t} - C_2 \cdot \frac{Q_2}{\tau_0}\left(\frac{1}{\nu_1^2} - \frac{1}{\nu_2^2}\right)^{-1} \cos \overline{n_2 t} - C_3 \cdot \frac{Q_2}{\tau_0}\left(\frac{1}{\nu_1^2} - \frac{1}{\nu_3^2}\right)^{-1} \cos \overline{n_3 t},$$

$$\phi_0' = -C_1 \cdot \frac{Q_3}{\tau_0'}\left(\frac{1}{\nu_2^2} - \frac{1}{\nu_1^2}\right)^{-1} \cos \overline{n_1 t} + C_2 \cdot \cos \overline{n_2 t} - C_3 \cdot \frac{Q_1}{\tau_0'}\left(\frac{1}{\nu_2^2} - \frac{1}{\nu_3^2}\right)^{-1} \cos \overline{n_3 t},$$

$$\phi_0'' = -C_1 \cdot \frac{Q_2}{\tau_0''}\left(\frac{1}{\nu_3^2} - \frac{1}{\nu_1^2}\right)^{-1} \cos \overline{n_1 t} - C_2 \cdot \frac{Q_1}{\tau_0''}\left(\frac{1}{\nu_3^2} - \frac{1}{\nu_2^2}\right)^{-1} \cos \overline{n_2 t} + C_3 \cos \overline{n_3 t}.$$

It follows that

$$\dot{\phi}_0 = -n_1 C_1 \sin \overline{n_1 t} - n_2 C_2 \frac{Q_3}{\tau_0}\left(\frac{1}{\nu_1^2} - \frac{1}{\nu_2^2}\right)^{-1} \sin \overline{n_2 t} - n_3 C_3 \frac{Q_2}{\tau_0}\left(\frac{1}{\nu_1^2} - \frac{1}{\nu_3^2}\right)^{-1} \sin \overline{n_3 t},$$

with like quantities for $\dot{\phi}_0'$ and $\dot{\phi}_0''$.

Since, when ν_1 is more nearly $= \nu_2$, $\left(\frac{1}{\nu_1^2} - \frac{1}{\nu_2^2}\right)^{-1}$ is a large quantity, we deduce the following theorem: the more nearly the pulsation-periods of two atoms agree, the intenser will be the vibrations of the system compounded of the two. For example, in the supposed spectrum given above, if the lines $\nu_1 \nu_2$ are close, then the lines $n_1 n_2$ of the combined spectrum will be very bright, and the line n_1 will be very much to the left of ν_1, while n_2 will be to the right of ν_2.

On the other hand, when ν_1 and ν_2 are very different, the intensity of the vibrations will be diminished. [Cf. Lockyer, *Studies in Spectrum Analysis*, p. 140, for a somewhat similar analogy in the case of vibrations damped by the encounters (? *approach*) of dissimilar molecules.]

β. Suppose two of the atoms to be alike, or $\left\{\begin{matrix}\nu_1 = \nu_2 \\ \tau_0 = \tau_0\end{matrix}\right\}$, and ν_1 and ν_3 to be not nearly equal, then the discriminating cubic becomes:

$$\tau_0^2 \tau_0'' \left(\frac{1}{\nu_1^2} - \frac{1}{n^2}\right)^2 \left(\frac{1}{\nu_3^2} - \frac{1}{n^2}\right) - \tau_0 (Q_1^2 + Q_2^2) \left(\frac{1}{\nu_1^2} - \frac{1}{n^2}\right) - Q_3^2 \tau_0'' \left(\frac{1}{\nu_3^2} - \frac{1}{n^2}\right) + 2 Q_1 Q_2 Q_3 = 0.$$

We find approximately

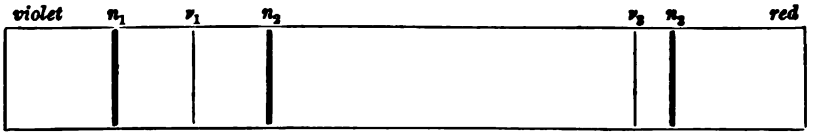
$$\frac{1}{n_1^2} = \frac{1}{\nu_1^2} - \frac{Q_2}{\tau_0}, \qquad \frac{1}{n_2^2} = \frac{1}{\nu_1^2} + \frac{Q_2}{\tau_0}, \qquad \frac{1}{n_3^2} = \frac{1}{\nu_3^2} - \frac{Q_1^2 + Q_3^2}{\tau_0 \tau_0''} \cdot \frac{\nu_3^2 \nu_1^2}{\nu_3^2 - \nu_1^2}.$$

The following are the spectra of the molecules composed of two like atoms and one unlike—when,

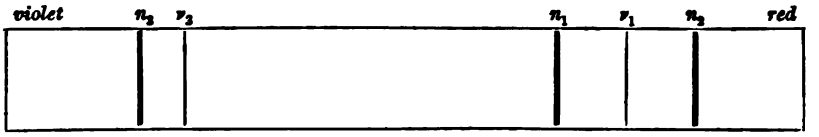
First, the pulsation of the two like atoms is more rapid than that of the unlike,

Secondly, the same pulsation is less rapid.

Digitized by Google



$\nu_1 > \nu_2$.



$\nu_1 < \nu_2$.

We see from the values of $n_1 n_2$ that the corresponding lines will be *farther from* ν_1 *than* n_2 *is from* ν_2.

γ. Let us consider the case of three like atoms within the sphere of their mutual influence.

The discriminating cubic now becomes

$$m^3 - \frac{Q_1^2 + Q_2^2 + Q_3^2}{\tau_0^2} \cdot m + \frac{2Q_1 Q_2 Q_3}{\tau_0^3} = 0,$$

where
$$m = \left(\frac{1}{\nu_1^2} - \frac{1}{n^2}\right).$$

Hence we see that m has three values of the forms

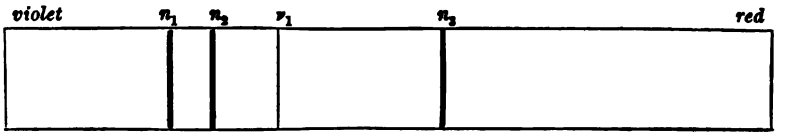
$$\alpha^2, \ \beta^2, \text{ and } -(\alpha^2 + \beta^2),$$

or
$$\frac{1}{n_1^2} = \frac{1}{\nu_1^2} - \alpha^2,$$

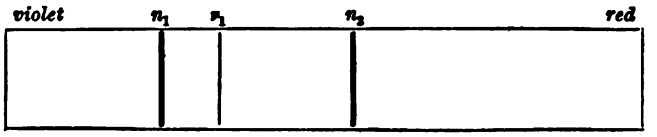
$$\frac{1}{n_2^2} = \frac{1}{\nu_1^2} - \beta^2,$$

$$\frac{1}{n_3^2} = \frac{1}{\nu_1^2} + \alpha^2 + \beta^2.$$

We have then the following spectrum:



In the case where the three atoms are at equal distances from each other the lines $n_1 n_2$ coincide:



Digitized by Google

Here we have
$$\frac{1}{n_1^2} = \frac{1}{n_2^2} = \frac{1}{\nu_1^2} - \frac{Q_1}{\tau_0},$$
$$\frac{1}{n_3^2} = \frac{1}{\nu_1^2} + 2\frac{Q_1}{\tau_0}.$$

The line n_1 however will probably be now much brighter.

16. *Second set of cases.* Suppose we have only two atoms mutually reacting, then

$$(4\pi a_1^2 + \lambda_0)\ddot{\phi}_0 + \tau_0\phi_0 + \frac{4\pi a_1^2 a_2^2}{\gamma}\ddot{\phi}_0' = 0,$$
$$\frac{4\pi a_1^2 a_2^2}{\gamma}\ddot{\phi}_0 + (4\pi a_2^2 + \lambda_0')\ddot{\phi}_0' + \tau_0'\phi_0' = 0.$$

Assume
$$\phi_0 = C_1 \cos \overline{nt}, \quad \phi_0' = C_2 \cos \overline{nt},$$
therefore
$$\{\tau_0 - (\lambda_0 + 4\pi a_1^2)\, n^2\}\, C_1 - Qn^2 C_2 = 0,$$
$$\{\tau_0' - (\lambda_0' + 4\pi a_2^2)\, n^2\}\, C_2 - Qn^2 C_1 = 0.$$

Hence, with the old notation,
$$\tau_0\tau_0'\left(\frac{1}{\nu_1^2} - \frac{1}{n^2}\right)\left(\frac{1}{\nu_2^2} - \frac{1}{n^2}\right) = Q^2,$$

or we have now a discriminating quadratic instead of the cubic. We find approximately, provided that ν_1 and ν_2 are not nearly equal,

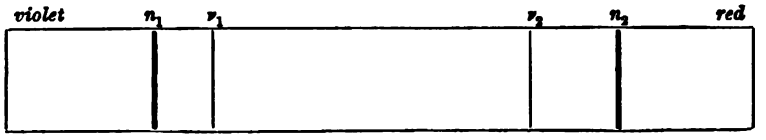
$$\frac{1}{n_1^2} = \frac{1}{\nu_1^2} - \frac{Q^2}{\tau_0\tau_0'}\frac{\nu_1^2\nu_2^2}{\nu_1^2 - \nu_2^2},$$
$$\frac{1}{n_2^2} = \frac{1}{\nu_2^2} - \frac{Q^2}{\tau_0\tau_0'}\frac{\nu_1^2\nu_2^2}{\nu_2^2 - \nu_1^2}.$$

Let ν_2, ν_1 be in order of magnitude, then $(\nu_1 > \nu_2)$

$$n_1 \text{ is } > \nu_1,$$
$$\text{and } \quad n_2 \text{ is } < \nu_2,$$

or n_1 and n_2 lie outside ν_2 to ν_1.

Hence we have the following spectrum for the combination:—



Suppose we substitute for n^2 to find the relation between C_1 and C_2, it easily follows that

$$\left.\begin{aligned} \phi_0 &= C_1 \cos \overline{n_1 t} - C_2 \cdot \frac{Q\nu_1^2\nu_2^2}{\tau_0(\nu_2^2 - \nu_1^2)} \cos \overline{n_2 t} \\ \phi_0' &= -C_1 \frac{Q\nu_1^2\nu_2^2}{\tau_0'(\nu_1^2 - \nu_2^2)} \cos \overline{n_1 t} + C_2 \cos \overline{n_2 t} \end{aligned}\right\},$$

Digitized by Google

where as before

$$\overline{n_1 t} = n_1 t + \alpha_1,$$
$$\overline{n_2 t} = n_2 t + \alpha_2.$$

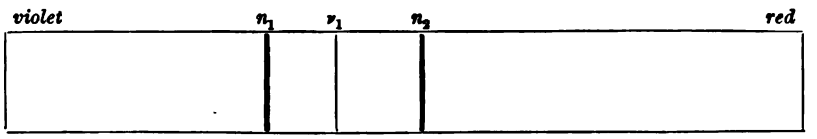
We note from these values that while the period of the free vibration of the first atom is changed from $\frac{2\pi}{\nu_1}$ to $\frac{2\pi}{n_1}$, yet the forced vibration having an amplitude varying with $\frac{1}{\gamma}$ will not rise into very much importance unless ν_2 be nearly equal to ν_1.

17. If $\left\{\begin{matrix}\nu_2 = \nu_1 \\ \tau_2 = \tau_1\end{matrix}\right\}$, the discriminating quadratic becomes

$$\tau_0^2 \left(\frac{1}{\nu_1^2} - \frac{1}{n^2}\right)^2 = Q^2,$$

$$\text{or} \quad \frac{1}{n_1^2} = \frac{1}{\nu_1^2} - \frac{Q}{\tau_0}, \qquad \frac{1}{n_2^2} = \frac{1}{\nu_1^2} + \frac{Q}{\tau_0}.$$

We then have a theoretical spectrum of the following kind:—



In both cases, that of two atoms and that of three atoms all equal, we get a double-lined theoretical spectrum, but with these distinctions: the line n_1 is situated in exactly the same position in both, in the three-atomed spectrum it will probably be much brighter; the line n_2 in the two-atomed spectrum is replaced in the three-atomed spectrum by a line n_3 still further removed towards the red. What, if any, is the difference between the spectra of an element in two different molecular conditions (e.g. oxygen and ozone)?*

In the case of two equal atoms we find

$$\left.\begin{aligned}\phi_0 &= \quad C_1 \cos \overline{n_1 t} + C_2 \cos \overline{n_2 t} \\ \phi_0' &= - C_1 \cos \overline{n_1 t} + C_2 \cos \overline{n_2 t}\end{aligned}\right\},$$

where

$$\overline{n_1 t} = n_1 t + \alpha_1,$$
$$\overline{n_2 t} = n_2 t + \alpha_2.$$

18. *Third Set of Cases.* Suppose there to be p mutually reacting atoms of different kinds.

In order to simplify the notation, as we are treating only of pulsations and not polar vibrations, we shall signify by $\phi_1, \phi_2 \ldots \phi_p$ the several pulsations, by $\lambda_1 \ldots \lambda_p$ the

* It may be shewn that the energy *lost* in the disassociation of a diatomic molecule $= \frac{\nu_1^2 (C_2^2 - C_1^2) Q}{2}$ approximately. This is positive, if $C_2 > C_1$, which is also the condition (Art. 33) that the force between the atoms should hold them together. Thus disassociation of a diatomic molecule ought to be accompanied by greater generation of heat the nearer the single bright line is to the violet end of the spectrum, e.g. the fundamental line of hydrogen should be nearer the violet than that of carbon.

Digitized by Google

coefficients of $\frac{1}{2}\dot{\phi}_1^2$, $\frac{1}{2}\dot{\phi}_2^2$, etc. in the kinetic energies of the respective atoms, and by $\tau_1 \dots \tau_p$ the coefficients of $\frac{1}{2}\phi_1^2$, $\frac{1}{2}\phi_2^2, \dots \frac{1}{2}\phi_p^2$ in the respective potential energies; $\frac{2\pi}{\nu_1} \dots\dots \frac{2\pi}{\nu_p}$ will be the periods of the free pulsations as formerly.

Then

$$L = \Sigma 2\pi a_r^2 \dot{\phi}_r^2 + \tfrac{1}{2}\Sigma\lambda_r\dot{\phi}_r^2 + \tfrac{1}{2}\Sigma\tau_r\phi_r^2 + \Sigma\frac{4\pi a_r^2 a_s^2}{\gamma_{rs}}\dot{\phi}_r\dot{\phi}_s.$$

Here as previously we only retain quantities of the order $\frac{a}{\gamma}$.

Applying the typical equation of motion and putting $Q_{rs} = \frac{4\pi a_r^2 a_s^2}{\gamma_{rs}}$, we find:

$$(\lambda_r + 4\pi a_r^2)\,\ddot{\phi}_r + \tau_r\phi_r + \Sigma Q_{rs}\ddot{\phi}_s = 0.$$

Assume $\phi_r = C_r \cos \overline{nt}$, etc., hence,

$$C_r\{\tau_r - n^2(\lambda_r + 4\pi a_r^2)\} - n^2\Sigma Q_{rs}C_s = 0,$$

$$\tau_r C_r\left(\frac{1}{\nu_r^2} - \frac{1}{n^2}\right) + \Sigma Q_{rs}C_s = 0.$$

Eliminating the C's by a determinant, we have the following equation to determine $\frac{1}{n^2}$.

$$\begin{vmatrix} \tau_1\left(\frac{1}{\nu_1^2} - \frac{1}{n^2}\right), & Q_{12}, & Q_{13}, & \dots & \dots & \dots & Q_{1p}, \\ Q_{21}, & \tau_2\left(\frac{1}{\nu_2^2} - \frac{1}{n^2}\right), & Q_{23}, & \dots & \dots & \dots & Q_{2p}, \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ Q_{p1}, & Q_{p2}, & \dots & \dots & \dots & \dots & \tau_p\left(\frac{1}{\nu_p^2} - \frac{1}{n^2}\right), \end{vmatrix} = 0$$

Assume $\frac{1}{\nu_1^2} - \frac{1}{n_1^2} = \mu$, and suppose μ is small, then

$$\begin{aligned} &\mu \cdot \tau_1\tau_2 \dots \tau_p \left(\frac{1}{\nu_2^2} - \frac{1}{\nu_1^2}\right)\left(\frac{1}{\nu_3^2} - \frac{1}{\nu_1^2}\right) \dots\dots \left(\frac{1}{\nu_p^2} - \frac{1}{\nu_1^2}\right) \\ &= Q_{12}^2 \cdot \tau_3\tau_4 \dots \tau_p \left(\frac{1}{\nu_3^2} - \frac{1}{\nu_1^2}\right)\left(\frac{1}{\nu_4^2} - \frac{1}{\nu_1^2}\right) \dots\dots \left(\frac{1}{\nu_p^2} - \frac{1}{\nu_1^2}\right) \\ &+ Q_{13}^2 \cdot \tau_2\tau_4 \dots \tau_p \left(\frac{1}{\nu_2^2} - \frac{1}{\nu_1^2}\right)\left(\frac{1}{\nu_4^2} - \frac{1}{\nu_1^2}\right) \dots\dots \left(\frac{1}{\nu_p^2} - \frac{1}{\nu_1^2}\right) \\ &+ \text{etc.} \end{aligned}$$

Hence, if the ν's are not any of them nearly equal,

$$\frac{1}{n_1^2} = \frac{1}{\nu_1^2} - \frac{Q_{12}^2}{\tau_1\tau_2}\,\frac{\nu_1^2\nu_2^2}{\nu_1^2 - \nu_2^2} - \frac{Q_{13}^2}{\tau_1\tau_3} \cdot \frac{\nu_1^2\nu_3^2}{\nu_1^2 - \nu_3^2} - \text{etc.},$$

and generally

$$\frac{1}{n_r^2} = \frac{1}{\nu_r^2} - \Sigma\frac{Q_{rs}^2}{\tau_r\tau_s} \cdot \frac{\nu_r^2\nu_s^2}{\nu_r^2 - \nu_s^2}.$$

Digitized by Google

We have then a means, given the free pulsations of any number of atoms, of calculating the pulsations peculiar to their combination, or of deducing the theoretical spectrum from the known single bright line spectra of the component atoms.

Adding together these expressions for $\frac{1}{n_r^2}$ we obtain the result

$$\Sigma \frac{1}{n_s^2} = \Sigma \frac{1}{\nu_s^2}.$$

From which we deduce the following conclusion:

$$\Sigma \frac{4\pi^2}{n_s^2} = \Sigma \frac{4\pi^2}{\nu_s^2};$$

or in words, *If any number p of different atoms* (we shall see later that they need not all be different) *be mutually reacting upon each other's pulsations in any manner whatever, or if some of them be reacting and some not reacting, the sum of the squares of the periods of the pulsations given by the p atoms is always a constant.* The constant of course being equal to the sum of the squares of the periods of the p atoms pulsating freely.

If the p atoms enter into a chemical combination (form a molecule) altogether or form several different chemical combinations (different molecules) the result is still true. This seems somewhat suggestive when applied to the bright lines in the spectrum of a compound.

Further we have by considering the minors of the above determinant when $n = n_1$,

$$C_r = -\frac{C_1 Q_{r1}}{\tau_r\left(\frac{1}{\nu_r^2} - \frac{1}{\nu_1^2}\right)} = -\frac{C_1 Q_{r1} \nu_1^2 \nu_r^2}{\tau_r(\nu_1^2 - \nu_r^2)}.$$

Hence we may write

$$\phi_1 = C_1 \cos \overline{n_1 t} - \Sigma . \frac{C_r Q_{1r} \nu_1^2 \nu_r^2}{\tau_1(\nu_r^2 - \nu_1^2)} . \cos \overline{n_r t}, \qquad \{r = 2 \text{ to } p\}$$

$$\phi_2 = C_2 \cos \overline{n_2 t} - \Sigma . \frac{C_r . Q_{2r} \nu_2^2 \nu_r^2}{\tau_2(\nu_r^2 - \nu_2^2)} . \cos \overline{n_r t}, \qquad \{r = 1, 3 \ldots p\}$$

$$\phi_3 = C_3 \cos \overline{n_3 t} - \Sigma . \frac{C_r . Q_{3r} \nu_3^2 \nu_r^2}{\tau_3(\nu_r^2 - \nu_3^2)} . \cos \overline{n_r t}, \qquad \{r = 1, 2, 4 \ldots p\}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots$$

$$\phi_p = C_p \cos \overline{n_p t} - \Sigma . \frac{C_r Q_{pr} \nu_p^2 \nu_r^2}{\tau_p(\nu_r^2 - \nu_p^2)} . \cos \overline{n_r t}, \qquad \{r = 1 \text{ to } p - 1\}.$$

The above equations then determine the periods and intensities of the pulsations of the combination of p atoms.

19. The whole question of deducing a compound spectrum from the single bright lines of given elements depends on the consideration of the above symmetrical determinant and its minors. The cases in which 2, 3, 4, ... all p atoms are of like kind, may be deduced by respectively putting $\begin{Bmatrix} \nu_1 = \nu_2 \\ \tau_1 = \tau_2 \end{Bmatrix}$, $\begin{Bmatrix} \nu_1 = \nu_2 = \nu_3 \\ \tau_1 = \tau_2 = \tau_3 \end{Bmatrix}$, etc. in the above determinant.

Digitized by Google

20. We may note the following general property of the combination spectrum, if $\nu_1 \dots \nu_p$ be supposed in order of magnitude, then $\frac{1}{n_1}$ is $< \frac{1}{\nu_1}$ and $\frac{1}{n_p}$ is $> \frac{1}{\nu_p}$, or the combination spectrum will (theoretically) have two bright lines, the one nearer the violet, the other nearer the red than any of the single bright lines of the elements.

21. One very important result of the above investigations is the following. If all the atoms are alike,

$$\tau_1 = \tau_2 = \tau_3 = \dots = \tau_p \text{ and } \nu_1 = \nu_2 = \dots = \nu_p,$$

and we may write the discriminating determinant

$$\begin{vmatrix} m & Q_{12} & Q_{13} & \dots & Q_{1p} \\ Q_{21} & m & Q_{23} & \dots & Q_{2p} \\ \dots & \dots & \dots & \dots & \dots \\ Q_{p1} & Q_{p2} & Q_{p3} & \dots & m \end{vmatrix} = 0,$$

where $m = \tau\left(\frac{1}{\nu^2} - \frac{1}{n^2}\right)$. We then note that in this equation for m, the power m^{p-1} does not occur. Hence the sum of the p values of m is zero, or

$$\frac{1}{\nu^2} - \frac{1}{n_1^2} + \frac{1}{\nu^2} - \frac{1}{n_2^2} + \frac{1}{\nu^2} - \frac{1}{n_3^2} + \dots = 0,$$

or

$$\frac{4\pi^2}{n_1^2} + \frac{4\pi^2}{n_2^2} + \dots \frac{4\pi^2}{n_p^2} = p \cdot \frac{4\pi^2}{\nu^2};$$

that is: *If owing to the effect of pressure or temperature an element gives p spectral lines instead of a single bright line, then the sum of the squares of the periods corresponding to the p bright lines is equal to p times the square of the period corresponding to the single bright line.*

22. Again, if Δ_{p-r} denote the determinant obtained from the discriminating determinant by omitting the first r rows and columns, then (Salmon's *Higher Algebra*, Lesson VI.) the roots of the equation $\Delta_{p-r} = 0$ separate the roots of the equation $\Delta_{p-r+1} = 0$. In other words, if there be a spectrum of q bright lines due to q mutually reacting atoms, and if another atom be introduced into the sphere of mutual reaction, then the first set of q lines will separate the second set of $q+1$ lines.

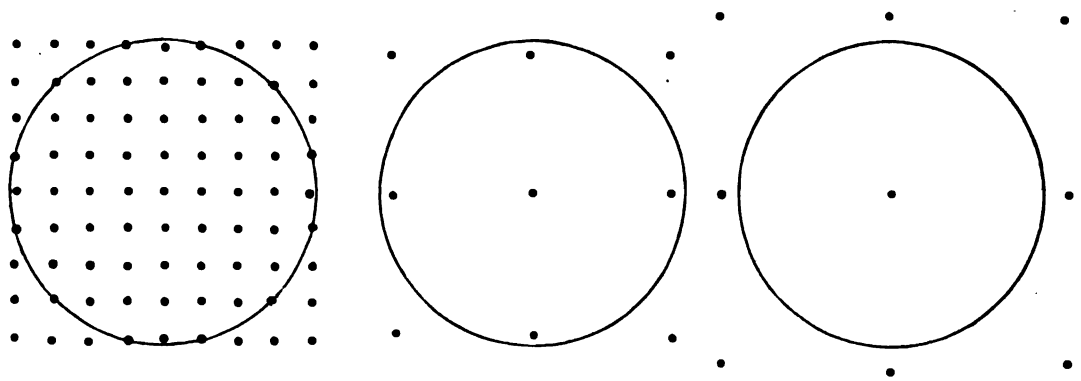
23. It must be noted that in all calculations as to the theoretical spectrum of a combination of atoms, we must not expect always to have as many *bright* lines as atoms in the combination; for first, some of the atoms even may have natural pulsations which correspond to ultra-red or ultra-violet rays, in which case, since the combination always has two periods, one greater than the greatest and one less than the least period of the free atoms, it follows that the combination will have at *least* two invisible (so-called heat or chemical) rays, it may have more. Next, if all the pulsations of the free atoms correspond to light rays, yet since the combination has at *least* two periods greater and less than these, it may have one or more periods which correspond to either chemical or heat rays.

Digitized by Google

24. Lastly, we may note that all the periods calculated for any combination of atoms depend upon the central distances between the atoms. If the atoms are changing their relative distances the periods also will change. Now suppose the distances of the atoms of a combination to vary about a mean position, then about each mean there would in any interval of time be a series of others differing slightly from it, there would be the appearance of bands in the spectrum; in other words, a combination of atoms which are oscillating about their mean distances would give a fluted spectrum. The oscillations of the combination of atoms could arise in various ways, as by the action of one combination upon another (one molecule on another), or even by impact if such became frequent owing to close proximity of the combinations, as in the case of a gas at considerable pressure, or in a solid or liquid body.

25. There is however another statical or average method of looking at the pulsations which may possibly give rise to a fluted spectrum. If a great number of atoms be at each instant within the sphere of mutual action, and we suppose their distances not to vary, then there would be a very great number of pulsations corresponding to all the different central distances; the very complexity of these lines may make it impossible to distinguish them from bands, and so a fluted spectrum arise. This would be more or less the case when we consider an element not in a molecular state and whose atoms were very close together.

In the accompanying figures let the atoms be supposed at rest and uniformly distributed, or else suppose the average number of atoms in a given volume to be uniformly distributed over the volume at their mean distance.



(i) (ii) (iii)

In the first figure the mean distance is one-fourth of what it is in the second, and one-fifth of what it is in the third figure. If the radius of the sphere of mutual action be a little more than four times the mean distance in the first case ($=\sqrt{18}$ times the mean distance), we find $3+7+7+9+9+9+7+7+3$ atoms in the plane of the paper, and about 340 atoms in the sphere of mutual action in figure (i), 7 atoms in figure (ii), and 1 atom in figure (iii). In the last case we should expect a single lined spectrum, in the second a 7 line spectrum, while the first case would present us with 340 lines. The densities in the three cases are as $1 : \frac{1}{64} : \frac{1}{125}$, or supposing the temperature to be the same the pressures are in the density ratio. For example, suppose the spectrum of some element to have at

Digitized by Google

ordinary pressure some 300 lines; hence, the temperature remaining the same, at $\frac{1}{64}$ the ordinary pressure it should have a spectrum of not more than 7 lines, and before the pressure reaches $\frac{1}{128}$th it should give only one line. This is not unsuggestive when we consider the spectrum of iron.

Here of course we suppose that the ultimate atom of the element (which is supposed to consist of uniform atoms) has only one free pulsation.

26. This method of considering the arrangement of the atoms of a body may perhaps be useful in explaining some peculiarities of the spectra of solid bodies or of elementary bodies whose atoms are all uniform, but when we come to the consideration of bodies composed of unlike atoms we are forced to consider the atoms grouped together in combinations or as they are termed molecules. The fact that a mixture of two elements gives a totally different spectrum to their chemical combination forces upon us the notion of atoms united in groups. We are then led to consider the forces which must exist between our pulsating atoms when they are united into groups or molecules, these we shall term *chemical forces;* the forces between groups of atoms or molecules will be termed *molecular forces;* while the forces which act between groups of molecules at considerable distances will be termed *gravitation forces.* Can our theory of atoms as spherical bodies pulsating in a fluid medium throw any light on these three kinds of forces? In order to investigate this point, let us return to the expression for the total kinetic energy of atoms and fluid. We shall consider it only so far as it refers to pulsations, leaving out of account all polar vibrations. From equation (ix) we have

$$K_1 = 2\pi a_1^2\phi_0^2 + 2\pi a_2^2\phi_0'^2 - \frac{2\pi a_1^2\phi_0}{\gamma}\left\{B_0 - \frac{B_1}{\gamma} + \underline{|2}\,\frac{B_2}{\gamma^2} - \text{etc.}\right\}$$
$$- \frac{2\pi a_2^2\phi_0'}{\gamma}\left\{A_0 - \frac{A_1}{\gamma} + \underline{|2}\,\frac{A_2}{\gamma^2} - \text{etc.}\right\},$$

where $A_0 = -\phi_0 a_1^2$, $B_0 = -\phi_0' a_2^2$, and generally

$$0 = \frac{(-1)^{p+1}\underline{|p+1}}{a_1^{p+2}} A_p + \frac{p}{\gamma}\left(\frac{a_1}{\gamma}\right)^{p-1}\left\{B_0 - (p+1)\frac{B_1}{\gamma} + (p+2)(p+1)\frac{B_2}{\gamma^2} - \text{etc.}\right\},$$

$$0 = \frac{(-1)^{p+1}\underline{|p+1}}{a_2^{p+2}} B_p + \frac{p}{\gamma^2}\left(\frac{a_2}{\gamma}\right)^{p-1}\left\{A_0 - (p+1)\frac{A_1}{\gamma} + (p+2)(p+1)\frac{A_2}{\gamma} - \text{etc.}\right\}.$$

27. We proceed to approximate to the values of the A's and B's.

First Approximation.

$$A_1 = \frac{a_1^3 a_2^2}{2\gamma^2}\phi_0', \qquad B_1 = \frac{a_2^3 a_1^2}{2\gamma^2}\phi_0,$$

$$A_2 = -\frac{1}{3}\frac{a_1^5 a_2^3}{\gamma^3}\phi_0', \qquad B_2 = -\frac{1}{3}\frac{a_2^5 a_1^2}{\gamma^3}\phi_0.$$

$$A_3 = \frac{a_1^7 a_2^2}{8\gamma^4}\phi_0', \qquad B_3 = \frac{a_2^7 a_1^2}{8\gamma^4}\phi_0.$$

Digitized by Google

Second Approximation.

$$A_1 = \frac{a_1^3 a_2^2}{2\gamma^2}\phi_0' + \frac{a_2^3 a_1^5}{2\gamma^5}\phi_0, \qquad A_2 = -\frac{a_1^5 a_2^2}{3\gamma^3}\phi_0' - \frac{a_2^3 a_1^7}{2\gamma^6}\phi_0,$$

$$B_1 = \frac{a_2^3 a_1^2}{2\gamma^2}\phi_0 + \frac{a_1^3 a_2^5}{2\gamma^5}\phi_0', \qquad B_2 = -\frac{a_2^5 a_1^2}{3\gamma^3}\phi_0 - \frac{a_1^3 a_2^7}{2\gamma^6}\phi_0'.$$

Third Approximation.

$$A_1 = \frac{a_1^3 a_2^2}{2\gamma^2}\phi_0' + \frac{a_2^3 a_1^5}{2\gamma^5}\phi_0 + \frac{a_2^5 a_1^5}{\gamma^7}\phi_0,$$

$$B_1 = \frac{a_2^3 a_1^2}{2\gamma^2}\phi_0 + \frac{a_1^3 a_2^5}{2\gamma^5}\phi_0' + \frac{a_1^5 a_2^5}{\gamma^7}\phi_0'.$$

Lastly we may note that A_4 and B_4 are of the order $\frac{1}{\gamma^5}$, and hence appear only in K_1 in terms of order $\frac{1}{\gamma^{10}}$; which we may very reasonably neglect.

28. For possible future use it may be as well to put on record the value of these constants in the case of *polar* vibrations so far as terms of the order $\frac{1}{\gamma^5}$.

$$A_1 = \frac{a_1^3}{2}\phi_1 + \frac{a_1^3 a_2^2}{2\gamma^2}\phi_0' + \frac{a_1^3 a_2^3}{2\gamma^3}\phi_1' + \frac{a_1^3 a_2^4}{2\gamma^4}\phi_2' + \frac{a_1^5 a_2^3}{2\gamma^5}\phi_0 + \frac{a_1^3 a_2^5}{2\gamma^5}\phi_2',$$

$$B_1 = \frac{a_2^3}{2}\phi_1' + \frac{a_2^3 a_1^2}{2\gamma^2}\phi_0 + \frac{a_2^3 a_1^3}{2\gamma^3}\phi_1 + \frac{a_2^3 a_1^4}{2\gamma^4}\phi_2 + \frac{a_2^5 a_1^3}{2\gamma^5}\phi_0' + \frac{a_2^3 a_1^5}{2\gamma^5}\phi_2,$$

$$A_2 = -\frac{a_1^4}{6}\phi_2 - \frac{a_1^5 a_2^2}{3\gamma^3}\phi_0' - \frac{a_1^5 a_2^3}{2\gamma^4}\phi_1',$$

$$B_2 = -\frac{a_2^4}{6}\phi_2' - \frac{a_2^5 a_1^2}{3\gamma^3}\phi_0 - \frac{a_2^5 a_1^3}{2\gamma^4}\phi_1,$$

$$A_3 = \frac{a_1^5}{24}\phi_2 + \frac{a_1^7 a_2^2}{8\gamma^4}\phi_0', \qquad B_3 = \frac{a_2^5}{24}\phi_2' + \frac{a_2^7 a_1^2}{8\gamma^4}\phi_0.$$

$$A_4 = -\frac{a_1^6}{120}\phi_4, \qquad B_4 = -\frac{a_2^6}{120}\phi_4', \text{ \&c., \&c.}$$

29. It remains to substitute the first set of constants in the expression for K_1; arranging the terms according to powers of γ we find:

$$\begin{aligned} K_1 = {} & 2\pi a_1^2\phi_0^2 + 2\pi a_2^2\phi_0'^2 + \frac{4\pi a_1^2 a_2^2}{\gamma}\phi_0\phi_0' \\ & + \frac{\pi a_1^3 a_2^3}{\gamma^4}(a_1\phi_0^2 + a_2\phi_0'^2) + \frac{4\pi a_1^4 a_2^4}{3\gamma^6}(a_2\phi_0^2 + a_1\phi_0'^2) \\ & + \frac{2\pi a_1^5 a_2^5}{\gamma^7}\phi_0\phi_0' + \frac{3\pi a_1^4 a_2^4}{2\gamma^6}.(a_2^2\phi_0^2 + a_1^2\phi_0'^2) \\ & + \frac{4\pi a_1^5 a_2^5(a_1^2 + a_2^2)}{\gamma^9}\phi_0\phi_0' + \text{etc.} \\ = {} & 2\pi a_1^2\phi_0^2 + 2\pi a_2^2\phi_0'^2 + U, \text{ say.} \end{aligned}$$

Digitized by Google

If we suppose the two atoms to have linear velocities q_1, q_2 in directions h_1, h_2, then as mentioned in Art. 2 we must add to the above value of K_1 the expression

$$-\pi q_1 q_2 a_1^2 a_2^2 \frac{d^2}{dh_1 dh_2}\left(\frac{1}{\gamma}\right) + 2\pi\phi_0' q_1 a_1^2 a_2^2 \frac{d}{dh_1}\left(\frac{1}{\gamma}\right)$$
$$+ 2\pi\phi_0 q_2 a_1^2 a_2^2 \frac{d}{dh_2}\left(\frac{1}{\gamma}\right) + \text{higher powers of } \frac{1}{\gamma}.$$

Now if q_1, q_2 are small compared with ϕ_0, ϕ_0', those terms involving only products of the forms $q_1 q_2$ and $q\phi_0$ may be neglected in the value of K_1 as compared with those which involve squares and products of the ϕ's. In other words we shall suppose for the present that the motion of the atoms about their mean positions or in space has nothing like the magnitude at which a point on their surfaces is moving. (We might express this more concisely by saying, that the linear velocity of the atoms has no magnitude like that which by our theory is supposed to produce the phenomena of magnetism.)

We have to introduce a term $\frac{1}{2}m_1 q_1^2 + \frac{1}{2}m_2 q_2^2$ into the expression for the total energy of the system. We shall also suppose some system of forces to act on the two atoms capable of expression by the force function W, so that the force in direction k on the first atom is given by $\frac{dW}{dk}$, so that the equations of motion for any variable χ say, take the form,

$$\frac{d}{dt}\left(\frac{dL}{d\dot{\chi}}\right) - \frac{dL}{d\chi} = \frac{dW}{d\chi}.$$

30. We have now to consider two totally different cases which correspond to the following circumstances:—

(1) Two atoms which affect each other's pulsations or atoms in the same molecule,

(2) Two atoms which do not affect each other's pulsations or atoms of different molecules.

Now
$$L = K_2 + 2\pi a_1^2 \phi_0^2 + \frac{\pi a_1^3}{3} q_1^2 - V$$
$$+ K_2' + 2\pi a_2^2 \phi_0'^2 + \frac{\pi a_2^3}{3} q_2^2 - V'$$
$$+ U$$
$$= K_2 + K_2' - V - V' + U,$$

if we suppose the coefficients in K_2 and K_2' altered so as to include the terms which denote apparent increase of mass. Now let χ be a coordinate of the first atom then the equation of motion may be written

$$\frac{d}{dt}\left(\frac{dK_2}{d\dot{\chi}}\right) + \frac{dV}{d\chi} = \frac{dU}{d\chi} + \frac{dW}{d\chi} - \frac{d}{dt}\left(\frac{dU}{d\dot{\chi}}\right).$$

Case (i). Let two atoms be mutually influencing each other (i.e. atoms of the same molecule), but uninfluenced so far as pulsation is concerned by any other external forces, then $W = 0$ or the atoms will appear to move as if they were not in a fluid medium but were with altered mass and subject to a force function U.

Digitized by Google

If $\chi = h$ we have the equation

$$\frac{d}{dt}\left(\frac{dK_2}{dq_1}\right) = \frac{dU}{dh},$$

to calculate translation so far as influenced by pulsation, for the velocity of the atom in direction h.

Case (ii). Let the two atoms not be mutually influencing each other's pulsations (i.e. atoms of different molecules at some distance).

The pulsations of the atoms will now be looked upon as completely independent; these pulsations will in fact be considered either as absolutely free or as the forced pulsations due to other members only of the molecules to which the atoms under consideration respectively belong. In order to find the force between two atoms of different molecules, we shall then assume that in the expression for the force-function, the pulsation velocities are fully determined by molecular construction as in *Case* (i). We in fact neglect $\frac{dV}{d\chi}$ and $\frac{dV}{d\dot{\chi}}$ so far as they depend on intermolecular distances, but retain $\frac{dV}{dh}$ as giving intermolecular force.

31. I. *Atomic Forces.* The forces between atoms of the same molecule.

The force function between *two* such atoms is given by

$$U = \frac{4\pi a_1^2 a_2^2}{\gamma}\phi_0\phi_0' + \frac{\pi a_1^3 a_2^3}{\gamma^4}(a_1\phi_0^2 + a_2\phi_0'^2)$$
$$+ \frac{4\pi a_1^4 a_2^4}{3\gamma^6}(a_2\phi_0^2 + a_1\phi_0'^2) + \frac{2\pi a_1^5 a_2^5}{\gamma^7}\phi_0\phi_0'$$
$$+ \frac{3\pi a_1^4 a_2^4}{2\gamma^8}(a_2^3\phi_0^2 + a_1^3\phi_0'^2) + \frac{4\pi a_1^5 a_2^5}{\gamma^9}(a_1^2 + a_2^2)\phi_0\phi_0' + \text{etc.}$$

Now ϕ_0 is of the form $\quad R\cos\overline{n_1 t} + S\cos\overline{n_2 t}$, and

ϕ_0' ..... $\quad R'\cos\overline{n_1 t} + S'\cos\overline{n_2 t}$.

Thus the product $\phi_0\phi_0'$ will vary with the time, but it must be remembered that $\frac{2\pi}{n_1}$ and $\frac{2\pi}{n_2}$ are infinitely small—each pulsation of any atom being gone through billions of times a second, if we are to judge from the analogy of light vibrations. Hence we shall be justified in taking the mean value of the product $\phi_0\phi_0'$, or the sum of the mean values of its terms. If m_1 be not equal to m_2 the mean value of $\cos\overline{m_2 t}\,.\cos\overline{m_2 t} = 0$; if however $m_1 = m_2$ the mean value of $\cos^2\overline{m_1 t} = \frac{1}{2}$. It follows that the mean value of

$$\phi_0\phi_0' = \frac{RR' + SS'}{2},$$

Digitized by Google

and of
$$\phi_0^2 = \frac{R^2 + S^2}{2},$$
$$\phi_0'^2 = \frac{R'^2 + S'^2}{2}.$$

Now R, R', S, S' are functions of $\frac{1}{\gamma}$ which have already been calculated to a first degree of approximation, i.e. when the first term of U was included in our equations of motion. In order however to see more generally the character of the force function we shall calculate its value in this simple case on the supposition that its second term is also considered.

The equations of motion may then be written:

$$\ddot{\phi}_0(\lambda_0 + 4\pi a_1^3) + \tau_0\phi_0 + \frac{2\pi a_1^4 a_2^2}{\gamma^4}\ddot{\phi}_0 + \frac{4\pi a_1^2 a_2^2}{\gamma}\ddot{\phi}_0' = 0,$$

or
$$\ddot{\phi}_0\left(\frac{\tau_0}{\nu_1^2} + Pa_1\right) + \tau_0\phi_0 + Q\ddot{\phi}_0' = 0,$$
$$\ddot{\phi}_0'\left(\frac{\tau_0'}{\nu_2^2} + Pa_2\right) + \tau_0'\phi_0' + Q\ddot{\phi}_0 = 0,$$

where $P = \frac{2\pi a_1^3 a_2^3}{\gamma^4}$ and the other symbols have the meanings already assigned them.

Let
$$\phi_0 = C_1 \cos \overline{nt}, \quad \phi_0' = C_2 \cos \overline{nt},$$

Then,
$$\left\{\tau_0\left(\frac{1}{n^2} - \frac{1}{\nu_1^2}\right) - Pa_1\right\} C_1 - Q C_2 = 0,$$
$$\left\{\tau_0'\left(\frac{1}{n^2} - \frac{1}{\nu_2^2}\right) - Pa_2\right\} C_2 - Q C_1 = 0,$$

or,
$$\left\{\tau_0\left(\frac{1}{n^2} - \frac{1}{\nu_1^2}\right) - Pa_1\right\}\left\{\tau_0'\left(\frac{1}{n^2} - \frac{1}{\nu_2^2}\right) - Pa_2\right\} = Q^2.$$

Hence we find approximately, ν_1 and ν_2 being supposed not very nearly equal:—

$$\frac{1}{n_1^2} = \frac{1}{\nu_1^2} - \frac{Q^2}{\tau_0\tau_0'}\frac{\nu_1^2\nu_2^2}{\nu_1^2 - \nu_2^2} + \frac{Pa_1}{\tau_0} + \frac{Q^4}{\tau_0^2\tau_0'^2}\frac{\nu_1^6\nu_2^6}{(\nu_1^2 - \nu_2^2)^3},$$
$$\frac{1}{n_2^2} = \frac{1}{\nu_2^2} - \frac{Q^2}{\tau_0\tau_0'}\frac{\nu_1^2\nu_2^2}{\nu_2^2 - \nu_1^2} + \frac{Pa_2}{\tau_0'} + \frac{Q^4}{\tau_0^2\tau_0'^2}\frac{\nu_1^6\nu_2^6}{(\nu_2^2 - \nu_1^2)^3},$$
$$C_1\left\{\frac{Q}{\tau_0'}\frac{\nu_1^2\nu_2^2}{\nu_1^2 - \nu_2^2} - \frac{Q^3}{\tau_0\tau_0'^2}\frac{\nu_1^6\nu_2^6}{(\nu_1^2 - \nu_2^2)^3}\right\} = -C_2,$$
$$C_2'\left\{\frac{Q}{\tau_0}\frac{\nu_1^2\nu_2^2}{\nu_2^2 - \nu_1^2} - \frac{Q^3}{\tau_0'\tau_0^2}\frac{\nu_1^6\nu_2^6}{(\nu_2^2 - \nu_1^2)^3}\right\} = -C_1'.$$

We have at once
$$\left.\begin{aligned}\phi_0 &= -(n_1 C_1 \sin \overline{n_1 t} + n_2 C_1' \sin \overline{n_2 t})\\ \phi_0' &= -(n_1 C_2 \sin \overline{n_1 t} + n_2 C_2' \sin \overline{n_2 t})\end{aligned}\right\},$$

Digitized by Google

and
$$U = +Q\frac{n_1^2C_1C_2 + n_2^2C_1'C_2'}{2} + P\frac{a_1(n_1^2C_1^2 + n_2^2C_1'^2) + a_2(n_2^2C_2^2 + n_2^2C_2'^2)}{4}$$
$$= -\frac{Q^2}{2}\frac{\nu_1^2\nu_2^2}{\nu_1^2 - \nu_2^2}\left\{\frac{C_1^2\nu_1^2}{\tau_0'} - \frac{C_2'^2\nu_2^2}{\tau_0}\right\} + P\frac{(a_1\nu_1^2C_1^2 + a_2\nu_2^2C_2'^2)}{4}$$
$$-\frac{Q^4}{2\tau_0\tau_0'}\frac{\nu_1^4\nu_2^4}{(\nu_1^2 - \nu_2^2)^3}\left\{\frac{C_1^2\nu_1^4(\nu_1^2 - 2\nu_2^2)}{\tau_0'} - \frac{C_2'^2\nu_2^4(\nu_2^2 - 2\nu_1^2)}{\tau_0}\right\}$$
$$= -\frac{\mu_1}{\gamma^2} + \frac{\mu_2}{\gamma^4},$$

where
$$\mu_1 = 8\pi^2a_1^4a_2^4 \cdot \frac{\nu_1^2\nu_2^2}{\nu_1^2 - \nu_2^2}\left\{\frac{C_1^2\nu_1^2}{\tau_0'} - \frac{C_2'^2\nu_2^2}{\tau_0}\right\},$$
$$\mu_2 = \tfrac{1}{2}\pi a_1^3a_2^3(a_1\nu_1^2C_1^2 + a_2\nu_2^2C_2'^2)$$
$$-\frac{128\pi^4a_1^3a_2^3}{\tau_0\tau_0'}\frac{\nu_1^4\nu_2^4}{(\nu_1^2 - \nu_2^2)^3}\left\{\frac{C_1^2\nu_1^4(\nu_1^2 - 2\nu_2^2)}{\tau_0'} - \frac{C_2'^2\nu_2^4(\nu_2^2 - 2\nu_1^2)}{\tau_0}\right\},$$

and as before $\nu_1 > \nu_2$.

32. Now unless we make some assumptions as to the internal construction of atoms it is not easy to predict anything with regard to the nature of μ_1 and μ_2. We see however that μ_1 will be positive if $\frac{C_1^2\nu_1^2}{\tau_0'}$ is $> \frac{C_2'^2\nu_2^2}{\tau_0}$. If $C_1 = C_2'$ and $\tau_0' = \tau_0$ this will certainly be the case; or again if $C_1 = C_2'$, since $\nu_1^2 = \frac{\tau_0}{\lambda_0 + 4\pi a_1^3}$, $\nu_2^2 = \frac{\tau_0'}{\lambda_0' + 4\pi a_2^3}$ it seems not improbable, that if $\nu_1 > \nu_2$, τ_0 will be $> \tau_0'$, and hence that $\nu_1^2\tau_0$ will be $> \nu_2^2\tau_0'$. In like manner μ_2 consists of two parts, the first of which is always positive and the second negative or positive according as $\frac{C_1^2\nu_1^4(\nu_1^2 - 2\nu_2^2)}{\tau_0'}$ is greater or less than $\frac{C_2'^2\nu_2^4(\nu_2^2 - 2\nu_1^2)}{\tau_0}$. If $C_1 = C_2'$ and $\tau_0 = \tau_0'$ the latter part of μ_2 will certainly be negative; as again if $\frac{C_1^2\nu_1^2}{\tau_0'}$ be $> \frac{C_2'^2\nu_2^2}{\tau_0}$; but whether μ_2 as a whole is positive or negative we cannot determine till we know the relation of the potential and kinetic energies of an atom to its radius.

We have however obtained the following result:—*The force between two atoms which are mutually influencing each other's pulsations, i.e. the 'atomic force' between two atoms, varies partly as the inverse cube, partly as the inverse fifth power of the central distance.* In order that two atoms may combine to form a molecule, the force between them must be attractive; hence as a first condition for a molecular union of *two* atoms it is necessary that μ_1 be negative, or
$$\frac{C_1^2\nu_1^2}{\tau_0'} \text{ be } < \frac{C_2'^2\nu_2^2}{\tau_0}.$$

But if γ becomes very great we have
$$\phi_0 = -C_1\nu_1 \sin \overline{\nu_1 t},$$
$$\phi_0' = -C_2\nu_2 \sin \overline{\nu_2 t},$$

Digitized by Google

or $C_1^2\nu_1^2$ represents the 'intensity' of the free pulsation of the first atom, while $C_2^2\nu_2^2$ is the like quantity for the second.

In order then that the atoms may form a molecular union, it is necessary that the product of the potential energy coefficient and the intensity of the free pulsation should be greater in that atom whose pulsation has the greater free period.

We have of course only calculated the lowest powers of $\frac{1}{\gamma}$ in the force function, but these will be the most important. When γ is of any magnitude compared with the atomic radii the first of these is really the all important term, and we might say that: *Forces of chemical combination vary as the inverse cube.*

33. In order to throw more light on these forces, let us suppose the atoms equal; then we find:

$$\frac{1}{n_1^2}=\frac{1}{\nu_1^2}-\frac{Q}{\tau_0}+\frac{Pa_1}{\tau_0},$$

$$\frac{1}{n_2^2}=\frac{1}{\nu_1^2}+\frac{Q}{\tau_0}+\frac{Pa_1}{\tau_0},$$

and

$$\phi_0=-n_1C_1\sin\overline{n_1t}-n_2C_2\sin\overline{n_2t},$$

$$\phi_0'=\quad n_1C_1\sin\overline{n_1t}-n_2C_2\sin\overline{n_2t}.$$

Hence

$$U=-\frac{Q}{2}\nu_1^2\left\{(C_1^2-C_2^2)+\frac{Q\nu_1^2}{\tau_0}(C_1^2+C_2^2)+\frac{Q^2\nu_1^4}{\tau_0^2}(C_1^2-C_2^2)+\frac{Q^3\nu_1^6}{\tau_0^3}(C_1^2+C_2^2)\right\}+\frac{Pa\nu_1^2}{2}(C_1^2+C_2^2)$$

$$=-\frac{Q\nu_1^2}{2}(C_1^2-C_2^2)-\frac{Q^2\nu_1^4}{2\tau_0}(C_1^2+C_2^2)-\frac{Q^3\nu_1^6}{2\tau_0^2}(C_1^2-C_2^2)-\left(\frac{Q^4\nu_1^6}{\tau_0^3}-Pa\right)\frac{\nu_1^2}{2}(C_1^2+C_2^2).$$

Hence if C_1 be *not* equal to C_2, two equal atoms within their sphere of mutual reaction act upon each other with the following forces:

(*a*) A force varying as the inverse square of the distance, which is attractive if $C_1<C_2$, repulsive if $C_1>C_2$.

(*b*) A force always repulsive varying as the inverse cube of the distance.

(*c*) A force varying as the inverse fourth power of the distance and attractive or repulsive according as $C_1^2<$ or $>C_2^2$.

(*d*) A force varying as the inverse fifth power of the distance which is attractive or repulsive according as $-\frac{Q^4\nu_1^6}{\tau_0^3}+Pa$ is positive or negative; or adopting the previous notation according as $\frac{\lambda_0}{4\pi a^3}>$ or $<\sqrt[3]{2}-1$.

If $C_1=C_2$ the first and the third forces vanish, and exactly as in the case of two unequal atoms we have central forces as the inverse cube and inverse fifth.

Digitized by Google

34. We may here make some observations referring to both kinds of atom-pairs. If an increase in ν_1, ν_2 or a decrease in the periods $\frac{2\pi}{\nu_1}$, $\frac{2\pi}{\nu_2}$ does not denote a decrease in the radii of the atoms, we see that these forces of atomic union or chemical forces are greater the greater are the quantities ν_1, ν_2 or the less the periods $\frac{2\pi}{\nu_1}$, $\frac{2\pi}{\nu_2}$; in other words a molecule which gives lines towards the violet end of the spectrum or in the ultra violet will be a stronger chemical combination than one which gives lines further removed towards the red end of the spectrum.

35. It may then be questioned whether the nature of the equilibrium in a diatomic molecule is statical or dynamical? There are two kinds of statical equilibrium conceivable, either the atoms are in contact or it is possible that the repulsive force between them may at some distance greater than the sum of their radii be equal to the attractive force. If the equilibrium is dynamical we must suppose one atom describing an orbit relative to the other; in this case we may compare a molecule to a planetary system and the action of one molecule on another to that of one planetary system on another. In one case the disturbance due to the approach of two molecules will be from the statical position of equilibrium, in the other case it will be from the mean orbit.

When the atoms are equal, and supposing $C_1 = C_2$, we find the force between them

$$= -\frac{\nu_1^2 C_1^2}{\gamma}\left(2\frac{Q^2\nu_1^2}{\tau_0} - 4\frac{Q^4\nu_1^6}{\tau_0^3} - 4Pa_1\right),$$

and this vanishes, or there will be a non-contact position of equilibrium, if

$$\frac{Q^2\nu_1^2}{\tau_0} - 2\frac{Q^4\nu_1^6}{\tau_0^3} - 2Pa_1 = 0,$$

that is, if

$$\frac{1}{1+\frac{\lambda_0}{4\pi a_1^3}}\frac{\gamma^2}{a^2} + \frac{2}{\left(1+\frac{\lambda_0}{4\pi a_1^3}\right)^3} - 1 = 0$$

$\left(\text{remembering } \nu_1^2 = \frac{\tau_0}{4\pi a_1^3 + \lambda_0}\right)$.

Now γ must be greater than $2a_1$ for such a position; hence,

$$1 - \frac{2}{\left(1+\frac{\lambda_0}{4\pi a_1^3}\right)^3} > \frac{4}{1+\frac{\lambda_0}{4\pi a_1^3}},$$

or

$$\left(1+\frac{\lambda_0}{4\pi a_1^3}\right)^3 - 4\left(1+\frac{\lambda_0}{4\pi a_1^3}\right)^2 - 2 > 0, \quad \frac{\lambda_0}{4\pi a_1^3} > 3.12\ldots$$

Whether or not this is possible depends on the internal construction of an atom concerning which we have made no assumptions. It must be noted however that if the atoms be supposed to have no mass (λ's = zero), this non-contact position of equilibrium becomes impossible.

36. Let us now turn to the case of a relative orbit. In order to find its form we may suppose that forces equal and opposite to those acting on one atom are imposed on the

Digitized by Google

other. This really amounts to no more than altering in a certain ratio the constants of the force function, and hence to find the relative orbit we may still assume that the force function is of the form $\frac{\mu_1}{\gamma^2} - \frac{\mu_2}{\gamma^4}$:—supposing that the atoms are not similar or, if they are, that $C_1 = C_2$.

Let $\frac{1}{\gamma} = u$ and θ be the angle γ makes with some fixed straight line, then we have to integrate

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{h^2a^2}(2\mu_1 u^3 - 4\mu_2 u^5)$$

where h^2 is a constant.

Or ultimately,

$$\left(\frac{du}{d\theta}\right)^2 = a^2 + bu^2 - cu^4,$$

$$a\theta + f = \int \frac{du}{\sqrt{(1-u^2)(1+k^2u^2)}},$$

where a, f and k are constants.

Hence

$$\alpha\theta + \beta = \int \frac{d\phi}{\sqrt{1-\kappa^2 \sin^2\phi}},$$

where $u = \cos\phi$ and α, β, κ are other constants.

It follows that

$$\phi = am\,(\alpha\theta + \beta),$$

$$\text{or} \quad u = \cos am\,(\alpha\theta + \beta).$$

Since $u = 0$ when $\alpha\theta + \beta = (2n+1)K$, it follows that the form of the orbit thus given is not as a general rule a closed curve; in other words there is no permanent union of the atoms to form a molecule. Hence in this particular case of two unlike atoms a dynamical stability of the molecule does not seem probable.

Whether the atoms of a molecule are in statical or dynamical equilibrium seems a question of considerable interest; it is possible that they change the nature of their equilibrium as the molecular state of the body is changed. For instance, the atoms of the molecule in a solid body may be in statical, while the atoms of a molecule in a rare gas may be in dynamical equilibrium.

37. We pass now to the forces between p unlike atoms forming a molecule. Here we are certain that the first term of the force function will be of the form

$$U = \Sigma \frac{4\pi a_r^2 a_s^2}{\gamma_{rs}} \phi_r \phi_s = \Sigma Q_{rs} \phi_r \phi_s,$$

where ϕ_r and ϕ_s are the pulsations of the rth and sth atom respectively, and are given by

$$\phi_r = -n_r C_r \sin \overline{n_r t} + \Sigma \frac{C_m Q_{rm} \nu_m^2 \nu_r^2}{\tau_r(\nu_m^2 - \nu_r^2)} n_m \sin \overline{n_m t},$$

$$\phi_s = -n_s C_s \sin \overline{n_s t} + \Sigma \frac{C_m Q_{sm} \nu_m^2 \nu_s^2}{\tau_s(\nu_m^2 - \nu_s^2)} n_m \sin \overline{n_m t}.$$

Digitized by Google

Now we need only the terms of most importance in the product; these are

$$-n_r^2 C_r^2 \sin^2 \overline{n_r t} \frac{Q_{sr}\nu_r^2\nu_s^2}{\tau_s(\nu_r^2-\nu_s^2)} - n_s^2 C_s^2 \sin^2 \overline{n_s t} \frac{Q_{sr}\nu_r^2\nu_s^2}{\tau_r(\nu_s^2-\nu_r^2)},$$

or with mean values, if $\nu_r > \nu_s$

$$-\frac{Q_{sr}\nu_r^2\nu_s^2}{2(\nu_r^2-\nu_s^2)}\left(\frac{\nu_r^2 C_r^2}{\tau_s} - \frac{\nu_s^2 C_s^2}{\tau_r}\right).$$

Hence

$$U = -\Sigma \frac{Q_{rs}^2\nu_r^2\nu_s^2}{2(\nu_r^2-\nu_s^2)}\left(\frac{\nu_r^2 C_r^2}{\tau_s} - \frac{\nu_s^2 C_s^2}{\tau_r}\right).$$

We see at once then that the force between any two unlike atoms in a molecule varies inversely as the cube of their distance, and that if the quantity $\tau\nu^2 C^2$ is greater for the atom whose free pulsation has the greater period, then the force is attractive; if however it is greater for the atom which has the less period in its free pulsation the force is repulsive.

38. The expression $\tau\nu^2 C^2$, or the product of the potential energy coefficient and the 'intensity' of the free pulsation, may be termed the *chemical intensity* of an atom. The expression $\dfrac{\nu_1^2 C_1^2 \tau_1 - \nu_2^2 C_2^2 \tau_2}{\dfrac{4\pi^2}{\nu_1^2} - \dfrac{4\pi^2}{\nu_2^2}}$, or the difference of the chemical intensities of two atoms divided by the difference of the squares of their periods, may be termed the *chemical coefficient* of the two atoms. If the chemical coefficient is *positive* the two atoms will attract each other, if *negative* they will repel. Its sign thus determines the possibility of two atoms entering into combination. If F_{12} represent the chemical coefficient, the quantity $\dfrac{64\pi^4 a_1^4 a_2^4}{\tau_1\tau_2} \cdot F_{12}$ measures the strength of the force between the two atoms. The force between them $= \dfrac{64\pi^4 a_1^4 a_2^4}{\tau_1\tau_2} \cdot F_{12} \cdot \dfrac{1}{\gamma_{12}^3}$, tending to decrease γ, and is therefore attractive if F_{12} be positive. It might perhaps then be better to adopt as our definition of the chemical coefficient the product of $\dfrac{\text{difference of chemical intensities}}{\text{difference of squares of periods}}$ and the quantity $\dfrac{64\pi^4 a_1^4 a_2^4}{\tau_1\tau_2}$ *. A knowledge then of the chemical coefficients of every two of a set of p atoms would enable us to determine the possibility of p atoms uniting to form a molecule, and also the *stability* of the molecule so determined. Again, if r other molecules were added to the p already combined it would enable us to determine whether this action would destroy the combination of p atoms.

39. Let us consider the simple but very suggestive case of a third atom being added to two already in combination.

Let $\dfrac{2\pi}{\nu_1}$, $\dfrac{2\pi}{\nu_2}$ be the periods of those in combination. Let also ν_1 be $> \nu_2$, and therefore for combination $I_1 < I_2$. If $\dfrac{2\pi}{\nu_3}$ be the free period of the atom added to them, we obtain the following results.

* This chemical coefficient might perhaps be termed the "chemical affinity."

Digitized by Google

Case (i). $\nu_3 > \nu_1$.

(*a*) If the 'chemical intensity' of the third atom is less than those of the other two ($I_3 < I_2$ and $< I_1$), it is capable of entering into a combination with the molecule formed of the two, because it is attracted to both.

(*b*) If the chemical intensity of the third atom is greater than either of the other two ($I_3 > I_2$ and $> I_1$), it is incapable of uniting with them to form a molecule because it is repelled from both.

(*c*) If the chemical intensity of the third atom is greater than the one and less than the other chemical intensity of the already combined atoms (I_3 lies between I_1 and I_2), it attracts the second atom and repels the first. Then in order that it may remain in the combination we must have

$$-\frac{A_{13}}{\gamma_{13}^3} < \frac{A_{23}}{\gamma_{23}^3} \text{ (where the } A\text{'s represent } \textit{chemical affinities}\text{).}$$

The first element will then be turned out of the combination, if

$$\frac{A_{12}}{\gamma_{12}^3} < -\frac{A_{31}}{\gamma_{31}^3}.$$

We shall then have the second and third atoms left together attracting each other irrespective of whatever changes may, owing to external disturbance, take place in their distance. In other words we have a new *stable* compound of two atoms.

If, on the other hand, $$\frac{A_{12}}{\gamma_{12}^3} > -\frac{A_{31}}{\gamma_{31}^3},$$

the first atom will not be forced out of the combination, but we shall have two mutually repelling atoms united by means of a third which attracts both. In this case the forces uniting the various atoms of the molecule are not all attractive; it is obvious that the molecule requires less to disassociate it, and we may say that we have an *unstable* compound of three atoms. Such theoretically unstable compounds seem to throw light on the nature of easily decomposed or even explosive substances.

The other cases may be treated more briefly as they present no further novelty.

Case (ii). $\nu_3 < \nu_1$ and $> \nu_2$. Remembering that since the first two atoms form a molecule and $\nu_1 > \nu_2$, therefore $I_2 > I_1$ we find:

(*a*) $I_3 > I_1$ and $< I_2$.

The three atoms will unite to form a new molecule.

(*b*) $I_3 > I_2$ and therefore $> I_1$.

The third atom will attract the first atom and repel the second.

$$\left.\begin{array}{ll} \text{If,} & -\dfrac{A_{32}}{\gamma_{32}^3} > \dfrac{A_{12}}{\gamma_{12}^3} \\ \text{and} & \dfrac{A_{13}}{\gamma_{13}^3} > -\dfrac{A_{32}}{\gamma_{32}^3} \end{array}\right\} \text{ the third atom will enter into a combination with the first atom and turn out the second.}$$

Digitized by Google

If, $\left.\begin{array}{l} -\dfrac{A_{32}}{\gamma_{32}^2} < \dfrac{A_{12}}{\gamma_{12}^2} \\ \dfrac{A_{13}}{\gamma_{13}^2} > -\dfrac{A_{32}}{\gamma_{32}^2} \end{array}\right\}$ an unstable compound of the three atoms will be formed.

If $-\dfrac{A_{32}}{\gamma_{32}^2} > \dfrac{A_{21}}{\gamma_{21}^2}$, then, since γ_{13} can never get very small, $-\dfrac{A_{32}}{\gamma_{32}^2}$ will be less than $\dfrac{A_{21}}{\gamma_{21}^2}$, and the third atom will be forced away from the compound of the first two atoms. If (as by the application of some mechanical force to bring the third atom and the molecule of the compound together, perhaps, for example, extreme pressure) γ_{13} could, notwithstanding the repulsive force on the third atom from the diatomic compound, get so small that $-\dfrac{A_{32}}{\gamma_{32}^2} > \dfrac{A_{21}}{\gamma_{21}^2}$, then the result of adding the third atom to the diatomic molecule would be that all three atoms would be disassociated. Yet it seems probable that even this would only be possible in the case of a limited number of molecules, and we might almost venture to state it as a general law: that, 'no element in its disassociated state (i.e. when its own atoms are not in molecular union) could possibly so break up a diatomic compound of two other elements that all three elements would appear disassociated.' This law might be very easily generalized.

(*c*) If $I_3 < I_1$ and therefore $< I_2$, the third atom repels the first and attracts the second. Possible relations may be considered as in (*b*).

Case (iii). $\nu_3 < \nu_2$.

(*a*) $I_3 > I_2$ and therefore $I_3 > I_1$.

Stable union of all three atoms.

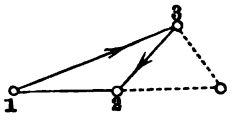
(*b*) $I_3 < I_1$ and therefore $I_3 < I_2$.

The diatomic molecule will not be broken up by the presence of the third atom.

(*c*) $I_3 > I_1$ and $I_3 < I_2$, the third atom attracts the first and repels the second.

Variations may be considered as in *Cases* (i), (*c*) and (ii), (*b*).

40. It will be noted that while considering the relation of the three atoms in the cases of forces partly repulsive and partly attractive we have deduced our results by supposing the atoms in a straight line. This is the position in which the repulsive force is least, and therefore that in which we may suppose the atom to approach the compound. Suppose, for example, 3 is attracted by 2 and repelled by 1. Then, if 3 describe a circle about 2, the repulsive force between 3 and 1 will be least when 3, 2, 1 lie in a straight line, and therefore 3 will tend to take up such a position.



41. The above example may perhaps show how this theory of atoms may be made to throw light on the laws of chemical combination and decomposition. The question of chemical combination and decomposition would become one of calculation were we able to observe the period of free pulsation of every elementary atom and to tabulate chemical intensities and chemical coefficients or the equivalent chemical affinities.

Digitized by Google

42. II. *Molecular Forces.*

We have now to consider the force-function between two molecules or combinations of atoms. In the case of a gas we may assume that the atoms of a molecule at ordinary pressure and density while mutually affecting each other's pulsations do not affect the pulsations of atoms of other molecules. When two molecules are brought near together, they probably affect the distances between the atoms first, and afterwards the atoms in different molecules will begin directly to affect each other's pulsations. In the case of a solid or liquid we should expect great complexity in the pulsations of each atom and continuous variation of the atomic distances (the continuous oscillation of atoms?); these phenomena lead to the continuous spectrum. Let us consider the force between two molecules whose atoms may be supposed not to be reciprocally affecting their pulsations. In this case an atom of the one will be pulsating independently of an atom of the other, and so far as they are individually concerned their pulsations may be looked upon as *forced.* Hence, if $\dot{\phi}_0$, $\dot{\phi}_0'$ be the respective atomic pulsation velocities, these velocities will be supposed fully determined by the inter-action of the atoms of one molecule, and to be unaffected by those of the other. Thus the resulting force-function will be treated as affecting only the *distance* between two atoms of different molecules (intermolecular distance). It follows that:

$$U = \frac{4\pi a_1^2 a_2^2}{\gamma}\dot{\phi}_0\dot{\phi}_0' + \frac{\pi a_1^3 a_2^3}{\gamma^4}(a_1\dot{\phi}_0^2 + a_2\dot{\phi}_0'^2) + \frac{4\pi a_1^4 a_2^4}{3\gamma^6}(a_2\dot{\phi}_0^2 + a_1\dot{\phi}_0'^2) + \ldots.$$

gives this force-function. In order then to obtain the force-function between two molecules we must sum the force-function between every pair of atoms in either. If we only wish the first terms of this force-function, since the molecules are at very great distances compared with those of their component atoms, we may assume that γ is the same for every pair of atoms; this value of γ will be looked upon as the distance between the two molecules. Suppose we have two p-atomic molecules, then the rth atom in the first molecule will (see page 90) have a velocity pulsation given by

$$\dot{\phi}_r = -n_r C_r \sin(n_r t + \alpha_r) + \Sigma \frac{n_m C_m Q_{rm} \nu_m^2 \nu_r^2}{\tau_r(\nu_m^2 - \nu_r^2)} \cos(n_m t + \alpha_m),$$

and the sth atom in the second molecule:

$$\dot{\phi}_s' = -n_s C_s' \sin(n_s t + \alpha_s') + \Sigma \frac{n_m C_m' Q_{sm} \nu_m^2 \nu_s^2}{\tau_s(\nu_m^2 - \nu_s^2)} \cos(n_m t + \alpha_m').$$

We see then that the value of the product $\dot{\phi}_r\dot{\phi}_s'$, when we replace the products of sines and cosines by their means, will not rise into importance unless r is equal to s, in which case its value becomes

$$\frac{n_r^2 C_r C_r'}{2}(\cos\alpha_r \cos\alpha'_r + \sin\alpha_r \sin\alpha'_r)$$

$$= \frac{n_r^2 C_r C'_r}{2}\cos(\alpha_r - \alpha'_r).$$

We can then write the force-function between the two molecules

$$U = \frac{\mu_1}{\gamma} + \frac{\mu_2}{\gamma^4},$$

Digitized by Google

where
$$\mu_1 = \sum_1^p 2\pi a_r^4 n_r^2 C_r C'_r \cos(\alpha_r - \alpha'_r),$$
$$\mu_2 = \sum_1^p \sum_1^p \frac{\pi a_r^3 a_s^3}{2} (a_r n_r^2 C_r^2 + a_s n_s^2 C'^2_s).$$

We see that μ_2 is always positive—i.e. *two molecules always attract each other with a force varying inversely as the fifth power of their distance.*

43. But with regard to μ_1 we see that its sign depends on the difference of phases of the atoms of like kind in the two molecules. If these differences can be anything whatever there may or may not be a force between any two molecules varying as the inverse square of their distance, and that from molecule to molecule this force may, if it exists, be attractive or repulsive. We should then have the following result—the molecules of a gas would be in part attracting and in part repelling each other. In this case it seems possible that a quantity of gas might be obtained whose particles had a disassociative tendency:—in other words, that a gas on expanding would lose internal (molecular configurational) energy, but the experiments of Joule and Thomson seem to show that a gas always does some however small amount of work in overcoming the cohesion of its own molecules when it is expanding. If this be granted we are compelled to consider that the differences of phase of two atoms of like kind in two different molecules are not purely arbitrary, and to assert that the differences of phase of any two like atoms does not generally exceed $\frac{\pi}{2}$. If we get over this difficulty by supposing that in any finite portion of a gas such as it is possible to experiment on, we shall have as many repelling as attracting molecules, and these in *all* varieties of difference of phase, we find that in the expression for the internal energy of a gas (due to molecular configuration) there will be no terms whatever varying inversely as the first power of the distances, and this internal energy will, varying entirely as the fourth power of the distances, be a very small quantity as shown by Joule and Thomson's experiments.

44. But even this assumption does not free us from new difficulties. These molecular forces must in the cases of a solid or liquid become the forces of cohesion or of viscosity, if we do not suppose in either case the atoms of different molecules to be as near each other as the atoms of the same molecule, and so the forces between different molecules to become in reality what we have termed chemical forces. Yet if these chemical forces become the chief forces of cohesion in the case of a solid it is difficult to understand how the molecules are preserved in such a case. All the molecules would be forced into chemical combination with one another, and there would seem no reason why a solid should always present us with a gas composed of the same kind of molecules. We are compelled then to consider a solid as composed of molecules whose distances, if small compared with the gases, are yet great compared with the atomic distances. That the atoms of one molecule influence those of another either by slightly affecting their respective pulsations, or still more probably their respective distances, the continuous spectrum of a solid forces us to believe, but if the molecules are to remain distinct we must suppose the principal forces between them are what we have termed molecular forces and not chemical

Digitized by Google

forces. If this however be so how can the forces between a number of molecules be partly attractive and partly repulsive? Under such a state of things cohesion seems impossible. This leads us to the conclusion that in any substance the difference of phase between two like atoms in different molecules cannot exceed $\frac{\pi}{2}$; or rather it will be sufficient if for every pair of molecules

$$\Sigma a_r^4 n_r^2 C_r C'_r \cos(\alpha_r - \alpha'_r)$$

is *positive.*

In the case of monatomic molecules we must have the difference of phase of the atoms always less than $\frac{\pi}{2}$.

In the case of diatomic molecules we must have

$$a_1^4 n_1^2 C_1 C'_1 \cos(\alpha_1 - \alpha'_1) + a_2^4 n_2^2 C_2 C'_2 \cos(\alpha_2 - \alpha'_2)$$

always *positive,* and so forth.

45. Now it must be acknowledged that these seem very arbitrary distinctions, and one is inclined to demand a general law which would include them all. The difficulty lies in this: What is it that determines the difference of phase between two atoms?

It is obvious that we may consider the Cs as all positive, since any negative sign would be brought into the argument by alteration of the phase, the condition then that the differences of phase shall not exceed $\frac{\pi}{2}$ is sufficient although not necessary. We may then ask: Is it necessary that two like atoms in different molecules should have any difference of phase? May not the state of pulsation in every molecule both as to amplitude and phase *or at least as to the latter* be the same for their like atoms?

Is there anything which at all confirms this view?

Suppose we attempted to find the result of a number of molecules whose pulsations of like period had every conceivable variety of phase disturbing the ether at any point. Then if one molecule produced at that point what might be termed a positive disturbance, another could be found owing to the difference of phase giving an equal and opposite negative disturbance, and so the total effect of all the disturbing molecules might be almost zero. In the Undulatory Theory of Light we are compelled to suppose at the same instant every point in the plane face of a small wave to have the same phase. It is hard to conceive how such is possible, if at the same instant every molecule helping to form that wave has its corresponding pulsation in a different phase.

For example, if one molecule produced a disturbance given by

$$D \sin(nt - \beta x + \kappa)$$

at any point of the ether, another owing to its difference of phase would produce a disturbance of the form $D \sin(nt - \beta x + \kappa')$,

and the total value of such disturbances would be zero if we gave κ all possible values.

Digitized by Google

Hence we have suggested to us the following conclusion:—*Like atoms in like molecules are pulsating in the same phase.*

We can then deduce the following law of molecular force. The force between two molecules consists in its more important terms of a part varying as the inverse square and a part varying as the inverse fifth power of the molecular distance; both these components are attractive.

46. We may draw the following conclusions:—

Every term of the intensity of this attractive force varies inversely as some (period)^2 of the molecular pulsations (i.e. has a factor n^2). Hence those substances which give bright lines nearest to the violet (supposing $a_r^4 n_r^2$ to *increase* with n_r) will have their molecular forces greatest. In other words, they will do more internal work in expanding. Or again, when in a solid state their forces of cohesion will be strongest or they will be what might be termed 'tough' substances. According to this *theory* then the toughness of *pure* metals would be in the same order as the frequencies of their single bright lines.

Again the relative orbit of two molecules so far as they are uninfluenced by other molecules will be very approximately (retaining only the first term of the force-function) a conic section about the other molecule in the focus.

It may be noted however that Clerk Maxwell's experiments on the viscosity of gases led him to suppose a repulsive and not an attractive force varying as the inverse fifth power of the distance between two very near molecules, a result with which unfortunately our theory does not appear to coincide (cf. Clifford's *Lectures and Essays*, I. p. 241, however).

Let us return to the statement that like atoms in like molecules are pulsating in the same phase. Supposing the atoms not in molecular union with others or monatomic molecules, we are led to the result that all like atoms when they are not portions of molecular compounds are vibrating in the same phase.

47. It is however not necessary to suppose that the phase of the free pulsation when altered to a forced pulsation is itself not altered. It is sufficient if the alteration be a function of $\frac{a}{\gamma}$, and vanish when $\frac{a}{\gamma}$ becomes insensible. The following example will perhaps cast light on the meaning of this statement.

Let us suppose four atoms two and two alike, and that they are about to unite one of each kind to form two diatomic molecules. Let the free pulsations of the first like pair be given by

$$\left.\begin{aligned}\phi_1 &= D_1 \cos(\nu_1 t + \alpha_1)\\ \phi_1' &= D_1' \cos(\nu_1 t + \alpha_1)\end{aligned}\right\} \text{ and } \left\{\begin{aligned}\phi_2 &= E_2 \cos(\nu_2 t + \alpha_2)\\ \phi_2' &= E_2' \cos(\nu_2 t + \alpha_2)\end{aligned}\right.$$

be the free pulsations of the second like pair, each pair being in the same phase.

Then the effect of atom ϕ_1 on atom ϕ_2 is as follows:—

(1) to alter the free period in a manner given by the following equation:—

$$\frac{1}{n_2^2} = \frac{1}{\nu_2^2} - \frac{Q^2}{\tau_0 \tau'_0} \frac{\nu_1^2 \nu_2^2}{\nu_2^2 - \nu_1^2};$$

Digitized by Google

(2) to alter E_2 say to C_2;

(3) to alter α_2 to $\alpha_2 + Q\mu_2$ say.

Then we find $$\phi_2 = C_2 \cos(n_2 t + \alpha_2 + Q\mu_2).$$

But to this we must add a forced pulsation with the altered period of the first atom, namely: $$-\frac{C_1 Q \nu_1^2 \nu_2^2}{\tau'_0(\nu_1^2 - \nu_2^2)} \cos(n_1 t + \alpha_1 + Q\mu_1),$$

the intensity of which is also a function of Q.

We thus have finally

$$\phi_2 = C_2 \cos(n_2 t + \alpha_2 + Q\mu_2) - \frac{C_1 Q \nu_1^2 \nu_2^2}{\tau'_0(\nu_1^2 - \nu_2^2)} \cos(n_1 t + \alpha_1 + Q\mu_1).$$

If γ were very great however the second term would vanish, $n_2 = \nu_2$, and C_2 must become $= E_2$,

$$\text{or } \phi_2 = E_2 \cos(n_2 t + \alpha).$$

Thus C_2 can only differ from E_2 by quantities of order Q.

The same reasoning gives us

$$\phi'_2 = C'_2 \cos(n_2 t + \alpha_2 + Q'\mu'_2) - \frac{C'_1 Q \nu_1^2 \nu_2^2}{\tau'_0(\nu_1^2 - \nu_2^2)} \cos(n_1 t + \alpha_1 + Q'\mu'_1).$$

Hence so far as these terms are concerned the attractive force between the molecules is given by $$\frac{2\pi a_2^2 n_2^2 C_2 C'_2 \cos(Q'\mu'_2 - Q\mu_2)}{\gamma^2},$$

and since $Q'\mu'_2 - Q\mu_2$ is always very small $\left(\text{of the order } \frac{1}{\gamma^2},\right)$ it is less than $\frac{\pi}{2}$, and the force as given by each pair of terms will be attractive.

In order that the pulsation periods of each molecule may be the same, it is necessary that $Q = Q'$. In order to obtain a wave of light from a number of different molecules it appears not improbable, though perhaps not absolutely necessary, that $\mu_2' = \mu_2$.

There is still another point to be noted: the chemical affinity of the molecules formed from the atoms ϕ_1 and ϕ_2 is a function of D_1 and E_2; the chemical affinity of the molecule formed from the atoms ϕ_1' and ϕ_2' is a function of D_1' and E_2', or as it would be perhaps better to say, of C_1, C_2 and of C_1', C_2'. Now when a substance is under constant conditions we cannot suppose the chemical affinity to vary much from molecule to molecule, and hence we must have C_1 and C_1' nearly equal to C_2 and C_2'. Hence we may say that when the molecules of a substance are subject to much the same external conditions, not only the phases, but the amplitudes of the atomic pulsations of the molecules are for the corresponding atoms of different molecules very nearly alike.

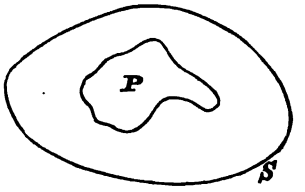
48. We have seen that the action of one atom on another is a vanishing quantity when the atoms are at any distance great compared with their linear dimensions; if however a great number of atoms are all tending to influence another atom even at a great distance in a like fashion, especially if they are all tending to force upon it a pulsation of equal or nearly equal period to its free pulsation, then the sum of these vanishing

Digitized by Google

individual actions becomes so magnified as to be of importance. One atom in the sun does not in the very slightest affect the pulsations of an earth-atom, but an infinite number of such atoms in the sun may affect the pulsations of an earth-atom. We cannot in that case treat the action of individual atoms upon each other, we must rather look at the outcome of the infinite number of atoms in producing a wave, and at the effect of this disturbance on an individual atom in its neighbourhood. Even then it seems improbable that it will be much affected unless there is a period of the wave of equal or nearly equal magnitude with the free period of the atom. In this case we must suppose the amplitude of the free pulsation of the atom—or in the case of the atom being in molecular union its principal pulsation or the pulsation corresponding to the free pulsation but with somewhat altered period—to be most affected. Such a change of amplitude however will affect the chemical intensity of the atom, and thus the chemical affinity of two atoms. Hence we note how it may happen that waves of different period may tend to alter the constitution of a body either as to outward appearance or molecular construction; waves—termed for convenience chemical, light and heat waves—may affect the body in different ways because they alter the chemical intensity of its atoms, the chemical affinity between the atoms of the same molecule, and lastly the molecular forces between the parts of a body.

49. If magnetism and electricity consist in the oscillations of the ultimate atoms of a body, they will not directly affect the pulsations, but by altering the atomic distances they will indirectly affect them. When however the atoms are pulsating freely, it is impossible any longer for alterations in the mutual distances to affect their pulsations. Hence if once a body is completely disassociated, magnetic or electric action according to our theory ought not to affect its spectrum.

50. We may note then that in order to fully grasp the 'state' of a body we must know not only its atoms and the particular manner in which they are united into molecules, but also the exact condition of the ether in the spot where the body is situated. For instance if P be a body, and S a closed surface surrounding it, then the condition of the body P depends at every instant on what is happening in the ether over the surface S. Its so-called mechanical chemical and physical properties vary with what is happening in the ether at the surface S. For example, the molecular theory of gases supposes the molecules of a gas only to be affected by the close neighbourhood of other molecules, when they are supposed to exert force upon each other. In the above theory the condition of a molecule at any time must be looked upon as a function of the disturbance in the ether at the spot where it may chance to be. If we knew the exact characteristics of the disturbance of the ether at that neighbourhood at the given time, we should be able to determine every property of the molecule without reference to other existing molecules.



51. III. *Gravitation Forces.*

If this theory of a continuous medium producing by its disturbance all the phenomena of the physical world be at all a true one we must expect it to give some explanation of the

Digitized by Google

so-called principle of gravitation. If two bodies P and Q are apparently drawn together by a certain force, it can only be because they are both disturbing the ether, and the effect of their disturbance can be represented by that force. Whatever "force Q exerts on P" must be the result of the state of the ether as produced by Q over a surface enclosing P; there is not such a thing as a force "inherent" in either P or Q. Now the only method whereby P can affect the ether is by its motion, whether of translation (including oscillation), rotation, or vibration. Now the motion of translation of two spherical atoms produces between them a force of a magnetic character, a sort of inverse magnetism if the translations or oscillations of the spheres are forced upon them, but a direct magnetism if they are free translations or oscillations of mutually reacting spheres. Hence the force of gravitation cannot have its origin in the translatory or oscillatory movement of the atoms. With regard to a motion of rotation it is difficult to see what effect the rotation of a sphere could have in a perfect fluid, if on the other hand we suppose the atoms ellipsoidal it will be found on calculating the kinetic energy of the fluid due to two ellipsoids rotating about their axes that no term occurs in it of the order $\frac{1}{\gamma}$, in other words two rotating ellipsoids do not mutually act on each other as if subject to a gravitation force. We are then thrown back on vibration or pulsation as the cause of so-called gravitation, and, as we have seen, the pulsation of spherical atoms does introduce terms corresponding to the ordinary gravitation potential. If two atoms be pulsating with periods *not mutually dependent*, we have seen that the force-function is given by

$$U = \frac{4\pi a_1^2 a_2^2}{\gamma} \phi_0 \phi_0' + \frac{\pi a_1^3 a_2^3}{\gamma^4} (a_1 \phi_0^2 + a_2 \phi_0'^2) + \text{etc.}$$

Now if the atoms be at sensible distances, the second term may be neglected as compared with the first. *The first will be all important, at the same time the gravitation term will only be the first of a long series, the rest of which may be safely neglected as insensible.*

52. We see then that two spheres with pulsation velocities ϕ_0 and ϕ_0' have a potential given by

$$V = \frac{4\pi a_1^2 a_2^2 \phi_0 \phi_0'}{\gamma}.$$

Suppose ϕ_0 and ϕ_0' to have terms

$$m_1 \beta_1 \cos(m_1 t + \alpha_1) \text{ and } m_2 \beta_2 \cos(m_2 t + \alpha_2),$$

then the potential will vary with the time, unless $\frac{2\pi}{m_1}$, $\frac{2\pi}{m_2}$, the pulsation periods, be so small that we may take the mean of ϕ_0 and ϕ_0'. We then find as a first condition for a gravitation potential that two atoms must be vibrating in the same period, for if not the mean of $\cos(m_1 t + \alpha_1) \cos(m_2 t + \alpha_2) = 0$.

Furthermore, in order that the force may always be *attractive*, it is necessary that the two terms should have the same phase, or β_1, β_2 being taken positive, that $\alpha_1 = \alpha_2$. (See however Arts. 44 and 45.) In other words we are led to the following conclusion: *Either every atom has one pulsation at least of common period and phase with every other, or else the law of gravitation does not hold for all atoms.*

Digitized by Google

53. Let us consider the first case. If we suppose that all atoms are not alike, we are naturally led to the conclusion that their free vibrations are different; or even if we were to suppose all atoms alike or all their free vibrations alike, then in the state of combination these periods would be changed, and every atom would thus cease to necessarily attract every other atom. We must then suppose a sort of 'gravitation beat' which is distinct from the free pulsation of the atoms. Each atom, besides its free pulsation and pulsations arising from combination, must be looked upon as pulsating in a definite unchangeable manner, amplitude, period and phase being unalterable by any mutual action of the atoms. Let $m\beta_1 \cos(mt+\alpha)$ represent this pulsation for one atom, $m\beta_2 \cos(mt+\alpha)$ for another, then the force-function between the two atoms will be

$$\frac{2\pi a_1^3 a_2^3 m^2 \beta_1 \beta_2}{\gamma}.$$

If we take $$\rho_1 = \frac{3\beta_1}{2\sqrt{2}a_1}, \quad \rho_2 = \frac{3\beta_2}{2\sqrt{2}a_2}, \text{ and } m^2 = \mu,$$

we may write the above $$U = \frac{\mu \cdot (\frac{4}{3}\pi a_1^3 \rho_1)(\frac{4}{3}\pi a_2^3 \rho_2)}{\gamma},$$

and might define μ as the constant of attraction and ρ_1 and ρ_2 as the densities of the atoms, where of course the densities of the atoms are not the densities of any substances formed of them, but if n be the number of atoms in a volume p times the volume of a single atom the density of the substance would be equal to $\frac{n\rho}{p}$.

Such a 'gravitation beat' common to all atoms might perhaps be called upon to explain the phenomenon of gravitation. Unfortunately it does not seem in harmony with our previous investigations. It is inconsistent with the alteration of the period of an atom owing to the influence of other atoms in its neighbourhood. Besides which it is difficult to conceive what supply of energy an atom could contain which would enable it to continue this 'gravitation beat' notwithstanding that the pulsations peculiar to its position with regard to neighbouring molecules, etc., were something very different. If we reject the notion of a gravitation beat, we seem thrust on a conclusion which seems to deny the principle of gravitation itself, namely, that it is not always true for all bodies. The result would appear to many so startling as to drive them back to the conception of 'inherent' force. Let us endeavour to mark the results to which our theory leads us when we endeavour to explain gravitation by it.

54. *First.* Two different atoms freely pulsating do not gravitate towards each other. If two atoms are capable of influencing each other's pulsations, there is a force between them varying as the inverse cube and not as the inverse square, but such influence only takes place at completely insensible distances. Two like atoms freely pulsating would attract each other according to the inverse square, granting that they are in the same phase as we have assumed above. Their attraction however would depend upon the amplitudes of their free pulsations. [It must be remembered throughout these remarks on gravitation forces, that they are of a totally different character and degree to chemical forces; of the same character, but of a totally different degree to molecular forces. In

Digitized by Google

other words the molecular force (or cohesive force) between *two* molecules is often sensible, the gravitating is always utterly insensible; only when billions and billions of molecules are attracting a like number does gravitation become a sensible force.]

Secondly. Suppose the molecules of a gas to be vibrating freely, that is, suppose the gas at such a temperature that the atoms of one molecule are not producing pulsations in another, then the molecules of such a gas will exert no gravitating force on the molecules of another gas freely vibrating unless some one or other of the free vibrations of these two sets of molecules have a common period. Hence, if we could be sure that the molecules of two gases had no common period, the following might theoretically be adopted as an experiment to shew their respective molecules had no attraction for each other. Let W be the internal work done by the first gas in expanding to double its previous volume V to $2V$. Let W' be the amount of work done when the second gas expands from V to $2V$, then if the two gases be mixed (the same amounts as above) in a volume V, the work done by this mixture of gases in expanding to $2V$ ought to equal $W+W'$. If however the molecules of one gas do attract those of the other, then the amount will be greater than $W+W'$, for the molecules of each gas have not only to overcome resistance in being separated from like molecules, but also from molecules of the other gas.

Thirdly. Let us consider the case of a solid or liquid. Since such give continuous spectra it is obvious that there is a force varying as the inverse square between every solid or liquid and every other solid or liquid; or again between a solid or liquid and a gas. We obtain however the apparently startling result,—that if the solid in its disassociated state had no period common with a certain gas, it would upon being liquefied, then rendered gaseous and finally absolutely disassociated, cease to exert any force upon this gas!

55. Since, however small a portion of a solid we take, we always get a continuous spectrum, we must suppose in any the smallest element of volume of a solid, wherever we take that element, the like number of atoms having the same period. For all these atoms, γ, the distance from any atom not at atomic or molecular distance may be supposed the same. Hence for every little element of a solid we may write a term in the force-function between it and any atom (a_s) of the form

$$U=-\frac{4\pi a_s^2\dot{\phi}_s\Sigma n_r C_r \sin(n_r t+\alpha_r)}{\gamma},$$

where n_r is to be given a continuous range of values.

Now the atom ϕ_s may be either freely pulsating or be under the influence of neighbouring atoms. In the latter case we may write

$$\dot{\phi}_s=-n_s B_s \sin(n_s t+\beta_s)+\frac{\Sigma B_p Q_{sp}\nu_s^2\nu_p^2}{\tau_s(\nu_p^2-\nu_s^2)}n_p \sin(n_p t+\beta_p).$$

The all-important term here is the first, as involving no Q factor. It is that which corresponds to the free pulsation of the atom with altered period. Hence in the summation in U we must take $n_r=n_s$, taking the mean we find for U the quantity

$$\frac{2\pi a_s^2 n_s^2 B_s C_s \cos(\alpha_s-\beta_s)}{\gamma}.$$

Digitized by Google

Now this will not give rise to an attractive force unless $\alpha_s - \beta_s$ be $< \frac{\pi}{2}$. Let us examine what this means. Precisely as in the case of molecular forces, we note that the term $B_s \cos(n_s t + \beta_s)$ in ϕ_s has arisen from a term $A_s \cos(\nu_s t + \kappa_s)$ or the free pulsation of the atom. Here B_s will be very nearly equal to A_s, and

$$\frac{1}{n_s^2} = \frac{1}{\nu_s^2} - \Sigma \frac{Q_{rs}^2}{\tau_r \tau_s} \frac{\nu_r^2 \nu_s^2}{\nu_r^2 - \nu_s^2}.$$

So that n_s does not differ so very much from ν_s, only in fact by terms of the order Q^2. The like holds for β_s and κ_s, so that as γ increases β_s equals more and more nearly κ_s.

Now the argument $n_s t + \alpha_s$ in the pulsation of the atoms of an element of the solid must have arisen by alteration of the free periods of some set of atoms in the element, say among others perhaps of one with the argument $\nu_p t + \kappa_p$. Hence α_s can only differ from κ_p by a small quantity. It follows therefore that

$$\alpha_s - \beta_s = \kappa_s - \kappa_p + \text{ terms of small magnitude.}$$

If then $\alpha_s - \beta_s$ is to be less than $\frac{\pi}{2}$, we may ask whether it be not possible that $\kappa_s = \kappa_p$? Hitherto we have only assumed as necessary that all like atoms when pulsating freely should be in the same phase: that is, they appear to have begun pulsating at the same instant. We seem now led to the conclusion that: *All atoms in the universe of whatever kind appear to have begun pulsating at the same instant.* A result due we may suppose to some not yet fully grasped physical cause. With this assumption again we have the term of the force-function, (since $\cos(\alpha_s - \beta_s) = 1$ nearly)—

$$= \frac{2\pi a_s^2 n_s^2 B_s C_s}{\gamma}.$$

If we sum this for all atoms with a period $\frac{2\pi}{n_s}$ in an element of the second solid, we get an expression of the form $\frac{C_s' C_s}{\gamma}$.

Now C_s, C_s' are evidently proportional to the volumes $\delta v \delta v'$ of the elements of the solids we have taken. Again, so long as the two solids are pulsating through a continuous range of beats, we must sum for all values of s in that range. We may then write the force between two elements which give a continuous spectrum

$$= \frac{\Sigma D_s' D_s \delta v \delta v'}{\gamma^3}.$$

56. Now in order that this may be identical with the ordinary law of gravitation we must suppose $\Sigma D_s D_s'$ to break up into two factors, each dependent only on the constitution of one of the elements. These factors would then be defined as proportional to the 'densities' of the bodies. In order that this breaking up into factors should occur, it appears necessary to suppose that D_s and D_s' are independent of s, and functions only of the total number of atoms and their kinds in either respective element. In other words, that the attractive force between two elements so far as it is dependent upon common

Digitized by Google

pulsations of one period is precisely equal to that portion of the force which depends upon common pulsations of another period. In this case we might write the attractive force between two elements $= \frac{\mu\rho\rho'\delta v\delta v'}{\gamma^2}$,

where ρ would be defined as 'density.'

There seem however many objections to such a supposition as that:—the portions of the attractive force arising from the same period occurring in both elements are the same for all periods;—foremost among them being the fact that as one element changes its state, the number of atoms giving a definite period may grow less and vanish ultimately altogether. We are then led to reconsider the expression

$$\frac{\Sigma D_s' D_s \delta v \delta v'}{\gamma^2}.$$

If we were to retain the one element the same but vary the other, the first set of constants D_s would remain fixed, while the second D_s' would change. There would then be no inconsistency in defining $\Sigma D_s' D_s$ as proportional to the density of the one body which gave the set D_s'. The only result would be that the density of a body if measured by means of the attraction of another body would vary according to the attracting body we might select. For example, let the earth be taken as standard, then if we determine the relative densities of two bodies by means of their weights, these relative densities will be in the ratio of $\Sigma D_s' D_s$ to $\Sigma D_s'' D_s$. But if we proceed to determine them by means of the attraction of any other body with attraction-constants D_s''', their relative densities will be in the ratio of

$$\Sigma D_s' D_s''' \text{ to } \Sigma D_s'' D_s''',$$

and this by no means need be the same ratio as that given by their weights. It is probable that the attraction-constants for most solid bodies which have been taken as standards may have been nearly proportional, so that

$$\frac{D_r}{D_r'''} = \frac{D_s}{D_s'''} = \frac{D_p}{D_p'''} = \text{etc.},$$

and that the density of any given substance as found by different methods may be nearly the same.

Such a result however, seemingly affecting the very basis of physics, is perhaps sufficient to shew that these pulsations are not sufficient to explain the phenomenon of gravitation. On the other hand it may be noted what very different results have been obtained by different methods for the mean density of the earth, ranging from less than 5 to 6·6. Even in Baily's experiments the average mean density as obtained from a set of experiments with one kind of balls on the torsion-rod differed from that obtained from experiments with balls of a different substance.

57. We are not however compelled to consider the 'mass,' whatever that may mean, of a body as varying. All we must suppose is that the force of gravitation is not that of the product of the masses divided by the square of the distance. It is a certain function of the constitution of both bodies divided by the square of the distance. This function *may be* in the case of most solid bodies almost exactly equal to the product of what

Digitized by Google

we term their masses, but it is in itself a function of their relative atomic conditions, and can in certain cases be conceived as vanishing. For example, in the case of two gases whose molecules are vibrating freely and which have no common pulsation period.

58. We may remark at once that such a theory does not suppose it possible for a body to be without weight, because a solid body like the earth (or a liquid) with a continuous range of beats will attract all other bodies whatever may be their atomic condition. Nor is it difficult to understand how it is possible that change in the atomic condition may not affect the weight.

Let C_s be the attraction-coefficient of an element of the earth for the period $\frac{2\pi}{n_s}$. Then the attraction of that element of the earth on an atom (a_s) pulsating with that period is (as above, Art. 55):—

$$\frac{2\pi a_s^2 n_s^2 B_s C_s}{\gamma^2}.$$

Now suppose in some (say solid) state we had m equal atoms all influencing each other, then we should have to sum the quantity $n_s^2 B_s C_s$ for a practically continuous set of values of s. On the other hand, if the solid becomes completely disassociated we may suppose its m atoms to be pulsating with some period $2\pi/n_r$. In order that the weight may be the same we must suppose:—

$$\Sigma n_s^2 B_s C_s = m \,.\, n_r^2 B_r C_r,$$

or that

$$n_r^2 B_r C_r = \frac{\Sigma n_s^2 B_s C_s}{m},$$

or that the attraction on one atom pulsating freely is the mean of the attractions on all m atoms pulsating with mutually enforced periods. Now if we remember how n_s is formed from n_r, differing first above and then below, and the like holding probably for B_s, B_r and C_s, C_r, the above relation seems perfectly intelligible. If B_s, C_s, etc., vary as $\frac{1}{n_s^2}$, the condition would become

$$\frac{1}{n_r^2} = \frac{1}{m} \Sigma \frac{1}{n_s^2},$$

a relation proved before to be true. (Art. 21.)

Again, supposing the solid or liquid to pass into that state in which its molecules are freely vibrating, and that these molecules consisting of p different atoms are then disassociated. The weight of the p different atoms freely pulsating will be the same as that of the molecule if

$$\sum_1^p a_r^2 \nu_r^2 B_r C_r = \sum_1^p a_r^2 n_r^2 B_r' C_r',$$

where

$$\frac{1}{n_r^2} = \frac{1}{\nu_r^2} - \Sigma \,.\, \frac{Q_{rs}^2 \nu_r^2 \nu_s^2}{\nu_r^2 - \nu_s^2}.$$

Hence, if

$$B_r \propto \frac{1}{a_r^2 \nu_r^2} \text{ and } C_r \text{ as } \frac{1}{\nu_r^2},$$

$$B_r' \propto \frac{1}{a_r^2 n_s^2} \text{ and } C_r' \text{ as } \frac{1}{n_s^2},$$

we must have

$$\Sigma \frac{1}{\nu_r^2} = \Sigma \frac{1}{n_r^2}$$

which follows at once from the values of $\frac{1}{n_r^2}$, etc. (Art. 18.)

Digitized by Google

59. The above are only taken as hypothetical cases to shew how it might be possible that the weight of a body or a mixture of bodies might remain unaltered notwithstanding molecular or chemical changes. Such changes alter the period and perhaps the amplitude of each atom's pulsation (B_r), but if this amplitude varies as $\frac{1}{a_r^2 n_r^2}$ they will not alter the 'weight.' We have supposed however that waves of like period to the pulsation affect its amplitude. In other words, if an atom pulsating freely were placed in a disturbance of the ether of its own period that the amplitude of its pulsation would be increased. The absorption of light and heat forces this conclusion upon us. In such a case our theory would compel us to believe that the gravitating power of the atom or its weight had been increased. It is possible that such increase of the amplitude may be *very minute*, even when a great number of atoms are put together, still, however minute, it is capable of presenting to our senses all the different stages which iron passes through, for example from an ordinary temperature to a white heat, before it begins to change its atomic condition. Some trace, however slight, we should expect to find of this alteration in the weight. In the above theory a red-hot iron ball ought to be heavier than one at ordinary temperature. If no such trace, *however minute*, can be found, it would seem an additional argument against a theory of gravitation as due to pulsation of spherical atoms in the ether.

Pulsating spherical atoms as the basis of physical phenomena seem capable of throwing considerable light on chemical and molecular physics. We may have to look further for an explanation of the principle of gravitation, either in the shape of the atoms themselves, in their method of vibrating, or still more probably in the nature of the medium in which these atoms are vibrating. If the ether be not a 'perfect fluid,' if there be something of the nature of 'skin friction' between atom and ether, it is possible that a rotatory motion of the atoms may be the origin of gravitation. Be this as it may the explanation of gravitation even as of magnetism and electricity must be sought in the translatory, rotatory or vibratory motion of the ultimate atoms.

Appendix.

We have supposed each free atom to have in its pulsation only one period. The free period is given by the equation

$$\tau_1 - (4\pi a_1^2 + \lambda_1) n^2 = 0.$$

Now it is possible that τ_1 and λ_1 are functions of n, as for example if the atom be looked upon as a solid elastic sphere. In this case let

$$\tau_1 = \chi_1(n), \quad \lambda_1 = \psi_1(n) \cdot \chi_1(n) - 4\pi a_1^2,$$

then we have the following equation to determine the free periods of the atom:

$$1 - \psi_1(n) \cdot n^2 = 0.$$

Let ${}_1\nu_1$, ${}_2\nu_1$, ${}_3\nu_1$ ... etc. be the roots. The number of these roots will probably be infinite (as in the case of an elastic sphere). If then we suppose every gas or vapour of an element when completely disassociated to give only one bright line, we must suppose

Digitized by Google

only one of the roots to give a period which falls within the light spectrum, and the existence of the other roots will not be disproved by the phenomenon of single bright line spectra.

In the case of two atoms the equations which determine the alteration of their periods due to mutual influence are

$$\{\tau_1 - (\lambda_1 + 4\pi a_1^3)\, n^2\}\, C_1 - \frac{4\pi a_1^3 a_2^3}{\gamma} n^2 C_2 = 0,$$

$$\{\tau_2 - (\lambda_2 + 4\pi a_2^3)\, n^2\}\, C_2 - \frac{4\pi a_1^3 a_2^3}{\gamma} n^2 C_1 = 0,$$

or

$$\chi_1(n) \left\{\frac{1}{n^2} - \psi_1(n)\right\} C_1 - Q C_2 = 0,$$

$$\chi_2(n) \left\{\frac{1}{n^2} - \psi_2(n)\right\} C_2 - Q C_1 = 0;$$

or

$$\chi_1(n)\, \chi_2(n) \left\{\frac{1}{n^2} - \psi_1(n)\right\} \left\{\frac{1}{n^2} - \psi_2(n)\right\} = Q^2.$$

Now Q^2 is small, and hence $\frac{1}{n^2}$ cannot differ much from $\frac{1}{{}_1\nu_1^2}$, $\frac{1}{{}_2\nu_1^2}$, etc., or from $\frac{1}{{}_1\nu_2^2}$, $\frac{1}{{}_2\nu_2^2}$, etc.

Let

$$\frac{1}{n^2} = \frac{1}{{}_\nu\nu_1^2} - m;$$

therefore

$$\chi_1({}_\nu\nu_1)\, \chi_2({}_\nu\nu_1)\, m\, \{\tfrac{1}{2}\, {}_\nu\nu_1^2\, \psi_1'({}_\nu\nu_1) + 1\} \left\{-\frac{1}{{}_\nu\nu_1^2} + \psi_2({}_\nu\nu_1)\right\} = Q^2,$$

or

$$m = \frac{Q^2}{\chi_1({}_\nu\nu_1)\, \{\tfrac{1}{2}\, {}_\nu\nu_1^2\, \psi_1'({}_\nu\nu_1) + 1\} \cdot \chi_2({}_\nu\nu_1) \left\{-\frac{1}{{}_\nu\nu_1^2} + \psi_2({}_\nu\nu_1)\right\}}.$$

Now in the case when there was only one free period we found

$$m = \frac{Q^2}{\tau_1 \tau_2 \left(\frac{1}{\nu_2^2} - \frac{1}{\nu_1^2}\right)}.$$

Hence in our results we have merely to replace

$$\tau_1 \tau_2 \left(\frac{1}{\nu_2^2} - \frac{1}{\nu_1^2}\right)$$

by the expression $\chi_1({}_\nu\nu_1)\, \chi_2({}_\nu\nu_1)\, \{\tfrac{1}{2}\, {}_\nu\nu_1^2\, \psi_1'({}_\nu\nu_1) + 1\} \left\{-\frac{1}{{}_\nu\nu_1^2} + \psi_2({}_\nu\nu_1)\right\}$

where it occurs in the vibration of the first atom; and

$$\tau_1 \tau_2 \left(\frac{1}{\nu_1^2} - \frac{1}{\nu_2^2}\right)$$

by

$$\chi_1({}_\nu\nu_2)\, \chi_2({}_\nu\nu_2)\, \{\tfrac{1}{2}\, {}_\nu\nu_2^2\, \psi_2'({}_\nu\nu_2) + 1\} \left\{-\frac{1}{{}_\nu\nu_2^2} + \psi_2({}_\nu\nu_2)\right\}$$

where it occurs in the vibration of the second atom.

Digitized by Google

We find
$$C_2 = \frac{QC_1}{\chi_2(\nu_1) \cdot \left\{\frac{1}{\nu_1^2} - \psi_2(\nu_1)\right\}}.$$

We have then to sum for all values of ν_1, and we can at once write down the general expression for the pulsation in either atom.

The result may be at once extended to any number of atoms. It will be noted that it does not differ *in character* from the result in the case of only one free pulsation. About *every* free pulsation we may draw conclusions very similar to those we have formed concerning the single free pulsation.

In the case of an elastic sphere both τ_1 and λ_1 are *transcendental* functions of
$$na_1\epsilon_1,$$
where ϵ_1 is a certain function of the elasticity of the sphere. Hence there is an infinite number of roots corresponding to an infinite number of free pulsations.

If we suppose the free atoms to pulsate with only a single period or a definite number of free periods, we cannot suppose their nature to be akin to that of a solid elastic sphere.

Indeed such a supposition does not carry us further towards a final solution of the problem; we are compelled to suppose an elastic sphere composed of atoms, and we only get one step further back in what would then bear the aspect of an infinite chain of atoms of different orders. We naturally ask the question, whether it is possible to conceive a spherical portion of the ether in any way differentiated from the rest? The only method hitherto suggested by which atoms might be considered as differentiated portions of the ether is, so far as I am aware, the vortex ring of Sir William Thomson. It is not easy again to see how our spherical atom could be considered as built up of vortex rings or to be differentiated off from the rest of the ether by any kind of vortex motion. On the other hand, if it be not a difference of motion which distinguishes an atom from the rest of the ether, we are compelled to suppose two primary substances, ether-substance and atom-substance, or to start with a dualistic physical conception. It does not simplify matters to suppose that the one ether substance can have two different states, or that atoms are akin to steam bubbles in water, for the very notion of change of state seems to point to an atomic construction, and we should be explaining our atoms by means of an ether which would in itself be atomic.

March 11, 1883.

NOTE.

This paper was written before Mr Leahy's interesting memoir on the pulsation of spheres in an elastic medium had been published, and was printed before a copy of that paper reached my hands. It appears to me that to treat the ether as a 'perfect fluid,' however far such may be from the actual state of affairs, leads to not uninteresting or unsuggestive results in relation to chemical and molecular forces. There are of course

Digitized by Google

very considerable difficulties in imagining the propagation of light under such circumstances. If the 'jelly' theory of the ether be accepted with the additional assumption that it acts as a perfect fluid for displacement velocities incomparably less than those of light vibrations, it would not seem satisfactory to obtain atomic and molecular forces from pulsations whose periods we have identified with those of light waves. On the other hand it may be noted that Sir William Thomson (*Lectures on Molecular Dynamics*, pp. 277, 278) finds it probable that the vibrations of the molecules which produce light have a velocity commensurable with that of those translations which we consider in the kinetic theory of gases. Hence it would seem necessary for the same magnitude of velocity to treat the ether both as a perfect fluid and as an elastic solid. That the ether itself must by its constitution explain gravitation is a proposition not to be lightly rejected, nor is it satisfactory to suppose two media, the one carrying light vibrations being an elastic solid, and the other explaining gravitation and chemical forces being a perfect fluid. It is difficult to understand how these media could be superposed. If we, however, assume space capable of a varying curvature and explain rays of light by waves of space-distortion, we are able to fill space with a perfect fluid such as we require. An atom would then have to be treated as an element of given curvature capable of transferring its position in space, but of such a curvature that it could not be penetrated by the fluid medium. The vibrational motion of such an element would appear as normal pulsations at the atomic surface in the fluid medium and as waves of space-distortion (light-waves) in space—both being vibratory motions of the same period.

April, 1885.

Digitized by Google

VI. *Some Applications of Generalized Space-Coordinates to Differential Analysis:—Potentials and Isotropic Elasticity.* By J. LARMOR.

[Read *April* 27, 1885.]

1. THERE are two well-known methods of dealing with physical questions of continuous differential analysis.

In the first and more ordinary one the differential equations satisfied by the quantities involved are investigated directly from the relations and properties of the system; and their transformation from simple rectangular to curvilinear coordinates may be effected either by direct transformation of the quantities and their differential coefficients, or by an independent investigation with the new variables. This latter process is easy where only differential coefficients of the first order are concerned, for these obey the laws of all vector quantities such as forces; but when, as is usual, differential coefficients of the second order occur it involves the use of vector differentiation or some similar process, which is of a more complicated character.

In the second method, which was invented by Lagrange, and developed and applied by Gauss, Green, and others, the equations are expressed as the conditions that a certain quadratic function of the differential coefficients of the first order, integrated over the system, shall retain a stationary value when small variations are imposed on the variables. In statical questions this function is the potential energy per unit volume at the place, and its integral is the total potential energy of the system. When its value is known, the equations of the system are obtained at once by application of the Method of Variations. Now this quadratic function, being a purely scalar quantity, will usually be expressible in a form which does not closely connect it with any special system of coordinate directions; and in any case, it may be transferred without difficulty from one system of coordinates to another by means of the vector laws obeyed by the fluxions.

The second method of procedure is therefore an easy and straightforward one when it is wished to express the equations in terms of special systems of coordinates. The first method involves a detailed examination of the internal properties of the system, and is therefore well suited to the clear exposition of the relations of a system whose internal structure is known, and to their expression in terms of the more simple coordinates. And the two are of course complementary to each other.

The object here proposed is to illustrate the use of the second method by its general application to some problems of common occurrence.

Digitized by Google

2. The character of a system of space-coordinates is completely determined by the nature of the expression for the square of the distance between two neighbouring points in terms of the differentials of the coordinates of those points: this will be a quadratic function of the differentials, with coefficients which may or may not be functions of the coordinates themselves.

There is another function related to this, and in a manner reciprocal to it, viz. the expression for the square of the resultant force or flux at a point corresponding to a given potential function, in terms of the rates of variation of that function with respect to the coordinates.

These two expressions can be made the basis of the whole analysis, when the relations considered are of an isotropic character.

3. Suppose the position of a point to be expressed in terms of ξ, η, ζ, which are given functions of the rectangular Cartesian coordinates x, y, z by which it may be originally specified. These coordinates will retain constant values over the surfaces

$$\xi = \text{constant}, \qquad \eta = \text{constant}, \qquad \zeta = \text{constant} \quad \text{.....(1)},$$

respectively.

The square of the distance ds between two consecutive points ξ, η, ζ and $\xi + d\xi$, $\eta + d\eta$, $\zeta + d\zeta$, in the space will be given by

$$ds^2 = Ad\xi^2 + Bd\eta^2 + Cd\zeta^2 + 2Dd\eta d\zeta + 2Ed\zeta d\xi + 2Fd\xi d\eta \quad \text{.....(2)},$$

where A, B, C, D, E, F are constants or functions of ξ, η, ζ, which are determined by the nature of the coordinate system.

If the system is determined by a triple series of parallel planes, the coefficients will be constants; and if α, β, γ be the angles between the directions in which these planes intersect respectively, i.e. between the axes of coordinates, we have

$$ds^2 = d\xi^2 + d\eta^2 + d\zeta^2 + 2d\eta d\zeta \cos\alpha + 2d\zeta d\xi \cos\beta + 2d\xi d\eta \cos\gamma.$$

If the system is determined by a triple series of orthogonal surfaces, D, E, F are zero, and, in Lamé's notation,

$$\left.\begin{aligned} A &= \frac{1}{h_1^2}, \text{ where } h_1^2 = \left(\frac{d\xi}{dx}\right)^2 + \left(\frac{d\xi}{dy}\right)^2 + \left(\frac{d\xi}{dz}\right)^2 \\ B &= \frac{1}{h_2^2}, \text{ where } h_2^2 = \left(\frac{d\eta}{dx}\right)^2 + \left(\frac{d\eta}{dy}\right)^2 + \left(\frac{d\eta}{dz}\right)^2 \\ C &= \frac{1}{h_3^2}, \text{ where } h_3^2 = \left(\frac{d\zeta}{dx}\right)^2 + \left(\frac{d\zeta}{dy}\right)^2 + \left(\frac{d\zeta}{dz}\right)^2 \end{aligned}\right\} \quad \text{.....(3)},$$

so that

$$ds^2 = \frac{d\xi^2}{h_1^2} + \frac{d\eta^2}{h_2^2} + \frac{d\zeta^2}{h_3^2} \quad \text{.....(4)}.$$

As examples, there are the common cases of rectangular, polar, and ellipsoidal coordinates.

Digitized by Google

If the system is determined by the triple series represented by (1), whose curves of intersection, drawn through the point ξ, η, ζ cut at angles α, β, γ, respectively, and if h_1, h_2, h_3 are defined as above, then

$$ds^2 = \frac{d\xi^2}{h_1^2} + \frac{d\eta^2}{h_2^2} + \frac{d\zeta^2}{h_3^2} + 2\frac{d\eta d\zeta}{h_2 h_3}\cos\alpha + 2\frac{d\zeta d\xi}{h_3 h_1}\cos\beta + 2\frac{d\xi d\eta}{h_1 h_2}\cos\gamma \quad\text{.....(5).}$$

4. The properties of the coordinate system ξ, η, ζ will in every case be completely specified if the coefficients of the expression in (2) are known.

Comparing it with (5), we find that the element of volume is

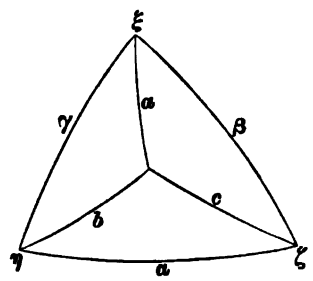
$$\begin{vmatrix} A & F & E \\ F & B & D \\ E & D & C \end{vmatrix}^{\frac{1}{2}} d\xi d\eta d\zeta \quad\text{.....(6),}$$

and the elements of area, on each of the coordinate surfaces $\xi =$ constant, $\eta =$ constant, $\zeta =$ constant, are respectively

$$\begin{vmatrix} B & D \\ D & C \end{vmatrix}^{\frac{1}{2}} d\eta d\zeta, \quad \begin{vmatrix} C & E \\ E & A \end{vmatrix}^{\frac{1}{2}} d\zeta d\xi, \quad \begin{vmatrix} A & F \\ F & B \end{vmatrix}^{\frac{1}{2}} d\xi d\eta \quad\text{.....(7).}$$

5. Suppose the potential function V to be expressed in terms of the coordinates ξ, η, ζ, and it is required to find the expression for the resultant force or flux at the point ξ, η, ζ in terms of V.

Let the direction of this resultant make angles a, b, c respectively with the lines of intersection of the coordinate surfaces at the point ξ, η, ζ; these angles will be related to α, β, γ, as in the annexed scheme supposed drawn on a spherical surface.



If R represent the magnitude of the required resultant, we shall have, by the fundamental property of the potential,

$$\left.\begin{aligned} h_1\frac{dV}{d\xi} &= R\cos a, \\ h_2\frac{dV}{d\eta} &= R\cos b \\ h_3\frac{dV}{d\zeta} &= R\cos c. \end{aligned}\right\} \quad\text{.....(8),}$$

But the relation between the mutual distances of four points on a sphere gives

$$\begin{vmatrix} 1 & \cos\gamma & \cos\beta & \cos a \\ \cos\gamma & 1 & \cos\alpha & \cos b \\ \cos\beta & \cos\alpha & 1 & \cos c \\ \cos a & \cos b & \cos c & 1 \end{vmatrix} = 0 \quad\text{.....(9).}$$

Digitized by Google

Therefore

$$\begin{vmatrix} 1 & \cos\gamma & \cos\beta & h_1\dfrac{dV}{d\xi} \\ \cos\gamma & 1 & \cos\alpha & h_2\dfrac{dV}{d\eta} \\ \cos\beta & \cos\alpha & 1 & h_3\dfrac{dV}{d\zeta} \\ h_1\dfrac{dV}{d\xi} & h_2\dfrac{dV}{d\eta} & h_3\dfrac{dV}{d\zeta} & R^2 \end{vmatrix} = 0 \quad\text{.....(10)},$$

or, what is the same,

$$\begin{aligned}(1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma)\,R^2 \\ = \sin^2\alpha\,.\,h_1^2\left(\frac{dV}{d\xi}\right)^2+\sin^2\beta\,.\,h_2^2\left(\frac{dV}{d\eta}\right)^2+\sin^2\gamma\,.\,h_3^2\left(\frac{dV}{d\zeta}\right)^2 \\ +2\,(\cos\beta\cos\gamma-\cos\alpha)\,h_2h_3\frac{dV}{d\eta}\frac{dV}{d\zeta}+2\,(\cos\gamma\cos\alpha-\cos\beta)\,h_3h_1\frac{dV}{d\zeta}\frac{dV}{d\xi} \\ +2\,(\cos\alpha\cos\beta-\cos\gamma)\,h_1h_2\frac{dV}{d\xi}\frac{dV}{d\eta} \quad\text{.....(11)},\end{aligned}$$

the expression for R^2 which was required.

In particular, if the coordinates form an orthogonal system,

$$R^2 = h_1^2\left(\frac{dV}{d\xi}\right)^2+h_2^2\left(\frac{dV}{d\eta}\right)^2+h_3^2\left(\frac{dV}{d\zeta}\right)^2 \quad\text{.....(12)},$$

as obviously should be true.

6. To complete the expressions (10) and (11), we should express $\cos\alpha$, $\cos\beta$, $\cos\gamma$ in terms of ξ, η, ζ. This is easily done, and it will be found—if we represent

the array $\left\|\begin{matrix} \dfrac{d\eta}{dx} & \dfrac{d\eta}{dy} & \dfrac{d\eta}{dz} \\ \dfrac{d\zeta}{dx} & \dfrac{d\zeta}{dy} & \dfrac{d\zeta}{dz} \end{matrix}\right|$ by $\dfrac{d(\eta,\ \zeta)}{d(x,\ y,\ z)}$,

and the array $\left\|\begin{matrix} \dfrac{d\xi}{dx} & \dfrac{d\xi}{dy} & \dfrac{d\xi}{dz} \end{matrix}\right|$ by $\dfrac{d(\xi)}{d(x,\ y,\ z)}$,

with similar notations for other cases (as already employed by Prof. Cayley)—that

$$\cos^2\gamma = \frac{\left\{\dfrac{d(\eta,\ \zeta)}{d(x,\ y,\ z)}\cdot\dfrac{d(\zeta,\ \xi)}{d(x,\ y,\ z)}\right\}^2}{\left[h_2^2h_3^2-\left\{\dfrac{d(\eta)}{d(x,\ y,\ z)}\cdot\dfrac{d(\zeta)}{d(x,\ y,\ z)}\right\}^2\right]\left[h_3^2h_1^2-\left\{\dfrac{d(\zeta)}{d(x,\ y,\ z)}\cdot\dfrac{d(\xi)}{d(x,\ y,\ z)}\right\}^2\right]} \quad\text{.....(13)},$$

with corresponding expressions for $\cos^2\alpha$ and $\cos^2\beta$.

If, however, we adopt (2) as the fundamental relation, we have the simple expressions of form

$$\cos\gamma = h_1h_2F = \left(\frac{F^2}{AB}\right)^{\frac{1}{2}} \quad\text{.....(14)},$$

as in the next section.

Digitized by Google

7. The reciprocal relation between these two fundamental scalar formulae (2), (10), will be clearly brought out by the following purely analytical method of deducing the second from the first:—

If ds denote an element of length in any direction through the point ξ, η, ζ, we have

$$\frac{dV}{ds}=\frac{d\xi}{ds}\frac{dV}{d\xi}+\frac{d\eta}{ds}\frac{dV}{d\eta}+\frac{d\zeta}{ds}\frac{dV}{d\zeta}\quad\text{.....(15).}$$

The value of R is the maximum value that this expression can have subject to the necessary relation (2) which connects $d\xi$, $d\eta$, $d\zeta$ and ds, viz. to

$$\phi\equiv A\left(\frac{d\xi}{ds}\right)^2+B\left(\frac{d\eta}{ds}\right)^2+C\left(\frac{d\zeta}{ds}\right)^2+2D\frac{d\eta}{ds}\frac{d\zeta}{ds}+2E\frac{d\zeta}{ds}\frac{d\xi}{ds}+2F\frac{d\xi}{ds}\frac{d\eta}{ds}=1\quad\text{.....(16).}$$

We are therefore to have $\delta\dfrac{dV}{ds}=0$ subject to the relation $\delta\phi=0$; and, proceeding by Lagrange's method of undetermined multipliers, we obtain

$$\left.\begin{aligned}
&A\frac{d\xi}{ds}+F\frac{d\eta}{ds}+E\frac{d\zeta}{ds}+\lambda\frac{dV}{d\xi}=0\\
&F\frac{d\xi}{ds}+B\frac{d\eta}{ds}+D\frac{d\zeta}{ds}+\lambda\frac{dV}{d\eta}=0\\
&E\frac{d\xi}{ds}+D\frac{d\zeta}{ds}+C\frac{d\zeta}{ds}+\lambda\frac{dV}{d\zeta}=0\\
\text{together with}\quad&\frac{dV}{d\xi}\frac{d\xi}{ds}+\frac{dV}{d\eta}\frac{d\eta}{ds}+\frac{dV}{d\zeta}\frac{d\zeta}{ds}-R=0
\end{aligned}\right\}\quad\text{.....(17).}$$

From the first three equations, we find immediately

$$1+\lambda R=0\quad\text{.....(18);}$$

and then, eliminating the unknowns $\dfrac{d\xi}{ds}$, $\dfrac{d\eta}{ds}$, $\dfrac{d\zeta}{ds}$, we have for R^2 the equation

$$\begin{vmatrix}
A & F & E & \dfrac{dV}{d\xi}\\
F & B & D & \dfrac{dV}{d\eta}\\
E & D & C & \dfrac{dV}{d\zeta}\\
\dfrac{dV}{d\xi} & \dfrac{dV}{d\eta} & \dfrac{dV}{d\zeta} & R^2
\end{vmatrix}=0\quad\text{.....(19),}$$

or

$$\begin{vmatrix} A & F & E\\ F & B & D\\ E & D & C\end{vmatrix}R^2=(BC-D^2)\left(\frac{dV}{d\xi}\right)^2+(CA-E^2)\left(\frac{dV}{d\eta}\right)^2+(AB-F^2)\left(\frac{dV}{d\zeta}\right)^2$$
$$+2(EF-AD)\frac{dV}{d\eta}\frac{dV}{d\zeta}+2(FD-BE)\frac{dV}{d\zeta}\frac{dV}{d\xi}$$
$$+2(DE-CF)\frac{dV}{d\xi}\frac{dV}{d\eta}\quad\text{.....(20),}$$

which agrees with (11).

Digitized by Google

This equation is also clearly that which determines an element of area in terms of its projections on the three coordinate surfaces at the place where it is situated.

I. 8. We now proceed to the expression of Laplace and Poisson's equation in terms of the coordinates ξ, η, ζ.

The potential energy of a system of attracting masses is well-known to be

$$U=\frac{1}{8\pi}\int R^2 . d\,\text{vol.} \quad \text{.....(21)},$$

extended throughout all space, where R is the expression for the resultant force. If we take Cartesian coordinates x, y, z

$$U=\frac{1}{8\pi}\iiint\left\{\left(\frac{dV}{dx}\right)^2+\left(\frac{dV}{dy}\right)^2+\left(\frac{dV}{dz}\right)^2\right\}dx\,dy\,dz \quad \text{.....(22)},$$

and the ordinary method of variation gives

$$\delta U=\frac{1}{4\pi}\iiint\left(\frac{dV}{dx}\frac{d\delta V}{dx}+\frac{dV}{dy}\frac{d\delta V}{dy}+\frac{dV}{dz}\frac{d\delta V}{dz}\right)dx\,dy\,dz$$

$$=-\frac{1}{4\pi}\int\left(\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}\right)\delta V . d\,\text{vol.} \quad \text{.....(23)},$$

on integration by parts; for the terms at the limits vanish, the attracting bodies being supposed to lie at finite distances. If we take generalized coordinates, R^2 is given by (10) or (11). Using the former expression, and, writing for shortness,

$$\Theta \text{ for } 1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma, \quad \text{.....(24)}$$

we have

$$U=\frac{1}{8\pi}\iiint\Theta^{-1}\left[\sin^2\alpha . h_1^2\left(\frac{dV}{d\xi}\right)^2+\sin^2\beta . h_2^2\left(\frac{dV}{d\eta}\right)^2+\sin^2\gamma . h_3^2\left(\frac{dV}{d\zeta}\right)^2\right.$$

$$+2(\cos\beta\cos\gamma-\cos\alpha)h_2h_3\frac{dV}{d\eta}\frac{dV}{d\zeta}+2(\cos\gamma\cos\alpha-\cos\beta)h_3h_1\frac{dV}{d\zeta}\frac{dV}{d\xi}$$

$$\left.+2(\cos\alpha\cos\beta-\cos\gamma)h_1h_2\frac{dV}{d\xi}\frac{dV}{d\eta}\right] . \Theta^{\frac{1}{2}}\frac{d\xi}{h_1}\frac{d\eta}{h_2}\frac{d\zeta}{h_3} \quad \text{.....(25)},$$

therefore

$$\delta U=\frac{1}{4\pi}\iiint\left[\Theta^{-\frac{1}{2}}\left\{\frac{h_1}{h_2h_3}\sin^2\alpha . \frac{dV}{d\xi}+\frac{1}{h_3}(\cos\alpha\cos\beta-\cos\gamma)\frac{dV}{d\eta}+\frac{1}{h_2}(\cos\gamma\cos\alpha-\cos\beta)\frac{dV}{d\zeta}\right\}\frac{d\delta V}{d\xi}\right.$$

$$+\Theta^{-\frac{1}{2}}\left\{\frac{1}{h_3}(\cos\alpha\cos\beta-\cos\gamma)\frac{dV}{d\xi}+\frac{h_2}{h_3h_1}\sin^2\beta . \frac{dV}{d\eta}+\frac{1}{h_1}(\cos\beta\cos\gamma-\cos\alpha)\frac{dV}{d\zeta}\right\}\frac{d\delta V}{d\eta}$$

$$+\Theta^{-\frac{1}{2}}\left\{\frac{1}{h_2}(\cos\gamma\cos\alpha-\cos\beta)\frac{dV}{d\xi}+\frac{1}{h_1}(\cos\beta\cos\gamma-\cos\alpha)\frac{dV}{d\eta}\right.$$

$$\left.\left.+\frac{h_3}{h_1h_2}\sin^2\gamma . \frac{dV}{d\zeta}\right\}\frac{d\delta V}{d\zeta}\right]d\xi\,d\eta\,d\zeta \quad \text{.....(26)}$$

$$=-\frac{1}{4\pi}\int\left[\frac{d}{d\xi}(\Theta^{-\frac{1}{2}}\{\ldots\})+\frac{d}{d\eta}(\Theta^{-\frac{1}{2}}\{\ldots\})+\frac{d}{d\zeta}(\Theta^{-\frac{1}{2}}\{\ldots\})\right]\Theta^{-\frac{1}{2}}h_1h_2h_3\,\delta V d\,\text{vol.} \quad \text{....(27)},$$

on integration by parts as before.

Digitized by Google

Comparing this result with (23), with which it must be identical, we see that

$$\nabla^2 V \text{ or } \frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}$$

is equivalent to

$$\Theta^{-\frac{1}{2}} h_1 h_2 h_3 \left[\frac{d}{d\xi}(\Theta^{-\frac{1}{2}}\{\ldots\})+\frac{d}{d\eta}(\Theta^{-\frac{1}{2}}\{\ldots\})+\frac{d}{d\zeta}(\Theta^{-\frac{1}{2}}\{\ldots\})\right] \ldots\ldots\ldots\ldots\ldots\ldots(28),$$

where $\{\ldots\}$ denotes the respective expressions within the $\{\}$ in (26).

The equation $\nabla^2 V = -4\pi\rho$ is thus transformed to the new coordinates; and the result agrees and may be compared with that obtained by the method of flux in Thomson and Tait's *Natural Philosophy*, 2 Edn., Appendix A_0.

If the coordinates form a rectangular system, we obtain Lamé's equation

$$h_1 h_2 h_3 \left[\frac{d}{d\xi}\left(\frac{h_1}{h_2 h_3}\frac{dV}{d\xi}\right)+\frac{d}{d\eta}\left(\frac{h_2}{h_3 h_1}\frac{dV}{d\eta}\right)+\frac{d}{d\zeta}\left(\frac{h_3}{h_1 h_2}\frac{dV}{d\zeta}\right)\right] = -4\pi\rho \ldots\ldots\ldots\ldots(29).$$

9. If in the general case (28) ξ, η, ζ are potential functions, i.e. are such that they satisfy

$$\nabla^2\xi = 0, \quad \nabla^2\eta = 0, \quad \nabla^2\zeta = 0, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(30)$$

throughout the space considered, then ξ, η, or ζ substituted for V in (28) makes the expression vanish; and by virtue of this simplification the expression for $\nabla^2 V$ reduces to

$$\frac{h_1 h_2 h_3}{\Theta}\cdot\left[\frac{h_1}{h_2 h_3}\sin^2\alpha\cdot\frac{d^2V}{d\xi^2}+\frac{h_2}{h_3 h_1}\sin^2\beta\cdot\frac{d^2V}{d\eta^2}+\frac{h_3}{h_1 h_2}\sin^2\gamma\cdot\frac{d^2V}{d\zeta^2}\right.$$
$$+\frac{2}{h_1}(\cos\beta\cos\gamma-\cos\alpha)\frac{d^2V}{d\eta d\zeta}+\frac{2}{h_2}(\cos\gamma\cos\alpha-\cos\beta)\frac{d^2V}{d\zeta d\xi}$$
$$\left.+\frac{2}{h_3}(\cos\alpha\cos\beta-\cos\gamma)\frac{d^2V}{d\xi d\eta}\right],$$

i.e. to $\Theta^{-1}\left[h_1^2\sin^2\alpha\cdot\frac{d^2V}{d\xi^2}+h_2^2\sin^2\beta\cdot\frac{d^2V}{d\eta^2}+h_3^2\sin^2\gamma\cdot\frac{d^2V}{d\zeta^2}\right.$

$$+2h_2h_3(\cos\beta\cos\gamma-\cos\alpha)\frac{d^2V}{d\eta d\zeta}+2h_3h_1(\cos\gamma\cos\alpha-\cos\beta)\frac{d^2V}{d\zeta d\xi}$$
$$\left.+2h_1h_2(\cos\alpha\cos\beta-\cos\gamma)\frac{d^2V}{d\xi d\eta}\right] \ldots\ldots\ldots(31).$$

When the coordinates form a rectangular system, this reduces again to Lamé's result

$$\nabla^2 V \equiv h_1^2\frac{d^2V}{d\xi^2}+h_2^2\frac{d^2V}{d\eta^2}+h_3^2\frac{d^2V}{d\zeta^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(32).$$

The more general result (31) may be expressed by saying that if, with the conditions of this section,

$$\left.\begin{aligned} R^2 &= f\left(\frac{dV}{d\xi}, \frac{dV}{d\eta}, \frac{dV}{d\zeta}\right), \\ \text{then} \quad \nabla^2 V &= f\left(\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}\right)V \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(33).$$

Digitized by Google

It may be observed that it is always possible to choose a coordinate, say ξ, so that it shall preserve a constant value over any bounding surface, and shall satisfy the equation $\nabla^2\xi = 0$; for it has only to be taken proportional to the potential of a free electric distribution on the surface, supposed conducting, in free space,—or it may be in the presence of other charged bodies, but in that case there will be discontinuity at the places occupied by those charges, if they are in the part of the field considered.

10. If we employ the other form (2) for R^2, and denote the determinant

$$\begin{vmatrix} A & F & E \\ F & B & D \\ E & D & C \end{vmatrix} \text{ by } \vartheta \text{.....(34)},$$

we have at once

$$\begin{aligned} U &= \frac{1}{8\pi}\int R^2 \,.\, d\,\text{vol.} \\ &= \frac{1}{8\pi}\iiint \vartheta^{-\frac{1}{2}}\Big[(BC-D^2)\Big(\frac{dV}{d\xi}\Big)^2 + \ldots + \ldots \\ &\qquad + 2\,(EF-AD)\frac{dV}{d\eta}\frac{dV}{d\zeta} + \ldots + \ldots\Big]\, d\xi d\eta d\zeta \text{ .....(35)},\end{aligned}$$

and it follows as before that $\nabla^2 V$ is equal to

$$\vartheta^{-\frac{1}{2}}\left[\frac{d}{d\xi}\vartheta^{-\frac{1}{2}}\begin{vmatrix} \frac{dV}{d\xi} & \frac{dV}{d\eta} & \frac{dV}{d\zeta} \\ F & B & D \\ E & D & C \end{vmatrix} + \frac{d}{d\eta}\vartheta^{-\frac{1}{2}}\begin{vmatrix} A & F & E \\ \frac{dV}{d\xi} & \frac{dV}{d\eta} & \frac{dV}{d\zeta} \\ E & D & C \end{vmatrix} + \frac{d}{d\zeta}\vartheta^{-\frac{1}{2}}\begin{vmatrix} A & F & E \\ F & B & D \\ \frac{dV}{d\xi} & \frac{dV}{d\eta} & \frac{dV}{d\zeta} \end{vmatrix}\right] \text{.....(36)};$$

and corresponding simplifications may occur.

II. 11. We now proceed to apply a similar analysis to the dynamical theory of an isotropic elastic solid.

We have first to determine the quadratic expression for the energy of deformation per unit volume. Following Kirchhoff*, let us take an element of the solid, and let f, g, h denote its three principal elongations, and F, G, H the three tensions which act on it per unit area in the directions of those elongations; we may conveniently take the element to be a right solid with its edges along these directions. In any case F, G, H represent the complete system of forces to which the element is subjected from the action of the contiguous parts, for with this specification there are no shears. Assuming, as usual, the truth of Hooke's law for the displacements considered, we have, from the isotropic character of the solid, equations of the form

$$\left.\begin{aligned} F &= af + bg + bh \\ G &= bf + ag + bh \\ H &= bf + bg + ah \end{aligned}\right\} \text{.....(37)},$$

* Crelle's *Journal*, Bd. 40; *Gesammelte Abhandlungen*, p. 247.

Digitized by Google

where a, b are two constants which specify the elastic qualities of the body, viz.

$$\left.\begin{array}{l} a+2b \text{ is the modulus of compression,} \\ \frac{1}{2}(a-b) \text{ is the modulus of shear,} \\ a-\dfrac{2b^2}{a+b} \text{ is Young's modulus.} \end{array}\right\} \quad \text{.....(38).}$$

The potential energy of deformation per unit volume at the point considered is

$$\tfrac{1}{2}(Ff+Gg+Hh) \text{.....(39),}$$

and is therefore $\quad \tfrac{1}{2}a(f^2+g^2+h^2)+b(fg+gh+hf),$

or, say $\quad V \equiv \tfrac{1}{2}a(f+g+h)^2+(b-a)(fg+gh+hf)$ .....(40).

The equations of equilibrium will be obtained as the conditions that the variation of the total potential energy

$$\int V d\,\text{vol.} + \text{the part due to external forces}$$

shall be zero.

12. Suppose now the position in space of a point to be given by the generalized coordinates ϕ, χ, ψ.

Consider in the undisturbed solid a point ϕ, χ, ψ,

and a consecutive point $\quad \phi+\xi,\ \chi+\eta,\ \psi+\zeta;$

when the solid is deformed these points will assume new positions

$$\left.\begin{array}{l}\phi+u\\ \chi+v\\ \psi+w\end{array}\right\} \text{ and } \left.\begin{array}{l}\phi+u+\xi+\dfrac{du}{d\phi}\xi+\dfrac{du}{d\chi}\eta+\dfrac{du}{d\psi}\zeta\\ \chi+v+\eta+\dfrac{dv}{d\phi}\xi+\dfrac{dv}{d\chi}\eta+\dfrac{dv}{d\psi}\zeta\\ \psi+w+\zeta+\dfrac{dw}{d\phi}\xi+\dfrac{dw}{d\chi}\eta+\dfrac{dw}{d\psi}\zeta\end{array}\right\} \text{.....(41).}$$

The square of the distance between these points will therefore, by (2), be given by

$$\begin{aligned}\mathfrak{L}^2 = {} & A'\left(\xi+\frac{du}{d\phi}\xi+\frac{du}{d\chi}\eta+\frac{du}{d\psi}\zeta\right)^2+B'\left(\eta+\frac{dv}{d\phi}\xi+\frac{dv}{d\chi}\eta+\frac{dv}{d\psi}\zeta\right)^2 \\ & +C'\left(\zeta+\frac{dw}{d\phi}\xi+\frac{dw}{d\chi}\eta+\frac{dw}{d\psi}\zeta\right)^2+2D'\left(\eta+\frac{dv}{d\phi}\xi+\frac{dv}{d\chi}\eta+\frac{dv}{d\psi}\zeta\right)\left(\zeta+\frac{dw}{d\phi}\xi+\frac{dw}{d\chi}\eta+\frac{dw}{d\psi}\zeta\right) \\ & +2E'\left(\zeta+\frac{dw}{d\phi}\xi+\frac{dw}{d\chi}\eta+\frac{dw}{d\psi}\zeta\right)\left(\xi+\frac{du}{d\phi}\xi+\frac{du}{d\chi}\eta+\frac{du}{d\psi}\zeta\right) \\ & +2F'\left(\xi+\frac{du}{d\phi}\xi+\frac{du}{d\chi}\eta+\frac{du}{d\psi}\zeta\right)\left(\eta+\frac{dv}{d\phi}\xi+\frac{dv}{d\chi}\eta+\frac{dv}{d\psi}\zeta\right) \text{.....(42),}\end{aligned}$$

where A', B', C', D', E', F' are the values of A, B, C, D, E, F at the point

$$\phi+u,\ \chi+v,\ \psi+w,$$

Digitized by Google

consecutive to ϕ, χ, ψ, and therefore

$$\left.\begin{aligned} A' &= A + \frac{dA}{d\phi}u + \frac{dA}{d\chi}v + \frac{dA}{d\psi}w \\ &= A + \mathfrak{d}A, \text{ say,} \end{aligned}\right\} \quad \text{.....(43)},$$

with similar expressions for B', ... F'.

Therefore, neglecting squares of small quantities $\frac{du}{d\phi}$, $\frac{du}{d\psi}$, ...

$$\begin{aligned} \mathfrak{L}^2 = \mathfrak{L}_0^2 &+ 2\left(A\frac{du}{d\phi} + E\frac{dw}{d\phi} + F\frac{dv}{d\phi} + \tfrac{1}{2}\mathfrak{d}A\right)\xi^2 + 2\left(\dots\right)\eta^2 + 2\left(\dots\right)\zeta^2 \\ &+ 2\left\{B\frac{dv}{d\psi} + C\frac{dw}{d\chi} + D\left(\frac{dw}{d\psi} + \frac{dv}{d\chi}\right) + E\frac{du}{d\chi} + F\frac{du}{d\psi} + \mathfrak{d}D\right\}\eta\zeta \\ &+ 2\left\{\dots\right\}\zeta\xi + 2\left\{\dots\right\}\xi\eta \quad \text{.....(44)}, \end{aligned}$$

where $\mathfrak{L}_0$ is the value of $\mathfrak{L}$ before deformation.

Since $\mathfrak{L}$ and $\mathfrak{L}_0$ differ by a quantity very small compared with either of them (i.e. of the second order), we have

$$\tfrac{1}{2}(\mathfrak{L}^2 - \mathfrak{L}_0^2) = \mathfrak{L}_0^2\epsilon \quad \text{.....(45)},$$

where ϵ is the elongation, per unit length, of the solid, in the direction of $\mathfrak{L}$, at the point ϕ, χ, ψ.

We may therefore write (44) in the form

$$\mathfrak{L}_0^2\epsilon = \mathfrak{A}\xi^2 + \mathfrak{B}\eta^2 + \mathfrak{C}\zeta^2 + 2\mathfrak{D}\eta\zeta + 2\mathfrak{E}\zeta\xi + 2\mathfrak{F}\xi\eta \quad \text{.....(46)}.$$

13. The values of f, g, h are the maximum and minimum and the stationary maximum-minimum value of this quantity ϵ; the value of $\mathfrak{L}_0$ being given by (2), or, what is the same, (5), viz.

$$\mathfrak{L}_0^2 = A\xi^2 + B\eta^2 + C\zeta^2 + 2D\eta\zeta + 2E\zeta\xi + 2F\xi\eta.$$

These values may be determined in the ordinary manner. Substituting from (2) in (46), we have

$$\begin{aligned} (\mathfrak{A} - A\epsilon)\xi^2 + (\mathfrak{B} - B\epsilon)\eta^2 + (\mathfrak{C} - C\epsilon)\zeta^2 + 2(\mathfrak{D} - D\epsilon)\eta\zeta + 2(\mathfrak{E} - E\epsilon)\zeta\xi \\ + 2(\mathfrak{F} - F\epsilon)\xi\eta = 0 \quad \text{.....(47)}. \end{aligned}$$

Differentiating with respect to ξ, η, ζ, and remembering that for the values in question the differentials of ϵ are zero, we obtain

$$\left.\begin{aligned} (\mathfrak{A} - A\epsilon)\xi + (\mathfrak{F} - F\epsilon)\eta + (\mathfrak{E} - E\epsilon)\zeta = 0 \\ (\mathfrak{F} - F\epsilon)\xi + (\mathfrak{B} - B\epsilon)\eta + (\mathfrak{D} - D\epsilon)\zeta = 0 \\ (\mathfrak{E} - E\epsilon)\xi + (\mathfrak{D} - D\epsilon)\eta + (\mathfrak{C} - C\epsilon)\zeta = 0 \end{aligned}\right\} \quad \text{.....(48)},$$

which lead to the eliminant

$$\begin{vmatrix} \mathfrak{A} - A\epsilon & \mathfrak{F} - F\epsilon & \mathfrak{E} - E\epsilon \\ \mathfrak{F} - F\epsilon & \mathfrak{B} - B\epsilon & \mathfrak{D} - D\epsilon \\ \mathfrak{E} - E\epsilon & \mathfrak{D} - D\epsilon & \mathfrak{C} - C\epsilon \end{vmatrix} = 0 \quad \text{.....(49)}.$$

Digitized by Google

The values of f, g, h are therefore the roots of this cubic equation in ϵ. Therefore

$$(f+g+h)\begin{vmatrix} A & F & E \\ F & B & D \\ E & D & C \end{vmatrix} = \begin{vmatrix} A & F & \mathfrak{E} \\ F & B & \mathfrak{D} \\ E & D & \mathfrak{C} \end{vmatrix} + \begin{vmatrix} A & \mathfrak{F} & E \\ F & \mathfrak{B} & D \\ E & \mathfrak{D} & C \end{vmatrix} + \begin{vmatrix} \mathfrak{A} & F & E \\ \mathfrak{F} & B & D \\ \mathfrak{E} & D & C \end{vmatrix}$$

$$= \mathfrak{A}(BC-D^2) + \mathfrak{B}(CA-E^2) + \mathfrak{C}(AB-F^2)$$
$$+ 2\mathfrak{D}(EF-AD) + 2\mathfrak{E}(FD-BE) + 2\mathfrak{F}(DE-CF) \ldots (50)$$

$$-(fg+gh+hf)\begin{vmatrix} A & F & E \\ F & B & D \\ E & D & C \end{vmatrix} = \begin{vmatrix} \mathfrak{A} & \mathfrak{F} & E \\ \mathfrak{F} & \mathfrak{B} & D \\ \mathfrak{E} & \mathfrak{D} & C \end{vmatrix} + \begin{vmatrix} \mathfrak{A} & F & \mathfrak{E} \\ \mathfrak{F} & B & \mathfrak{D} \\ \mathfrak{E} & D & \mathfrak{C} \end{vmatrix} + \begin{vmatrix} A & \mathfrak{F} & \mathfrak{E} \\ F & \mathfrak{B} & \mathfrak{D} \\ E & \mathfrak{D} & \mathfrak{C} \end{vmatrix}$$

$$= (\mathfrak{B}\mathfrak{C}-\mathfrak{D}^2)A + (\mathfrak{C}\mathfrak{A}-\mathfrak{E}^2)B + (\mathfrak{A}\mathfrak{B}-\mathfrak{F}^2)C$$
$$+ 2(\mathfrak{E}\mathfrak{F}-\mathfrak{A}\mathfrak{D})D + 2(\mathfrak{F}\mathfrak{D}-\mathfrak{B}\mathfrak{E})E + 2(\mathfrak{D}\mathfrak{E}-\mathfrak{C}\mathfrak{F})F \ldots (51),$$

and the value of V as given by equation (40) may be written down.

14. Having thus determined the value of V, the equations of internal and surface equilibrium may be deduced by the Method of Variations in the ordinary manner.

If Φ, X, Ψ denote the components (oblique) of the surface force, per unit area, which acts on the boundary of the solid, taken in the directions of the lines of intersection of the pairs of surfaces $\chi\psi$, $\psi\phi$, and $\phi\chi$ respectively, and so as when positive to tend to increase the values of ϕ, χ, ψ; and if W denote the potential energy per unit volume at any point in the solid, due to the action from a distance of external systems, —then the energy condition of equilibrium is that

$$\delta U \equiv \delta \int V d\,\mathrm{vol.} + \delta \int W d\,\mathrm{vol.} - \int (\Phi A^{\frac{1}{2}}\,\delta u + \mathrm{X} B^{\frac{1}{2}}\,\delta v + \Psi C^{\frac{1}{2}}\,\delta w)\, d\,\mathrm{surface} = 0 \ \ldots\ldots (52)$$

for all possible consistent variations of u, v, w.

Therefore, with the notation of § 10, we have

$$\iiint \mathfrak{H}^{\frac{1}{2}}\,\delta(V+W)\, d\phi\, d\chi\, d\psi - \iint (\Phi A^{\frac{1}{2}}\,\delta u + \mathrm{X} B^{\frac{1}{2}}\,\delta v + \Psi C^{\frac{1}{2}}\,\delta w)\, dS = 0 \ \ldots\ldots\ldots\ldots (53).$$

Now V is a function (quadratic) of the differential coefficients of u, v, w with respect to ϕ, χ and ψ. If A, B, C, D, E, F are constants, which corresponds to the case of oblique Cartesian coordinates, V is a function of those differential coefficients alone, with constant coefficients. If A, B, ... F are not constants, the terms $\mathfrak{d}A$, $\mathfrak{d}B$, ... $\mathfrak{d}F$ will introduce u, v, w themselves; so that V is now a quadratic function of u, v, w and their differential coefficients with respect to ϕ, χ, ψ.

Thus we have

$$\delta V \equiv \frac{dV}{d\frac{du}{d\phi}}\frac{d\delta u}{d\phi} + \frac{dV}{d\frac{du}{d\chi}}\frac{d\delta u}{d\chi} + \ldots + \frac{dV}{d\frac{dv}{d\phi}}\frac{d\delta V}{d\phi} + \ldots$$
$$+ \frac{dV}{du}\delta u + \frac{dV}{dv}\delta v + \frac{dV}{dw}\delta w \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (54).$$

Digitized by Google

And on substituting this value in (53), and integrating by parts in the usual manner the terms which contain differential coefficients of δu, δv, δw, we have, writing down explicitly only the terms depending on δu,

$$\iint \vartheta^{\frac{1}{2}} \frac{dV}{d\dfrac{du}{d\phi}} \delta u \,.\, d\chi d\psi + \iint \vartheta^{\frac{1}{2}} \frac{dV}{d\dfrac{du}{d\chi}} \delta u \,.\, d\psi d\phi + \iint \vartheta^{\frac{1}{2}} \frac{dV}{d\dfrac{du}{d\psi}} \delta u \,.\, d\phi\, d\chi - \iint \Phi A^{\frac{1}{2}} \delta u dS$$

$$- \iiint \left\{ \frac{d}{d\phi}\left(\vartheta^{\frac{1}{2}} \frac{dV}{d\dfrac{du}{d\phi}} \right) + \frac{d}{d\chi}\left(\vartheta^{\frac{1}{2}} \frac{dV}{d\dfrac{du}{d\chi}} \right) + \frac{d}{d\psi}\left(\vartheta^{\frac{1}{2}} \frac{dV}{d\dfrac{du}{d\psi}} \right) - \vartheta^{\frac{1}{2}} \frac{dV}{du} - \vartheta^{\frac{1}{2}} \frac{dW}{d\phi} \right\} \delta u \,.\, d\phi d\chi d\psi$$

$$+ \text{ terms containing } \delta v \text{ and } \delta w = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (55).$$

Now as the forms of δu, δv, δw are quite at our disposal, subject to conditions of continuity, we can choose them so as to make either the volume integral or the surface integral in this equation zero at pleasure; and we may also have any two of these variations of u, v, w zero at pleasure. This equation therefore cannot be satisfied unless the quantity under each separate type of volume or surface integral is zero; and we thus obtain the internal and surface equations of equilibrium.

15. To express the latter in their simplest form, it will be convenient to take the solid considered to be the element of volume $d\phi d\chi d\psi$ of a larger body. The surface element dS is now an element of area of a face of this solid, and is, by § 4,

$$\left(\frac{d\vartheta}{dA}\right)^{\frac{1}{2}} d\chi d\psi, \quad \left(\frac{d\vartheta}{dB}\right)^{\frac{1}{2}} d\psi d\phi, \quad \text{or} \quad \left(\frac{d\vartheta}{dC}\right)^{\frac{1}{2}} d\phi d\chi$$

according to its position.

The force Φ of (52) is the force exerted across this element of area by the matter on the other side of it, per unit area, in a direction such as to increase ϕ without altering χ or ψ, i.e. in a direction parallel to the intersection of χ, ψ. If we adopt the notation that $T_{\phi\psi}$ represent the component force per unit area acting from outside on that face of the element which lies on the surface ϕ in a direction such as to increase ψ without altering the values of ϕ, χ, i.e. in a direction parallel to the intersection of ϕ, χ, we have therefore from the surface integrals the relations

$$\left.\begin{aligned} T_{\phi\psi} &= A^{-\frac{1}{2}} \left(\frac{d\vartheta}{dA}\right)^{-\frac{1}{2}} \vartheta^{\frac{1}{2}} \frac{dV}{d\dfrac{du}{d\phi}} \\ T_{\chi\phi} &= A^{-\frac{1}{2}} \left(\frac{d\vartheta}{dB}\right)^{-\frac{1}{2}} \vartheta^{\frac{1}{2}} \frac{dV}{d\dfrac{du}{d\chi}} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (56),$$

with similar expressions for the other surface-stresses.

Digitized by Google

From the volume integral in (55) we may now obtain the equations of internal equilibrium. We may shorten their expression by making use of the values of the stresses just obtained in (56).

In the first place, we remark that, by (43) and (44),

$$\frac{dV}{du} = \frac{1}{2}\frac{dA}{d\phi}\frac{dV}{d\mathfrak{V}A} + \frac{1}{2}\frac{dB}{d\phi}\frac{dV}{d\mathfrak{V}B} + \ldots + \frac{dF}{d\phi}\frac{dV}{d\mathfrak{V}F} \quad \ldots\ldots\ldots\ldots(57).$$

We have then

$$\left.\begin{aligned}
&\frac{d}{d\phi}\left\{A^{\frac{1}{2}}\left(\frac{d\mathfrak{V}}{dA}\right)^{\frac{1}{2}}T_{\phi\phi}\right\} + \frac{d}{d\chi}\left\{A^{\frac{1}{2}}\left(\frac{d\mathfrak{V}}{dB}\right)^{\frac{1}{2}}T_{\chi\phi}\right\} + \frac{d}{d\psi}\left\{A^{\frac{1}{2}}\left(\frac{d\mathfrak{V}}{dC}\right)^{\frac{1}{2}}T_{\psi\phi}\right\} - \mathfrak{V}^{\frac{1}{2}}\frac{dV}{du} - \mathfrak{V}^{\frac{1}{2}}\frac{dW}{d\phi} = 0 \\
&\text{with the similar equations} \\
&\frac{d}{d\phi}\left\{B^{\frac{1}{2}}\left(\frac{d\mathfrak{V}}{dA}\right)^{\frac{1}{2}}T_{\phi\chi}\right\} + \frac{d}{d\chi}\left\{B^{\frac{1}{2}}\left(\frac{d\mathfrak{V}}{dB}\right)^{\frac{1}{2}}T_{\chi\chi}\right\} + \frac{d}{d\psi}\left\{B^{\frac{1}{2}}\left(\frac{d\mathfrak{V}}{dC}\right)^{\frac{1}{2}}T_{\psi\chi}\right\} - \mathfrak{V}^{\frac{1}{2}}\frac{dV}{dv} - \mathfrak{V}^{\frac{1}{2}}\frac{dW}{d\chi} = 0 \\
&\frac{d}{d\phi}\left\{C^{\frac{1}{2}}\left(\frac{d\mathfrak{V}}{dA}\right)\ T_{\phi\psi}\right\} + \frac{d}{d\chi}\left\{C^{\frac{1}{2}}\left(\frac{d\mathfrak{V}}{dB}\right)^{\frac{1}{2}}T_{\chi\psi}\right\} + \frac{d}{d\psi}\left\{C^{\frac{1}{2}}\left(\frac{d\mathfrak{V}}{dC}\right)^{\frac{1}{2}}T_{\psi\psi}\right\} - \mathfrak{V}^{\frac{1}{2}}\frac{dV}{dw} - \mathfrak{V}^{\frac{1}{2}}\frac{dW}{d\psi} = 0
\end{aligned}\right\} \ldots(58),$$

the values of $\frac{dV}{du}$, $\frac{dV}{dv}$, $\frac{dV}{dw}$ being most easily determined by means of (57).

16. In the case in which the coordinate surfaces ϕ, χ, ψ form an orthogonal system, the formulae become more simple.

We have now

$$\left.\begin{aligned}
&D = 0, \qquad E = 0, \qquad F = 0, \\
&\text{and, if we wish to revert to Lamé's notation, we have} \\
&A = \frac{1}{h_1^2}, \qquad B = \frac{1}{h_2^2}, \qquad C = \frac{1}{h_3^2}.
\end{aligned}\right\} \ldots\ldots\ldots\ldots(59).$$

By (44),

$$\begin{aligned}
\mathfrak{L}_0^2 e = &\left(A\frac{du}{d\phi} + \tfrac{1}{2}\mathfrak{V}A\right)\xi^2 + \left(B\frac{dv}{d\chi} + \tfrac{1}{2}\mathfrak{V}B\right)\eta^2 + \left(C\frac{dw}{d\psi} + \tfrac{1}{2}\mathfrak{V}C\right)\zeta^2 \\
&+ \left(B\frac{dv}{d\psi} + C\frac{dw}{d\chi}\right)\eta\zeta + \left(C\frac{dw}{d\phi} + A\frac{du}{d\psi}\right)\zeta\xi + \left(A\frac{du}{d\chi} + B\frac{dv}{d\phi}\right)\xi\eta \ldots\ldots\ldots(60),
\end{aligned}$$

where $$\mathfrak{L}_0^2 = A\xi^2 + B\eta^2 + C\zeta^2.$$

The cubic equation whose roots are f, g, h is now

$$\begin{vmatrix} \mathfrak{A} - Ae & \mathfrak{F} & \mathfrak{E} \\ \mathfrak{F} & \mathfrak{B} - Be & \mathfrak{D} \\ \mathfrak{E} & \mathfrak{D} & \mathfrak{C} - Ce \end{vmatrix} = 0 \ldots\ldots\ldots\ldots(61),$$

and therefore

$$f + g + h = \frac{\mathfrak{A}}{A} + \frac{\mathfrak{B}}{B} + \frac{\mathfrak{C}}{C} \ldots\ldots\ldots\ldots(62),$$

$$fg + gh + hf = \frac{\mathfrak{B}\mathfrak{C} - \mathfrak{D}^2}{BC} + \frac{\mathfrak{C}\mathfrak{A} - \mathfrak{E}^2}{CA} + \frac{\mathfrak{A}\mathfrak{B} - \mathfrak{F}^2}{AB} \ldots\ldots\ldots\ldots(63).$$

Digitized by Google

We have now, by (40),

$$V = \tfrac{1}{2}a\left(\frac{\mathfrak{A}}{A}+\frac{\mathfrak{B}}{B}+\frac{\mathfrak{C}}{C}\right)^2 - (a-b)\left(\frac{\mathfrak{B}\mathfrak{C}-\mathfrak{D}^2}{BC}+\frac{\mathfrak{C}\mathfrak{A}-\mathfrak{E}^2}{CA}+\frac{\mathfrak{A}\mathfrak{B}-\mathfrak{F}^2}{AB}\right) \quad \text{.....(64).}$$

Therefore, by (60),

$$\begin{aligned}
V = {} & \tfrac{1}{2}a\left(\frac{du}{d\phi}+\frac{dv}{d\chi}+\frac{dw}{d\psi}+\frac{1}{2}\frac{\mathfrak{d}A}{A}+\frac{1}{2}\frac{\mathfrak{d}B}{B}+\frac{1}{2}\frac{\mathfrak{d}C}{C}\right)^2 \\
& -(a-b)\left[\frac{dv}{d\chi}\frac{dw}{d\psi}+\frac{dw}{d\psi}\frac{du}{d\phi}+\frac{du}{d\phi}\frac{dv}{d\chi}\right. \\
& -\tfrac{1}{2}\left(\frac{dv}{d\psi}\frac{dw}{d\chi}+\frac{dw}{d\phi}\frac{du}{d\psi}+\frac{du}{d\chi}\frac{dv}{d\phi}\right) \\
& +\tfrac{1}{2}\frac{du}{d\phi}\left(\frac{\mathfrak{d}B}{B}+\frac{\mathfrak{d}C}{C}\right)+\frac{1}{2}\frac{dv}{d\chi}\left(\frac{\mathfrak{d}C}{C}+\frac{\mathfrak{d}A}{A}\right)+\frac{1}{2}\frac{dw}{d\psi}\left(\frac{\mathfrak{d}A}{A}+\frac{\mathfrak{d}B}{B}\right) \\
& +\tfrac{1}{4}\left(\frac{\mathfrak{d}B}{B}\frac{\mathfrak{d}C}{C}+\frac{\mathfrak{d}C}{C}\frac{\mathfrak{d}A}{A}+\frac{\mathfrak{d}A}{A}\frac{\mathfrak{d}B}{B}\right) \\
& \left.-\tfrac{1}{4}\left\{\frac{A}{C}\left(\frac{du}{d\psi}\right)^2+\frac{A}{B}\left(\frac{du}{d\chi}\right)^2+\frac{B}{C}\left(\frac{dv}{d\psi}\right)^2+\frac{B}{A}\left(\frac{dv}{d\phi}\right)^2+\frac{C}{B}\left(\frac{dw}{d\chi}\right)^2+\frac{C}{A}\left(\frac{dw}{d\phi}\right)^2\right\}\right] \quad \text{.....(65),}
\end{aligned}$$

wherein for brevity, as before,

$$\left.\begin{aligned}
\mathfrak{d}A &\equiv \frac{dA}{d\phi}u+\frac{dA}{d\chi}v+\frac{dA}{d\psi}w \\
\mathfrak{d}B &= \frac{dB}{d\phi}u+\frac{dB}{d\chi}v+\frac{dB}{d\psi}w \\
\mathfrak{d}C &= \frac{dC}{d\phi}u+\frac{dC}{d\chi}v+\frac{dC}{d\psi}w
\end{aligned}\right\}.$$

We have thus obtained the expression in general orthogonal coordinates, for the potential energy of deformation of a strained isotropic solid.

17. By (56), we find

$$\left.\begin{aligned}
T_{\phi\phi} &= \frac{dV}{d\dfrac{du}{d\phi}} \\
&= a\left(\frac{du}{d\phi}+\frac{1}{2}\frac{\mathfrak{d}A}{A}\right)+b\left(\frac{dv}{d\chi}+\frac{1}{2}\frac{\mathfrak{d}B}{B}\right)+b\left(\frac{dw}{d\psi}+\frac{1}{2}\frac{\mathfrak{d}C}{C}\right) \\
T_{\chi\phi} &= B^{\frac{1}{2}}A^{-\frac{1}{2}}\frac{dV}{d\dfrac{du}{d\chi}} \\
&= \tfrac{1}{2}(a-b)\left\{A^{\frac{1}{2}}B^{-\frac{1}{2}}\frac{du}{d\chi}+B^{\frac{1}{2}}A^{-\frac{1}{2}}\frac{dv}{d\phi}\right\} \\
&= T_{\phi\chi}, \text{ by symmetry}
\end{aligned}\right\} \quad \text{.....(66);}$$

with similar expressions for the other stresses.

Digitized by Google

To find the equations of internal equilibrium, we first observe that the values of $\frac{dV}{du}$, $\frac{dV}{dv}$, $\frac{dV}{dw}$ are obtained in a simple form by (57). It is clear from the way in which $\mathfrak{d}A$, $\mathfrak{d}B$, $\mathfrak{d}C$ enter into (60) that

$$2\frac{dV}{du}=\frac{1}{A}\frac{dA}{d\phi}\frac{dV}{d\frac{du}{d\phi}}+\frac{1}{B}\frac{dB}{d\phi}\frac{dV}{d\frac{dv}{d\chi}}+\frac{1}{C}\frac{dC}{d\phi}\frac{dV}{d\frac{dw}{d\psi}}$$

$$=\frac{1}{A}\frac{dA}{d\phi}T_{\phi\phi}+\frac{1}{B}\frac{dB}{d\phi}T_{\chi\chi}+\frac{1}{C}\frac{dC}{d\phi}T_{\psi\psi}\quad\text{.....(67)},$$

with similar expressions for

$$\frac{dV}{dv}\text{ and }\frac{dV}{dw}.$$

Therefore we have the equations for this case

$$\left.\begin{aligned}
&\frac{d}{d\phi}\{(BCA)^{\frac12}T_{\phi\phi}\}+\frac{d}{d\chi}\{(CAA)^{\frac12}T_{\chi\phi}\}+\frac{d}{d\psi}\{(ABA)^{\frac12}T_{\psi\phi}\}\\
&\qquad-\tfrac12\left[\left(\frac{BC}{A}\right)^{\frac12}\frac{dA}{d\phi}T_{\phi\phi}+\left(\frac{CA}{B}\right)^{\frac12}\frac{dB}{d\phi}T_{\chi\chi}+\left(\frac{AB}{C}\right)^{\frac12}\frac{dC}{d\phi}T_{\psi\psi}\right]-(ABC)^{\frac12}\frac{dW}{d\phi}=0\\
&\frac{d}{d\phi}\{(BCB)^{\frac12}T_{\phi\chi}\}+\frac{d}{d\chi}\{(CAB)^{\frac12}T_{\chi\chi}\}+\frac{d}{d\psi}\{(ABB)^{\frac12}T_{\psi\chi}\}\\
&\qquad-\tfrac12\left[\left(\frac{BC}{A}\right)^{\frac12}\frac{dA}{d\chi}T_{\phi\phi}+\left(\frac{CA}{B}\right)^{\frac12}\frac{dB}{d\chi}T_{\chi\chi}+\left(\frac{AB}{C}\right)^{\frac12}\frac{dC}{d\chi}T_{\psi\psi}\right]-(ABC)^{\frac12}\frac{dW}{d\chi}=0\\
&\frac{d}{d\phi}\{(BCC)^{\frac12}T_{\phi\psi}\}+\frac{d}{d\chi}\{(CAC)^{\frac12}T_{\chi\psi}\}+\frac{d}{d\psi}\{(ABC)^{\frac12}T_{\psi\psi}\}\\
&\qquad-\tfrac12\left[\left(\frac{BC}{A}\right)^{\frac12}\frac{dA}{d\psi}T_{\phi\phi}+\left(\frac{CA}{B}\right)^{\frac12}\frac{dB}{d\psi}T_{\chi\chi}+\left(\frac{AB}{C}\right)^{\frac12}\frac{dC}{d\psi}T_{\psi\psi}\right]-(ABC)^{\frac12}\frac{dW}{d\psi}=0
\end{aligned}\right\}\text{...(68)},$$

in which the values of the stresses may be substituted from (66).

18. The equations now determined are those that apply to the most general cases of curvilinear coordinates. In any special system, the analysis might be very much cut down by dealing at once with the special values of A, B, C... which will often be either constants of functions of only one variable.

Without here going back through the analysis, but merely by taking advantage of the general results already obtained, we may now apply the theory to some special cases.

1st. The case of oblique Cartesian coordinates possesses some interest. The coefficients A, B, C... are now constants, so that $\mathfrak{d}A$, $\mathfrak{d}B$,... are all zero, and the analysis of §§ 12—15 applies with this important simplification. The potential energy V of the strain is a quadric function of the differential coefficients of u, v, w, with coefficients which are constants: it is given by (50) and (51). The stresses are given in terms of its differential coefficients by (56). The equations of internal equilibrium are given in terms

Digitized by Google

of the stresses and the applied external forces *only* by equations (58); for $\frac{dV}{du}$, $\frac{dV}{dv}$, $\frac{dV}{dw}$ are now each zero, and the coefficients are all constants and can therefore be taken outside the sign of differentiation.

19. 2nd. As examples of orthogonal systems we may take the well-known cases of columnar and polar coordinates.

(α) Columnar coordinates r, θ, z. Here

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2,$$

and therefore $\quad A = 1, \quad B = r^2, \quad C = 1$

and $\quad \mathfrak{d}A = 0, \quad \mathfrak{d}B = 2ru, \quad \mathfrak{d}C = 0$ .....(69),

and the value of V is at once written down from (65).

It is however to be noticed that u, v, w are the increments of r, θ, z respectively, and that v is not here the increment of $rd\theta$ as is sometimes the case.

The values of the internal stresses may be written down by (66), viz.:

$$\left.\begin{aligned}
T_{rr} &= a\frac{du}{dr} + b\left(\frac{dv}{d\theta} + \frac{u}{r}\right) + b\frac{dw}{dz}, & T_{r\theta} &= \tfrac{1}{2}(a-b)\left(\frac{1}{r}\frac{du}{d\theta} + r\frac{dv}{dr}\right)\\
T_{\theta\theta} &= b\frac{du}{dr} + a\left(\frac{dv}{d\theta} + \frac{u}{r}\right) + b\frac{dw}{dz}, & T_{\theta z} &= \tfrac{1}{2}(a-b)\left(r\frac{dv}{dz} + \frac{1}{r}\frac{dw}{d\theta}\right)\\
T_{zz} &= b\frac{du}{dr} + b\left(\frac{dv}{d\theta} + \frac{u}{r}\right) + a\frac{dw}{dz}, & T_{zr} &= \tfrac{1}{2}(a-b)\left(\frac{du}{dz} + \frac{dw}{dr}\right)
\end{aligned}\right\} \quad (70).$$

The equations of internal equilibrium are by (68)

$$\left.\begin{aligned}
&\frac{d}{dr}(rT_{rr}) + \frac{d}{d\theta}(T_{r\theta}) + \frac{d}{dz}(rT_{rz}) - T_{\theta\theta} - r\frac{dW}{dr} = 0\\
&\frac{d}{dr}(r^2T_{r\theta}) + \frac{d}{d\theta}(rT_{\theta\theta}) + \frac{d}{dz}(r^2T_{z\theta}) - r\frac{dW}{d\theta} = 0\\
&\frac{d}{dr}(rT_{rz}) + \frac{d}{d\theta}(T_{\theta z}) + \frac{d}{dz}(rT_{zz}) - r\frac{dW}{dz} = 0
\end{aligned}\right\} \quad \text{..... (71).}$$

(β) Polar coordinates r, θ, ω. Here

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \,.\, d\omega^2,$$

and therefore $\quad A = 1, \quad B = r^2, \quad C = r^2 \sin^2\theta,$

$\quad \mathfrak{d}A = 0, \quad \mathfrak{d}B = 2ru, \quad \mathfrak{d}C = 2r\sin^2\theta u + 2r^2 \sin\theta\cos\theta v$ ..... (72),

wherein u, v, w are the displacements of r, θ, ω respectively.

By (65) the value of V may be at once written down.

Digitized by Google

The values of the internal stresses are, by (66),

$$\left.\begin{aligned}
T_{rr} &= a\frac{du}{dr} + b\left(\frac{dv}{d\theta}+\frac{u}{r}\right) + b\left(\frac{dw}{d\omega}+\frac{u}{r}+\tan\theta\, v\right), T_{r\theta} = \tfrac{1}{2}(a-b)\left(\frac{1}{r}\frac{du}{d\theta}+r\frac{dv}{dr}\right)\\
T_{\theta\theta} &= b\frac{du}{dr} + a\left(\frac{dv}{d\theta}+\frac{u}{r}\right) + b\left(\frac{dw}{d\omega}+\frac{u}{r}+\tan\theta\, v\right), T_{\theta\omega} = \tfrac{1}{2}(a-b)\left(\frac{1}{\sin\theta}\frac{dv}{d\omega}+\sin\theta\frac{dw}{d\theta}\right)\\
T_{\omega\omega} &= b\frac{du}{dr} + b\left(\frac{dv}{d\theta}+\frac{u}{r}\right) + a\left(\frac{dw}{d\omega}+\frac{u}{r}+\tan\theta\, v\right), T_{\omega r} = \tfrac{1}{2}(a-b)\left(r\sin\theta\frac{dw}{dr}+\frac{1}{r\sin\theta}\frac{du}{d\omega}\right)
\end{aligned}\right\}\quad(73).$$

The equations of internal equilibrium are, by (68),

$$\left.\begin{aligned}
&\frac{d}{dr}(r^2\sin\theta T_{rr}) + \frac{d}{d\theta}(r\sin\theta T_{r\theta}) + \frac{d}{d\omega}(rT_{r\omega}) - r\sin\theta T_{\theta\theta} - r\sin\theta T_{\omega\omega} - r^2\sin\theta\frac{dW}{dr} = 0\\
&\frac{d}{dr}(r^2\sin\theta T_{\theta r}) + \frac{d}{d\theta}(r^2\sin\theta T_{\theta\theta}) + \frac{d}{d\omega}(r^2 T_{\theta\omega}) - r^2\cos\theta T_{\omega\omega} - r^2\sin\theta\frac{dW}{d\theta} = 0\\
&\frac{d}{dr}(r^2\sin^2\theta T_{\omega r}) + \frac{d}{d\theta}(r^2\sin^2\theta T_{\omega\theta}) + \frac{d}{d\omega}(r^2\sin\theta T_{\omega\omega}) - r^2\sin\theta\frac{dW}{d\omega} = 0
\end{aligned}\right\}\quad(74).$$

20. The equations thus obtained for these cases, (α) and (β), may be simplified considerably by performing the differentiations. They are then the same as were obtained orginally by Lamé by transformation from Cartesian coordinates, and applied by him to the consideration of the elastic yielding of a sphere under given forces and other similar problems. They may also be obtained from first principles by a process involving vector differentiation (see Webb, *Messenger of Mathematics*, 1882, XI. pp. 146—155) which might be extended to the general theory of orthogonal coordinates, at the cost however of increasing the complication and the number of terms.

The general equations (66), (68) also lead to the equations which apply in the cases of ellipsoidal coordinates and elliptic columnar coordinates and such other orthogonal systems as have been used. These are the systems of equations which would naturally be applied to the case of elastic solids bounded by confocal quadric surfaces or by elliptic cylindrical surfaces respectively: but their degree of complication is such as to render them of not much practical value except in the more simple forms of strain, in which, for example, one or two of the three displacements are zero or very small. In these latter cases the general analysis is clearly much shortened by introducing the simplification at the beginning of the work.

Digitized by Google

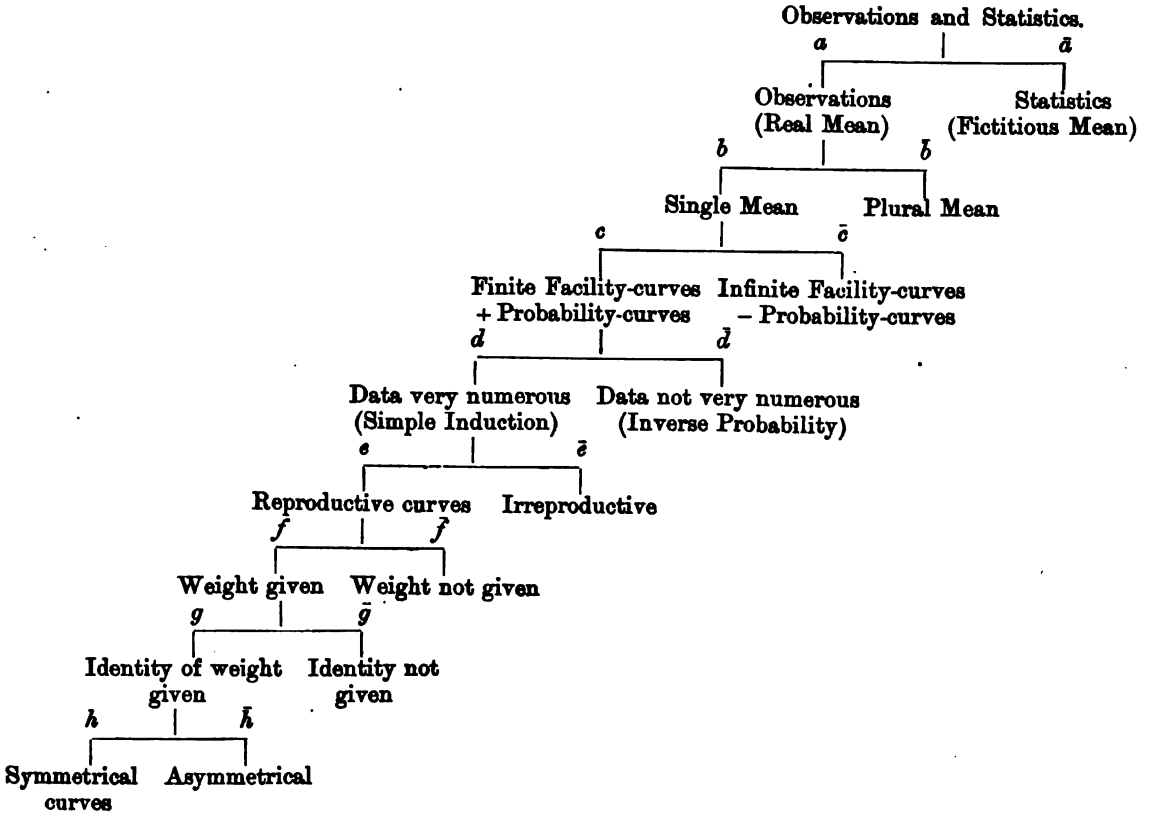
VII. *Observations and Statistics. An Essay on the Theory of Errors of Observation and the First Principles of Statistics.* By F. Y. Edgeworth, M.A. Communicated by J. W. L. Glaisher, M.A.

[Read *May* 25, 1885.]

CONTENTS.

	PAGES
Divisions of the subject (exhibited by the accompanying *tree*)	139
Distinction between real and fictitious Means. (Cp. pp. 161, 167)	139
Exponential law of Error not applicable to facility-curves extending to infinity	140
Proof of law of Error for Finite facility-curves	141
Simple Induction and Inverse Probability	142
Reproductive curves	142
Knowledge of Modulus unduly assumed by some writers (e.g. in case of ratio of male : female births)	143
Real Means sought by Simple Induction	144
Remarks on Mr Galton's method of ascertaining the number of elements in a binomial	145
Real Means sought by Inverse Probability	145
Probability-curves (treated inversely)	146
A priori Probabilities (cp. pp. 149, 157)	147
Most probable value of Modulus (inferred from observations) *not* $Se^2 \div (n-1)$	148
Corrected value of probable error in same case	149
New formula for reduction of observations not known to have same Modulus	149
General case of finite facility-curves (inversely treated)	150
Inversion Proper	151
Criticism of the *Method of Least Squares* as distinguished from Inversion Proper. Cp. p. 164	152
Facility-curve of average more divergent than that of primary	158
The mean of two observations *not* always the most probable value	159
Statistics proper. Subjective Means	160
Rationale of Laplace's *Method of Situation*, and of his assumption that the mean error is the measure of disadvantage	162
The "most advantageous" distinguished from the "most probable" value	163
Laplace's "*à posteriori*" method of reducing "observations *déjà faites*"	164
"Most advantageous" value of modulus inferred from observations	164
Difficulty of distinguishing subjective from real Means	166
The three principal Means compared	166

Digitized by Google



E.g. *abcdefgh* denotes the following problem: Given a not indefinitely numerous set of observations grouped about a single real Mean according to probability-curves, of which the weight is unknown, and which are not known to have the same weight, to find the best value of the real thing by way of inverse probability.

OBSERVATIONS AND STATISTICS.

THE object of this paper is to distinguish and examine the different cases which are presented by the problem*: What is the best Mean?

(*a*) The first distinction which offers itself is between observations and statistics. According to the definition here proposed, observations and statistics agree in being quantities grouped about a Mean; they differ, in that the Mean of observations is real, of statistics is fictitious. The mean of observations is a cause, as it were the source from which diverging errors emanate. The mean of statistics is a description, a representative quantity put for a whole group, the best representative of the group, that quantity which, if we must in practice put one quantity for many, minimizes the error unavoidably attending such practice. Thus measurements by the reduction of which we ascertain a real time, number, distance, are observations. Returns of prices, exports and imports, legitimate and illegitimate marriages or births and so forth, the averages of which constitute the premises of practical

* This problem involves the perhaps more important one: What is the worth (in particular, the precision) of a proposed Mean?

Digitized by Google

reasoning, are statistics. In short observations are different copies of one original; statistics are different originals affording one "generic portrait." Different measurements of the same man are observations; but measurements of different men, grouped round l'homme moyen, are primâ facie at least statistics.

This delicate distinction does not generally correspond to any strongly marked difference, either in the* external agencies which cause the phenomenon of quantities grouped about a Mean, or in our formula for determining the Mean. Accordingly the definition here suggested is not expressed in ordinary discourse. The popular or the technical meaning of the terms will be employed here according to the context.

(*b*) A second division is between cases where the given observations or statistics are grouped about a single mean, and where there are several such quaesita.

(*c*) A third division is between facility-curves which are either finite or probability-curves, and those which extend to infinity. The importance of this distinction lies in the proposition here submitted that, where facility-curves extending to infinity occur, there the exponential law of error does not hold. That is to say, if there be n facility-curves,

$$y = f_1(z), \quad y = f_2(z), \quad \&c.,$$

and E be put† for

$$\gamma_1 z_1 + \gamma_2 z_2 + \&c.,$$

where $z_1, z_2 \ldots$ are values taken at random from the respective series, then the values of E do *not* in general range under the probability-curve $\left(\frac{1}{\sqrt{\pi}c} e^{-\frac{x^2}{c^2}}\right)$ when the curves f extend to infinity.

Let us take the case most favourable to uniformity, where the facility-curves are symmetrical and identical [say $y = f(z)$]. By the usual reasoning it is shown that the sought expression, the ordinate proportionate to the number of times that E commits an error of the extent x, is equal to

$$\frac{1}{\pi}\int_0^\infty \left\{\int_{-\infty}^{\infty} f(z) \cos az\, dz\right\}^n \cos ax\, da.$$

This is inferred to be equal to $\frac{1}{\sqrt{\pi}c} e^{-\frac{x^2}{c^2}}$, where $\frac{c^2}{2}$ is the mean square of error for the facility-curve $f(z)$. The steps by which this inference‡ is usually reached are two. (1) The expression within the brackets being expanded in terms of a it is assumed that the terms above the second may for purposes of integration be neglected; when small, because they are small, and, when large, because the whole expression is then small. (2) The thus truncated series is for purposes of integration approximately equated to $e^{-\frac{c^2}{4}a^2}$. Both these steps seem precarious. They have been suspected by§ eminent authorities. They are proved to be erroneous by conspicuous examples. The family of facility-curves which is of the type

$$f(x) = \frac{1}{\pi}\int_0^\infty e^{-a^4} \cos ax\, da,$$

* See p. 166.

† Cp. Todhunter's *History of Probabilities*, Art. 1002.

‡ See Todhunter's *History of Probabilities*, Art. 1002, p. 556.

§ Glaisher, *Monthly Notices of Royal Astronomical Society*, 1873. Ellis, *Camb. Math. Journ.*, iv. p. 132.

Digitized by Google

where t is any positive quantity, does not produce the probability-curve as that under which the values of E range, except in the particular case when $t=2$.

Mr Ellis indeed in his proof of the law of error does not take those dangerous steps*. Yet upon his own showing his course of reasoning is theoretically insecure† and practically misleading. The example which he gives, the case where $f(z)$ is $\frac{1}{2}e^{-z}$ between infinite limits, seems even more damaging to his theory than he admits. For the final form in that case is $\frac{1}{\pi}\int_0^\infty\left(\frac{1}{1+a^2}\right)^s \cos ax$ which is known to be of the form $e^{-x}(a+bx+cx^2+\&c.)$.

The following reasoning shows that the received conclusion is admissible when the facility-curves $f(z)$ do not extend beyond finite limits. Supposing first that the elemental facility-functions are all identical and symmetrical and that E is the sum of s errors taken at random; the sought expression, the ordinate of the curve under which the values of E range, say u_{xs}, where x is the extent of error, is the $\frac{x}{\omega}$th term of the multiple

$$[f(z)\,t^{-\frac{z}{\omega}}+\&c.+f(2\omega)\,t^{-2}+f(\omega)\,t^{-1}+f(0)\,t^{0}+f(\omega)\,t^{+1}+f(2\omega)\,t^{+2}+\&c.+f(z)\,t^{+\frac{z}{\omega}}]^s.$$

Observing the formation of the coefficient of $t^{\frac{x}{\omega}}$ in the $(s+1)^{\text{th}}$ power of the expression within the brackets, we have (1) $u_{s+1,x}=\int_{-b}^{+b}f(z)\,u_{x+s,s}\,dz$ [where $\pm b$ are the limits of the elemental facility-curves $y=f(z)$]; that is, supposing only even powers of z are involved; for otherwise the above integral will have to be separated into three parts. Assuming such a tendency towards a limiting form that the effect of proceeding from the sth to the $(s+1)^{\text{th}}$ power is indefinitely small (for that part of the result, those values of x with which we are concerned) we may write the left-hand member of (1) $u+\frac{du}{ds}$. The right-hand member may be expanded into

$$\int_{-b}^{b}f(z)\,dz\,u_{xs}+\int_{-b}^{b}f(z)\frac{z^2}{2}\,dz\,\frac{d_2u_{xs}}{dx^2}+\&c.,$$

and the terms above the second neglected. For since, by hypothesis b is finite, if the linear unit be taken properly, the mean powers of error for the facility-curve $y=f(z)$ above the mean square may be neglected. An appropriate unit would be sb. Accordingly an approximate solution of (1) is afforded by a solution of the partial differential equation

$$(2)\quad \frac{du}{ds}=\frac{c^2}{4}\frac{d_2u}{dx^2}\left(\text{where } \frac{c^2}{2}=\int_{-b}^{+b}z^2f(z)\,dz\right),\ \text{if } b^4\frac{d_4u}{dx^4}\ \&c.\ \text{are small.}$$

The complete solution of this equation is $Ae^{-\frac{(x-a)^2}{sc^2}}$. The symmetry about the origin shows that $a=0$. And the condition of a facility-curve, that its integral should be equal to unity, gives $A=\frac{1}{\sqrt{\pi sc}}$. The sought final form is therefore $\frac{1}{\sqrt{\pi sc}}e^{-\frac{x^2}{sc^2}}$.‡ This reasoning is easily adapted to the cases where E is the weighted sum (as it may be called) or

* *Camb. Phil. Trans.*, Vol. VIII. p. 210. † Ibid. p. 215. ‡ x^2 not exceeding sc^2.

Digitized by Google

weighted mean of s observations; and where the elemental curves are not* identical and not symmetrical.

(*d*) The next distinction is between cases where the number of observations or statistical returns is practically infinite, large enough to completely exhibit the law of facility according to which they are reproduced, and cases where the number is not large enough to form the basis of a valid induction, where the data can only be regarded as samples taken at random from the facility-curves under which they range. The first method is followed by Quetelet when he determines the character of the curve under which the data range by taking as the *probable error* of that curve the point which just divides into two halves the number of observations or statistics which lie on one side of the mean, by Mr Galton in his method of quartiles and deciles, and very generally in all graphical methods of statistics. The second method is pursued by mathematicians in the discovery of the mean, when the facility-curves under which the observations range are known to be probability-curves, and would doubtless be more extensively employed but for the mathematical difficulties incidental to "l'ignorance entière où l'on est de la loi des erreurs de chaque observation†." The first procedure is pursued by Airy in his *Theory of Errors* when [§ 33] he says, "If we investigate the value of the modulus, first by means of the Mean Error, secondly by the Error of the Mean Square, we shall probably obtain discordant results. We cannot assert *à priori* which of these is the better." The second procedure is pursued by Mr Merriman when regarding‡ the given observations as having resulted from a source of error consisting of a probability-curve with a certain unknown modulus [*Method of Least Squares*, § 24] he determines by *inverse probability* what is the point and what the modulus from which the errors are most likely to have resulted. In short the first procedure is of the nature of ordinary induction. The second procedure belongs to that more methodical investigation of causes by way of Inverse Probability which Laplace in his introductory essay on Probabilities describes as "le principe fondamental de cette branche de l'Analyse des hasards qui consiste à remonter des événemens aux causes."

(*e*) The next division is between reproductive and other facility-curves; reproductive facility-curves being defined as such that if a couple or any finite number of values x_1, x_2, &c. be taken at random from the series represented by such a curve, then the sum or any arithmetical mean $\gamma_1 x_1 + \gamma_2 x_2 + \&c. \div s\gamma$ ranges under a facility-curve similar in type to the primary curve. The general type of such a facility-curve appears to be

$$\frac{1}{\pi}\int_0^\infty e^{-a^2}\cos az\, da.$$

(*f*) The next distinction is between cases where the *weight* or measure of divergence for the facility-curves involved by the problem is given, and where it has be to inferred from the data. Laplace and those who have followed him, §Herschell and ‖De Morgan, seem provisionally at least to have assumed that, when we are given the number of times

* In this case c^2 must be treated as a function of s.

† Laplace, *Théorie*, 3rd ed., p. 337.

‡ *On Least Squares*, p. 143.

§ Essay on Quetelet's *Probabilities*.

‖ *Encyc. Metrop.*

Digitized by Google

that an event has occurred in each of two alternative ways (e.g. births male and female), then we are given the modulus according to which statistics may be expected to diverge from the given ratio, the weight as it were of the given observation. It is assumed that the case may be compared to that of black and white balls, drawn at random from an urn containing as many balls as there are births. But this is to assume that all the births are *independent*. The more appropriate conception is surely that we are given indeed the ratio between the *mass* of ivory and of ebony in the urn, but that the number of balls, of independent causes, is not given, but to be inferred from the data. In short the case is similar to the case where we are given a set of observations x_1, x_2, &c., and have to infer from these data the modulus according to which they diverge*. The statistics of death rates considered in the light of this analogy show a divergence many times greater than that assigned by the current theory.

(*g*) The next division is between cases where the weight of the facility-curves is, or is not, given as identical. Mr Glaisher† has pointed out the confusion which results from ignoring this distinction.

(*h*) The next and last distinction is between symmetrical and asymmetrical facility-curves.

These divisions are exhibited in the accompanying *tree*, the branches of which have been completely developed only on one side. The work of examining the 256 *species infimæ* which are constituted by these eight principles of division will be greatly abbreviated by the facts that some branches stop short, some, though existent, are barren, and some, though real and important, are so exactly parallel to others that they do not require a separate description. In examining the different species the following order will be observed. The simpler cases will have precedence of those which are more complex and difficult. Accordingly the first member of each division will be considered before the second. The first species that will be considered is the simplest; namely that which is constituted by taking the first member of each division. It may be denoted by the symbol *abcdefgh*. Next will be (or would be, if it existed) considered the case $abcdefg\bar{h}$, where $\bar{h}$ denotes the presence of the second member of the eighth division. The ramifications of the class *abcdefg* being thus exhausted we shall (or would if possible) consider the class $abcdef\bar{g}$ subdivided into two species corresponding to the positive and negative values of *h*. When *abcdef* is exhausted we proceed to $abcde\bar{f}$, and so on.

abcdefgh. This is the case of a complete set of observations or statistics grouped about a single mean under a probability-curve whose modulus is known beforehand. One instance would be presented by throwing up *m* coins *n* times (as Prof. Jevons did) and recording each time the preponderance of heads and tails, with a view of discovering whether the coins are perfectly symmetrical or 'loaded' in favour of either side. Here we may assume the modulus to be given beforehand, namely $\sqrt{\frac{1}{2n}}$; though strictly the modulus

* When writing this passage I had not seen Prof. Lexis' *Massenerscheinungen* and other original writings; to which I have endeavoured to do justice in a paper "On the Methods of Statistics," published by the Statistical Society, 1885 (see pp. 11, 18, 32).

† *Memoirs of the Astronomical Society*, Vol. XL. pp. 102, 103.

Digitized by Google

is here dependent upon the sought mean. In fact the case is not usefully to be separated from *abcdefgh*. Of both it may be remarked that in determining the centre of the curve no one odd symmetrical function of the data has (apart from considerations of convenience) an advantage over another. To enquire which function would be oftenest right in the long run would be to introduce an element of Inverse Probability which is not appropriate to the present class (having the attribute *d*) *abcdefgh̄* and the other species of *abcdef* are either non-existent or unimportant.

abcdef̄. This case may be thus distinguished from *abcdef*. In the latter case we are given the law of divergence by experience *prior* to that which constitutes our present data; necessarily then,—since our present data are supposed extensive,—experience of a still higher degree of certainty, such as that which is embodied in the rules concerning games of chance. In the latter case we must determine the modulus from the observations. The former case is that which, as before remarked, mathematicians have too readily supposed. The latter case is by far the most usual in all important observations and statistics. As before indicated, in the present roughly inductive procedure, there is no ground for preferring before another any one function of the observations, which if the series were really complete might be equated to the mean or to the modulus. For instance, if the curve were $y = \frac{1}{\sqrt{\pi}c} e^{-\frac{x^2}{c^2}}$, then theoretically (1) $c = \sqrt{\pi}\, Se$, where Se is the sum of the errors taken positively on both sides of the mean, (2) $c^2 = 2Se^2$, (3) $c^4 = \frac{4}{3} Se^4$, &c. In short we assume here that the apparent errors may be considered as real. The ramifications of this class present no new difficulty.

abcdē is subject to similar remarks in so far as we know beforehand something of the character of the irreproductive curve under which the observations range. This case brings into view a difficulty which was hardly worth noticing in the preceding cases, namely that we must suppose given not only the complete curve of observations, but also the knowledge at what part of such curves the sought real point usually occurs; in what manner observations group themselves about the real point. This might be taken for granted in the case of symmetrical curves. But in asymmetrical curves the question arises, Is the real point for example at the centre of gravity, or the centre of area, or at the longest ordinate of the given curve? Probably the most important case under this heading is that of what Mr Galton* calls binomial curves, only the term need not be confined as it is by him to symmetrical curves.

abcdēfgh is exemplified by Mr Galton's binomial of seventeen (or fewer) elements with alternatives even and known *à priori* as in games of chance.

abcdēfgh̄ is a similar case with unequal alternatives; for instance the (discontinuous) facility-curve or series which is presented by developing $(\frac{10}{17} + \frac{7}{17})^{17}$ and taking 18 ordinates proportioned to the terms of developement. In this case it will be remarked that the real thing sought, the ratio from which the series results, is represented by the point at which occurs both the centre of gravity and greatest ordinate.

* *Phil. Mag.*, 1875.

Digitized by Google

abcdēf̄gh is exemplified by a binomial curve such as Mr Galton supposes to be presented by the height of a man or wall. The only difficulty here is concerning the determination of the *modulus* as it may be called, or, as Mr Galton puts it, the number of independent equal causes from which the curve may be considered to have resulted. If we knew all about the series we should know (1) the number of elements, (2) the extent of each and therefore (3) the total range or interval between maximum and minimum and (4) the probable error. Now we are not given (1) and (2). But we are given approximately (4) and more conjecturally (3), from which we may deduce (1) and, if necessary, (2). Mr Galton takes as the value of (3) the *mean* of the series of observations; for instance, in the case of measurements of different men, he would take the height of *l'homme moyen* for the total range of error; *OP* in Fig. 1. And this in fact constitutes a superior limit to that range. An inferior limit, and generally a much more accurate value, would be the interval between the measurements which are widest apart, between the tallest giant and the shortest dwarf; *QQ'* in Fig. 1. Thus in the case of a wall consisting of 17 courses of stones each having a probable error of $\frac{1}{5}^{\text{th}}$ of an inch, Mr Galton takes for our (3) the mean height of the wall 51 inches. But a much nearer value would be the difference between the greatest and least height of the wall as given by actual measurement, a difference in the case supposed not likely to exceed an inch.

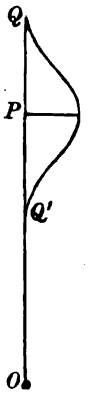


Fig. 1.

abcdēf̄gh̄. This case is exemplified by observations presumed to range under an unsymmetrical binomial curve the proportions of which are not given beforehand. The real point sought would presumably be either at the centre of gravity or largest ordinate positions theoretically identical*.

abcd̄. At this stage, as† before explained, Inverse Probability makes its appearance. The following is the simplest case.

abcd̄efgh. Given a set of observations x_1, x_2, &c., and given that they have been generated by divergence from an unknown point x according to one given law of error, a probability-curve of given modulus c, to find the most probable value of x. It is found to be the average of x_1, x_2, &c. by putting $\frac{dP}{dx}=0$ where

$$P=\frac{1}{\pi^{\frac{n}{2}}c^n}\,e^{-\frac{(x-x_1)^2+(x-x_2)^2+\&c.}{c^2}}.$$

This solution is sanctioned by the highest mathematical authority. There is however a philosophical difficulty which does not seem to have been noticed—with respect at least to observations in time and space; for the analagous difficulty with respect to numerical statistics has been insisted upon by ‡Cournot, §Ellis, and ‖Mr Venn. In the preceding proof it is tacitly assumed that prior to the data one value of the quaesitum is

* Dr Macalister's determination of the parameter appertaining to the law of error which he has discussed in the *Proceedings of the Royal Society* (1879) belongs to this category. His method is analogous to that of Airy noticed under (*d*), page 142.

† Above, p. 142.

‡ *Exposition de la théorie de Chance*, ch. VIII.

§ *Camb. Phil. Trans.*, ch. VIII., Remarks on law of succession.

‖ *Logic of Chance*, ch. V.

Digitized by Google

as likely as another. Correctly the expression P should be affected with a factor ρ, a function of x, representing the *à priori* probability that any value of x is the true one. Now if ρ is not constant, as it is usually assumed to be, the maximum point is given not by the above written equation but by the following

$$\rho \frac{dP}{dx} + \frac{d\rho}{dx} P = 0.$$

To this difficulty* two kinds of answer may be made. (1) It may be pointed out that, as n increases, the influence of ρ (being supposed finite) becomes ever relatively less. In considerable numbers therefore the received solution is accurate. To this it may be rejoined that, as the number of observations is increased, the case becomes merged in the (preceding) case where the observations are so copious as to reveal their law of generation by mere inspection or at least unmethodical induction. The more refined method only becomes accurate where it is not wanted. This rejoinder may perhaps be rebutted by the allegation that the inverse method, though not available when the number of observations are very small, nor wanted when they are very large, may yet be useful for intermediate numbers. The objection may be compared to the fallacious reasoning† in the ordinary proof of the law of error that certain terms of a development may be neglected because, both when they are very small and when very large, they are unimportant.

(2) This view is confirmed by the second answer which may be made to the objections against the use of *à priori* Probabilities. It is not true that we have *no* experience about these probabilities. We have a rough general experience that one value of the measureable occurs as often as another at any rate between the limits with which we are practically concerned. For instance we have some experience that one statistical ratio occurs about as often as another. For if the fact were widely different we should not have failed to notice it. This is certainly rather a rough foundation on which to build an exact science. Yet it is of a piece with the materials of not the least important science to which the calculus of Probabilities is applied, sociology. For in estimates of social utility there is continually assumed some such axiom as that things between which no difference has been noticed may be treated as equal‡.

To this defence it may be objected that the proposition that one event is as likely as another is self-contradictory; for that a double event is an event, and it is repugnant that both the single and plural event shall have the same probability.

This objection has been made with special reference to statistical frequency—e.g. as against Donkin's axiom that an unknown probability may be treated as $\frac{1}{2}$. The objection however is equally available against the assumption that values of space or time are *à priori* equally distributed between certain limits, as against the assumption that statistical

* See Article by the present writer on *à priori* Probabilities, *Phil. Mag.*, Sept. 1884.

† See above, p. 140.

‡ For instance when it is assumed that an increase in national wealth or in mean life is advantageous; or that an equal distribution of wealth or power is *primâ facie* a good. See upon the principle implied some remarks by the present writer in *Mind*, April, 1884, and *Hermathena*, 1884.

Digitized by Google

ratios are *à priori* equally distributed between 0 and 1. The objection in all three aspects may thus be illustrated*.

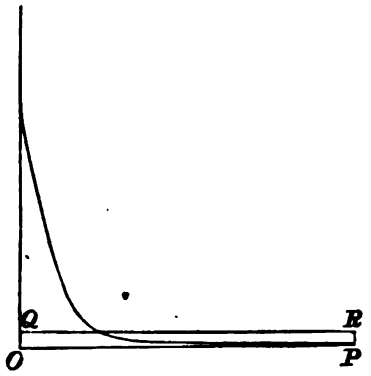


FIG. 2.

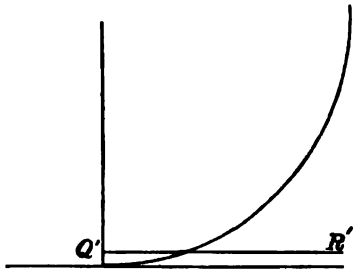


FIG. 3.

Let OP be an abscissa $=1$, and let the horizontal line QR be the facility-curve representing the *à priori* even distribution of x between 0 and 1. To find the distribution of the values obtained by raising each x to the power n, put $\xi = x^n$,

$$\left(x = \xi^{\frac{1}{n}},\ dx = \frac{1}{n}\xi^{\frac{1}{n}-1}\right).$$

Whence the facility-curve expressing the distribution of ξ is $\eta = k\xi^{\frac{1}{n}-1}$ (where k is a constant) represented by the left curve. Conversely, suppose that the values of ξ were equally distributed, and let the line $Q'R'$ represent that equal distribution. Put $\xi = x^n$ and transform to the curve $y = kx^{n-1}$; represented by the curve on the right.

It seems sufficient to reply to this objection that what is here asserted is not that all values are equally distributed, but only those values in whose distribution after copious experience no unevenness has obtruded itself. Take for instance statistical probabilities about deaths from different causes, or births, male and female, of twins, triplets and four at a time. In the immense variety of the Registrar General's returns we perceive no prevalence of one fraction, no clustering about one point. And therefore we assert that the *à priori* probability of single—*not plural*—social events may be treated as a constant.

Upon the whole the two arguments—that *à priori* probabilities may sometimes be dispensed with and are generally given—taken together, and corroborated by the practical success which has attended the calculus of probabilities, appear to establish the foundations of that calculus as rough but solid.

abcdefgh is non-existent.

abcdefgh. By reasoning on a par with that which has just been employed the solution of this case† is found to be the weighted mean of the observations.

abcdefgh. Given a set of observations x_1, x_2, &c., and given that they have been generated by divergence according to one and the same probability-curve from a single

* See note §, p. 148.

† Cp. Merriman, Part II. 27.

Digitized by Google

point, but given neither that point nor the modulus, to find* primarily the former and as a secondary end the latter. By parity of reasoning the central point and modulus are given by the maximum value of $P\rho\sigma$, where P and ρ have the signification above assigned, and σ signifies the *à priori* probability that c would have a particular value. Whence two equations $\frac{d}{dx}P\rho\sigma=0$ and $\frac{d}{dc}P\rho\sigma=0$. Whence, if ρ and σ are constant,

$$x=\frac{x_1+x_2+\&c.}{n} \text{ and } c=\sqrt{2\left[\frac{(x-x_1)^2+(x-x_2)^2+\&c.}{n}\right]}\dagger.$$

This solution is exposed to the same objections as that of the last but one case, and admits of the same defence. The function σ though unknown may be neglected when the observations are numerous. And it is not entirely unknown; for we have a rough general experience that one modulus occurs as often as another. It must be admitted that the rejoinder to this second plea has in this case a peculiar force. For (considering the importance of the quantity c^2) it may be plausibly maintained that we should treat as equally likely *à priori* all values not of c, but of c^2. And again it seems natural to treat as equally likely all values of h the reciprocal of c; as Mr Merriman‡ has done. And there is another at least equally plausible view. Considering the probability-curve as the limiting case of what has been called a binomial curve, we have $c \propto \sqrt{\frac{2pq}{N}}$, where N is the number of elements. And, if one value of N is as likely to occur as another, then h^2 not h should be what may be called the independent variable. These conflicting§ views seem equally plausible. The evidence which tells equally for all of them may seem not to tell much for any of them. The correct position seems to be that our rough general experience assures us not that one of these views is true but that none of them differ very materially from the truth. Whichever of them we adopt, we shall obtain the same value of x, and if n is considerable very nearly the same value of c.

It may be remarked that the value here found for the modulus differs from that assigned by Gauss‖ and many eminent mathematicians in that they would substitute $(n-1)$ for n in the above expression for c. A careful consideration of Gauss' argument will show that what he seeks and finds is *not* that value of c which, taken in connexion

* The case is still a case of single mean according to the definition of class *b*.

† The value of the parameter h in Dr MacAlister's curve of error *as determined by inverse probability* (cf. above, p. 145) is found by putting $h=\frac{1}{c}$; and, in the value of c above given, for x the *geometric* mean, and for

$$(x-x_r)^2 \quad \left[\frac{\log x_r}{x}\right]^2.$$

‡ *On Least Squares*, p. 143.

§ Attention to *à priori* probabilities is required in the transformation of facility-curves. The general rule for transforming $y=f(x)$ to a facility-curve in ξ, where $x=\phi(\xi)$, is put to η (the ordinate of the new curve)

$$=f[\phi(\xi)]\,\phi'(\xi).$$

But it must be remembered that, if the values of x are *à priori* equally probable, the *à priori* probability of ξ is not constant, but $\propto \frac{d\xi}{dx}$ or $\frac{1}{\phi'(\xi)}$. Thus to find from the equation in ξ the most probable (*à posteriori*) value of ξ, we must multiply η by $\frac{1}{\phi'(\xi)}$; whence we obtain $f'[\phi(\xi)]=0$ for the maximum, as was to be expected. These considerations should be borne in mind in studying Dr MacAlister's paper in *Proc. Royal Society*, 1873.

‖ *Theoria Combinationis*, § 38, followed by Airy and most writers.

Digitized by Google

with the proper value of x, constitutes that system of x, c which makes P a maximum, but that value of c which makes P a maximum, account being taken of *all* the possible values which x may have. It is as if any one, given $z = F(xy)$, should determine not the x and y which make z a maximum, but, integrating $F(xy)$ with regard to x between given limits, should determine the y which makes the resulting function a maximum. But surely the primary view of the quaesitum in this case is that value of x which makes P a maximum, account being taken of all possible values of c; and the next most natural view of the quaesitum is the *system* of values for x and c which makes P a maximum—alternatives which yield the same value of x. A third quaesitum is the *probable error* incurred by putting $x = \frac{x_1 + x_2 + \&c.}{n}$. But neither to this question is the answer given by Gauss. The probable error is found by taking for the ordinate of the facility-curve under which the values of x range the expression

$$\int_0^\infty Pdh \div \int_{-\infty}^{\infty} \int_0^\infty Pdh\,dx;$$

supposing the h's to be *à priori* equally likely. Whence it appears that the probable error of x is given by quartering a facility-curve of the form

$$y = \frac{A}{(Sx^2 + nx^2)^{\frac{n+1}{2}}};$$

not, as the followers of Gauss suppose, by quartering a probability-curve whose modulus is $\sqrt{\frac{Sx_r^2}{n(n-1)}}$. In short the value assigned by these mathematicians does not appear to correspond to the maximum of anything in particular. Nevertheless here, as so often in mathematics, genius may have flown on to a point whither method afterwards constructs a road. It will presently be submitted that, though the *most probable* value of the modulus is $s\rho^2 \div n$, yet the *most advantageous* value of the same is greater.

abcdēfḡh. This case differs from the preceding, in that the weight is not known beforehand to be the same for all the observations. In this case put

$$P = \frac{1}{\pi^{-\frac{n}{2}}} \frac{1}{c_1} \frac{1}{c_2} \ldots e^{-\frac{(x-x_1)^2}{c_1^2} - \frac{(x-x_2)^2}{c_2^2} - \&c.} \times \rho \times \sigma_1 \sigma_2 \sigma_3.$$

Whence

$$c_1^2 = 2(x - x_1)^2,$$
$$c_2^2 = 2(x - x_2)^2,$$
$$\&c. = \&c.$$

$$\frac{1}{c_1^2}(x - x_1)^2 + \frac{1}{c_2^2}(x - x_2)^2 + \&c. = 0.$$

Whence

$$\frac{1}{(x - x_1)} + \frac{1}{(x - x_2)} + \&c. = 0,$$

an equation of $(n-1)$ degrees, whose coefficients are symmetrical functions of the given observations, which might possibly be evaluated by a calculating machine. Approximative

Digitized by Google

values of x are afforded by the limiting functions of this equation; the first approximation being as might be expected the average.

This solution is exposed to the objections against the inverse method which have been considered with reference to simpler cases. Nor does this case admit of the defence that the unknown functions $\rho \ldots \sigma_1\sigma_2$ become extinct by the repetition of observations. Each new observation introduces a new *à priori* probability. Nevertheless there are two important sub-cases where this defence is applicable. (1) When the observations are divisible into a few groups and all the members of the same group are known beforehand to have the same modulus. In this case P is of the form

$$\frac{1}{\pi^{-\frac{m+n}{2}}}\frac{1}{c_1^m}+\frac{1}{c_2^n}\rho\sigma_1\sigma_2 e^{-\frac{(x-x_1)^2+(x-x_2)^2}{c_1^2}-\frac{(x-x_1)^2+(x-x_2)^2}{c_2^2}}.$$

Differentiating with regard to c_1, c_2 and x and eliminating c_1 and c_2, we have a cubic equation for x which affords as rigid a solution as any of the preceding cases. (2) The second favourable sub-case is where the moduli may be presumed not to be very widely apart. For in this case we may be confident that the influence of $\sigma_1\sigma_2\ldots$ will become extinct. We have at the same time confidence in employing a derived function of low degree instead of the general equation.

abcdē. Analogous reasoning is applicable to curves other than probability-curves, provided that we know beforehand about as much of their character—the family at least if not the parameter. The only important instance appears to be the binomial of Mr Galton. This when consisting of many elements merges into the probability-curve. When consisting of few elements it becomes a discontinuous small curve not amenable to a special method analogous to the preceding, but falling into the general class.

In general in class $\bar{e}$ the distinction between f and $\bar{f}$ is, not between knowledge and ignorance of a parameter, but, as the family of the facility-curves is in general unknown, between knowledge and ignorance of some mean function of error (as the inverse-mean-square) which is taken as the measure of a facility-curve's precision. Let us begin with the simpler division f and consider an important sub-class under this heading, namely *abcdēfḡh.* This case does not, like the case of probability-curves (e) admit of a perfect solution by way of inverse probability. But it may be shown that an approximate solution is afforded by a certain weighted arithmetic mean.

Any of the facility-curves (not supposed identical) may be put approximately in the form*

$$H_1 e^{-M_1x^2-P_1x^4\ldots-R_1x^r}.$$

Accordingly an approximate solution of the inverse problem is afforded by the maximum value of Y, where $\log\frac{1}{Y}(=Z)$

$$\begin{aligned}&= M_1(x-x_1)^2+M_2(x-x_2)^2+\&c.\\&+P_1(x-x_1)^4+P_2(x-x_2)^4+\&c.,\\&\ldots\ldots\ldots\ldots\ldots\ldots\\&+R_1(x-x_1)^r+R_2(x-x_2)^r+\&c.\end{aligned}$$

* Especially, when the curves may be regarded as probability-curves deformed at the extremities; as in the case of binomial curves.

Digitized by Google

(This reasoning is justified by mathematical precedent. It is allowable to treat a binomial of many elements as a probability-curve, and by reasoning similar to that which has just been employed to deduce that the weighted arithmetic mean is the most probable value.) An approximate solution of $\frac{dZ}{dx}$ is afforded by its last limiting function equated to zero: namely $R_1(x-x_1)+R_2(x-x_2)+\&c.=0$. Whence $x=$ the weighted arithmetic mean $R_1x_1+R_2x_2+\&c.\div R_1+R_2+\&c.$ It is to be observed that the weights here are not the same as in Laplace's solution; though they are apt to vary in the same direction as his weights. This may be seen in one important case: where the facility-curves differ only in respect of a parameter, being each of the type $y=\frac{1}{c}f\left(\frac{x}{c}\right)$. In that case Laplace's weights, the inverse-mean-squares, are proportional to $\frac{1}{c^2}$; R is proportional to $\frac{1}{c^r}$.

When the number of observations is considerable, the approximate value above found tends to coalesce with the correct value. For, since the observations are supposed to emanate from one source of error and to range under a finite facility-curve, as their number increases, it becomes increasingly probable that they will lie, not level, but in a heap. Considering then $Y=f(x-x_1)\times f(x-x_2)\times\&c.$, we see that there must be certain central values of x for which the majority of the factors $f(x-x_1)$, &c. are relatively large and the minority relatively small; but that for values of x nearer the limits of the facility-curves the majority of the factors are relatively small. Accordingly the effect of the continued multiplication must be to make the curve $Y=0$ spring up as represented in Fig. 4 (where each Y is supposed to be multiplied by such a factor that $\int Y\,dx=1$). At

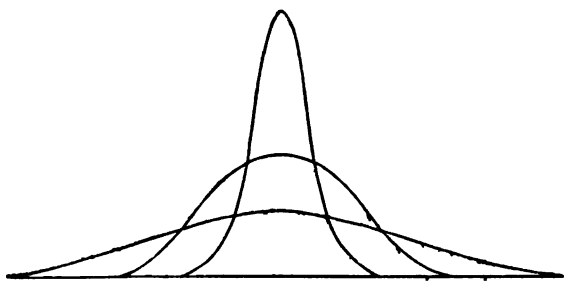


FIG. 4.

the same time $Z\left(=\log\frac{1}{Y}\right)$ tends to become infinite outside narrow limits. Therefore all the differentials of Z outside narrow limits tend to be infinite. Therefore the values of x for which $\frac{dZ}{dx}=0$ and that for which $\frac{d^{r-1}Z}{dx^{r-1}}=0$ lie close together.

The difficulties which beset the direct solution of the general case under consideration have induced Laplace and his followers to resort to an indirect procedure, which may be

Digitized by Google

distinguished from the *proper-inverse method as the *Method of Least Squares.* This celebrated method consists of two steps. (1) The first is to put for the quaesitum a weighted arithmetical mean, not as the most probable†, but as the most workable value. This sacrifice of precision to practicability seems quite justified by mathematical precedent. (2) The second step is to determine the constants of the assumed form, the weights of the arithmetical mean, by a principle which may be thus generalised. To test the accuracy of a *method of reduction,* i.e. function of the observations which it is proposed to put for the quaesitum: take at random any number of sets of values

$$\begin{array}{l} e_1,\ e_2 \ldots\ldots\ldots\ldots\ldots\ldots e_s, \\ e_1',\ e_2' \ldots\ldots\ldots\ldots\ldots\ldots e_s', \\ e_1'',\ e_2'' \ldots\ldots\ldots\ldots\ldots\ldots e_s'', \end{array}$$

which as it were ‡emanate from the same source of error, each e according to the law of error indicated by its subscript, the law of error for one of the observations. For instance the first set is thus formed; e_1 is taken at random from under the facility-curve $y=f_1(x)$; e_2 from under the facility-curve $y=f_2(x)$, and so on. Calculate the value of θ for each set and arrange the values so found under a facility-curve; then the divergence of the facility-curve so resulting may be taken as a measure of the inaccuracy of the method of reduction $\theta(x_1, x_2,\ldots)$. Accordingly, if the divergence of that resulting facility-curve should be greater for θ_1 than θ, then θ_1 is a less accurate reduction. Laplace has employed this principle in at least four other cases besides that which is now under consideration. The first instance is his reasoning about Bayes' theorem, where the process called by Mr Todhunter§ "assumed inversion" will be found to involve the principle just enounced. A second instance is in the problem of mean marriages also noticed by Mr Todhunter‖. A third instance is in the second¶ supplement where Laplace professes to test the accuracy of his *Method of Situation* by an application of the principle in question. A fourth instance is in the first supplement where Laplace argues upon his principle that the inaccuracy of his weighted mean is of a negligible order.

It is here submitted with great deference that this indirect method has no theoretical cogency and does not add anything to the presumptions in favour of the weighted arithmetical mean which are obtained from the inverse-proper, or *à posteriori,* point of view. To show this it may be sufficient to consider the instance in which the method of reduction, the $\theta(x_1, x_2\ldots)$ proposed to be tested, is that which the inverse-proper method assigns

* See note, p. 157.

† Cp. Glaisher, *op. cit.* p. 10.

‡ The inverse-proper method which has been hitherto pursued in this paper may be thus contrasted with the (second step of the) method of least squares now under consideration. The former, regarding observations as having emanated from a source of error, remounts to the source by the principles of inverse probability established by Laplace at the beginning of his introductory essay. The latter begins as it were at the source, and, taking at random any number of sets of errors emanating from the source, operates on them in the manner described in the text. Accordingly the former (or at least a particular species of it to be distinguished presently, see below, p. 164) is described by Laplace as *à posteriori,* relating to "observations deja faites"; the latter as *à priori* relating to "observations non faites encores."—*Théorie Analytique,* third ed., p. 333, national ed., p. 365.

§ *History of Probability,* pp. 554—556.

‖ P. 603.

¶ Pp. 6—11 (of the Supplement), noticed by Mr Todhunter at his p. 610.

Digitized by Google

as the most probable value of x, that is if Fig. 5 denote the curve

$$Y = f(x - x_1) \times f(x - x_2) \times \ldots\ldots,$$

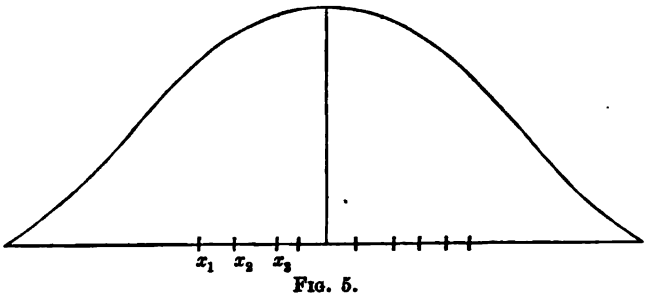


FIG. 5.

then $\theta(x_1, x_2 \ldots)$ is the point at which the longest ordinate occurs; and the curve Y represents truly the inaccuracy of θ, the number and extent of errors we shall commit if every time we obtained the given set of observations we put $\theta(x_1, x_2 \ldots)$ for the true value. Now let us consider, if we put ourselves at the source so to speak, if in the language of Laplace and De Morgan we want to construct a method of dealing with the observations "before they were made*," how we should obtain a facility-curve corresponding to Y. A little attention will show that we must take, not at random every set of s values which flow from the source of error, *but only those particular sets the constituents of which are situate relatively to each other at the same distances as the given observations* $x_1, x_2 \ldots$ But in general this last procedure is not the same as Laplace's. In fact it is only the same in the case of the probability-curve; in which case the inverse-proper requires no aid.

The inconclusiveness of the fundamental principle upon which Laplace's† "assumed inversion" rests may be further exhibited by example. Consider the method of reduction $\left(\frac{x_1^{\frac{1}{n}} + x_2^{\frac{1}{n}} + \&c.}{s}\right)^n$; n odd, weights of observations identical. According to the principle assumed, that value of n is best which is such that, if we take at random sets of s observations, and form with each set the mean $\left(\frac{x_1^{\frac{1}{n}} + x_2^{\frac{1}{n}} + \&c.}{s}\right)^n$, the values so obtained range under a facility-curve of least divergence. By a well-known theorem‡ when s is considerable the values of $\frac{x_1^{\frac{1}{n}} + x_2^{\frac{1}{n}} + \&c.}{s}$ may be regarded as ranging under the curve

$$\frac{\sqrt{s}}{\sqrt{\pi}k} \rho^{-\frac{t^2 s}{k^2}},$$

where $\frac{k^2}{2}$ = mean − twice $\frac{1}{n}$th power of error for the original curve

$$= \frac{\Gamma\left(\frac{\frac{2}{n}+1}{2}\right) \times c^{\frac{2}{n}}}{\sqrt{\pi}}.$$

* De Morgan, *Encycl. Metrop.*, § 89. † Mr Todhunter's phrase. ‡ Todhunter, p. 570.

Digitized by Google

The error then of $\frac{x_1^{\frac{1}{n}}+x_2^{\frac{1}{n}}+\&c.}{s}$, may be represented by $\frac{kT}{\sqrt{s}}$, where T corresponds to any assigned value of $2\int_0^T \frac{1}{\sqrt{\pi}} e^{-x^2}dx$, e.g. $\frac{1}{2}$, $\frac{1}{10}$, &c. Now, in order to transform the facility-curve in ξ representing the frequency of the values $\frac{x_1^{\frac{1}{n}}+x_2^{\frac{1}{n}}+\&c.}{s}$ to one in ξ' representing the values $(x_1^{\frac{1}{n}}+x_2^{\frac{1}{n}}+\&c.)^n$, the rule is* to put $\xi=\xi'^{\frac{1}{n}}$ and $\left(\text{since } d\xi=\frac{1}{n}\xi'^{\frac{1}{n}-1}\right)$ to multiply after substitution by $\frac{1}{n}\xi'^{\frac{1}{n}-1}$. What was $\frac{kT}{\sqrt{s}}$ in the primary curve becomes in the derived curve $\left(\frac{kT}{\sqrt{s}}\right)^n$, that is $\left\{\frac{2\Gamma\frac{\left(\frac{2}{n}+1\right)}{2}}{\sqrt{\pi s}}\right\}^{\frac{n}{2}} \times c \times T^n$. When n is large, this expression approaches $\left(\frac{2}{s}\right)^{\frac{n}{2}} \times cT^n$. But the error of the method of reduction $\frac{x_1+x_2+\&c.}{s}$ is $\frac{cT}{\sqrt{s}}$. Thus, s and n being considerable, the method of reduction $\left(\frac{x_1^{\frac{1}{n}}+x_2^{\frac{1}{n}}+\&c.}{s}\right)^n$ is better than

$$\frac{x_1+x_2+\&c.}{s};$$

since the *probable error*† and all errors up to $T=1$ are less for the facility-curve representing the former values than in the case of the latter. The two curves may be denoted by Fig. 6. But we know by inversion-proper, if the facility-curve under which the

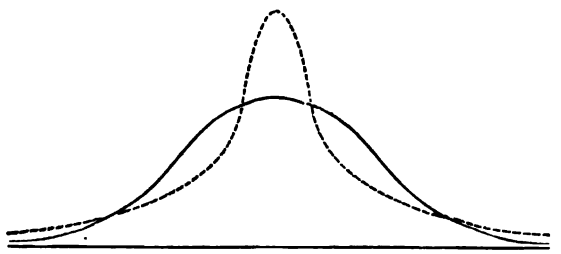


FIG. 6.

observations range is a probability-curve, that the most probable value is $\frac{x_1+x_2+\&c.}{s}$. The method of assumed inversion is therefore at variance with the method known to be right.

The incorrectness of the principle under consideration may be more generally demonstrated by pointing out the correction of which it stands in need. Let $\theta(x_1, x_2, \ldots c_1, c_2)$ be the method of reduction which is to be tested. Let $\Phi(x)$ be the facility-curve which

* See note § p. 148.

† This example relates to that species of the method of Least Squares, preferred by Mr Glaisher (*Memoirs of Astronom. Society*, Vol. XL. p. 101), in which the quæsitum is the *most probable* value. If we consider the species which aims at the *most advantageous* value, to be explained below, the example here adduced will not be so evidently at variance with the result of inversion proper.

Digitized by Google

is formed by taking at random sets of $x_1, x_2 \ldots$ &c. (from under the facility-curves of the observations, namely $y = f_1(x)$, $y = f_2(x) \ldots$) and substituting in θ the values of $x_1, x_2 \ldots$. Let $\psi(x_1, x_2 \ldots)$ be the most probable value as given (by way of inversion proper) by the solution of the equation

$$\frac{d}{dx} f_1(x - x_1) \times f_2(x - x_2) \ldots = 0, \qquad \text{or } \frac{dY}{dx} = 0.$$

If now our attention was confined to a single given set of observations x_1, x_2, &c., the problem of finding the best member of the θ family would not have much meaning. $\psi(x_1, x_2 \ldots)$ is the best value, and whatever value is arithmetically nearest this is best. It is only when we take account of the variations of value which x_1, x_2, &c., may assume consistent with their emanating from the same source, that the problem becomes significant. Now for any particular set of x_1, x_2, &c., the frequency with which an error of the exact extent $+e$ is committed by putting $\theta(x_1, x_2 \ldots c_1, c_2)$ for the quaesitum is

$$F(\theta - e) \div \int F(x) dx,$$

between limits including all the positive values of $F(x)$, where $F(x) = Y$. In order therefore to find the frequency of the error e in the long run, we must integrate

$$\frac{F(\theta - e) f(x_1) f(x_2) \ldots f(x_s)\, dx_1 dx_2 \ldots dx_s}{\int F(x) dx},$$

between extreme limits of $x_1, x_2 \ldots x_s$. The result, say $\chi(e, c_1, c_2 \ldots)$, gives the law of facility for the error committed in the long run by using the method of reduction θ. Now there is no ground for believing that the result of this corrected process is, in general, or in the particular case when $\theta = c_1 x_1 + c_2 x_2 +$ &c., identical with the result of the method of least squares, at least when the number of observations, though it may be considerable, is finite; for, if the number be really infinite, of course* any method of reduction will be admissible. Indeed it does not seem to be contended that plurality of observations makes the second step of the method of least squares more *secure;* but only more *practicable,* by allowing the *law of error* to be brought to bear upon it. The discrepancy between the method of least squares and the proper method may therefore be illustrated by a simple example. Let there be two observations x_1 and x_2; resulting from two facility-curves, $y = \frac{3}{4}(1 - x^2)$ and $y = \frac{3}{4}h(1 - h^2x^2)$, extending respectively between limits ± 1 and $\pm \frac{1}{h}$.

(1) According to the principle of the method of least squares, we have to find the facility-curve under which $\lambda e_1 + \mu e_2$ ranges; where e_1, e_2 are values taken at random from under the given facility-curves respectively, and λ, μ are undetermined multipliers whose sum $= 1$. To effect this operation we may proceed thus. First find the facility-curves under which λe_1 and μe_2 respectively range. They are respectively

$$\frac{3}{4} \frac{1}{\lambda} \left[1 - \frac{z^2}{\lambda^2}\right], \text{ between limits } z = \pm \lambda,$$

and

$$\frac{3}{4} \frac{h}{\mu} \left[1 - \frac{h^2}{\mu^2} y^2\right], \text{ between limits } y = \pm \frac{\mu}{h}.$$

* Cp. Glaisher, *op. cit.*

Digitized by Google

Then, to find the range of values $y+z$, proceed upon the principles laid down on a former page*. Integrate the expression

$$\frac{3}{4}\,\frac{1}{\lambda}\left(1-\frac{z^2}{\lambda^2}\right)\times\frac{3}{4}\,\frac{h}{\mu}\left[1-\frac{h^2}{\mu^2}(e+z)^2\right]$$

between proper limits, and having regard to the sign of the odd terms in z. (The integral breaks up into three parts: from $\frac{\mu}{h}$ to e, from e to 0, and from λ to e.) The resulting expression will be the ordinate of a facility-curve in e, $\frac{3}{4}H(1-H^2e^2)$, representing the frequency of the error $\lambda e_1+\mu e_2$. This curve will evidently be least dispersed when H is a maximum, or $dH=0$.

(2) The inverse-proper method puts

$$Y=\tfrac{3}{4}\{1-(x-x_1)^2\}\times\tfrac{3}{4}h\{1-h^2(x-x_2)^2\}.$$

The frequency of error e incurred by putting $x=\lambda x_1+\mu x_2\,(\lambda+\mu=1)$ is for a particular system of x_1 and x_2,

$$\frac{\frac{3}{4}[1-\{(\lambda-1)x_1+\mu x_2-e\}^2]\times\frac{3}{4}h[1-h^2\{\lambda x_1+(\mu-1)x_2-e\}^2]}{\int Y dx},$$

where Y is integrated between its extreme intersections with the ordinate. To find the frequency of the error e in the long run the above expression must be multiplied by $\frac{9}{16}(1-x_1^2)\,h\,(1-h^2x_2^2)$ and integrated with regard to x_1 and x_2 between limits ± 1, $\pm\frac{1}{h}$ respectively. There is evidently no correspondence between the expression in $\lambda:\mu$ obtained by this procedure and that of the method of least squares.

The value of the method of least squares will be more fully appreciated by marking the different degrees of its applicability. (α) It is theoretically right in the case of only a *single* observation being given. Suppose we had to make with a certain instrument an observation not capable of being repeated. The frequency and extent of error incurred by taking on each different occasion a single observation as the true value of the thing measured would exactly be represented by the facility-curve obtained by continued measurements of one and the same thing; *provided* at least we assume that the *à priori*† probability of the observed value being the true value is constant—a proviso which removes the only advantage which might seem to remain to the "assumed" as compared with the proper inversion. It should be observed that this case does not cover Laplace's application of his method to Bayes' theorem. If m white balls and n black have been drawn, then taking $m:n$ as the *real* proportion of the balls in the urn is doubtless analogous to taking a single observation in space or time as the true value of the thing measured. But the analogy fails in this respect that, when the real ratio of balls is different from $\frac{m}{n}$, the *real modulus* also is different from what it was in the case of $\frac{m}{n}$ being the real ratio. Whereas in our illustration we had supposed an instrument of constant precision.

* Above, p. 141.

† In one sense of the term, see above, p. 148 et sqq.

Digitized by Google

(β) A second case is that in which the facility-curves are of the form $\frac{1}{c\sqrt{\pi}} e^{-\frac{x^2}{c^2}}$. In this case the method of least squares* may be said to be accidentally right; agreeing as it does with inversion proper†. It is submitted that the apparent verification of the method of least squares by the practice of observers is due to the prevalence of probability-curves *in rerum naturâ* at least in an approximate form.

(γ) In a third case, the second, the insecure step of the method of least squares may be dispensed with. This is the case‡ where all the facility-curves are identical. In this case, if we may assume that the solution is an arithmetical mean, we may assume that it is the simple arithmetical mean—by a principle of symmetry which seems justified by precedent. As there is no reason why one weight should be different from another we may treat all the weights as the same. Nevertheless this reasoning must be admitted with caution. How do we know that it is desirable to take account of all the observations that, e.g. the best reduction of x_1, x_2, x_3 is not (the symmetrical one) *either*

$$\frac{x_1 + x_2}{2} \text{ or } \frac{x_1 + x_3}{2} \text{ or } \frac{x_2 + x_3}{2}?$$

There will be given below § instances in which the mean is the most improbable value. The instances no doubt are of *infinite* facility-curves; but this property may be supposed retained while the curves are slightly deformed so as to become finite. In such cases the average might not be the most accurate arithmetical mean.

(δ) The second step of the method of least squares seems not to be indispensable in one other case: when the facility-curves consist of clusters‖, the members of each cluster being identical. By parity of reasoning with the last case the solution may be written

$$\frac{\lambda(x_1 + x_2 + \&c. + x_s)}{s} + \frac{\lambda'(x_1' + x_2' + \&c. + x'_{s'})}{s'} + \&c.$$

Now since by this formula we restrict ourselves to taking account of the *average* of each cluster of observations the problem seems reduced to the following. Regarding each average $\frac{x_1 + x_2 + \&c.}{s}$ as a single observation ranging under a corresponding facility-curve to find the most probable value (or at least the most accurate arithmetic mean).

Now the quantities $\frac{x_1 + x_2 + \&c.}{s}$ are known by the *law of error* to range under probability-curves whose moduli are the respective inverse mean-squares. Whence the solution of the modified problem is the same as the solution of the original problem afforded by the method of least squares. Nevertheless the modified problem is not the original problem. We have no security that considering the whole series of sets x_1, x_2...; x_1', x_2'...&c. the arithmetical mean above found is the one which is the least inaccurate.

* Meaning, as throughout, the indirect method above characterised; for of course the direct (inverse-proper) method may in this particular case be called a method of least squares, since its solution is the minimum of a certain sum of squares ($-\log Y$).

† $abc\bar{d}e\bar{f}\bar{g}h$.

‡ $abc\bar{d}\bar{e}fgh$.

§ See p. 159 and note.

‖ The case considered by Laplace at the end of Art. 23, chap. IV.

Digitized by Google

For instance, let the facility-curves be of the form Je^{-Rx^t}. Is there any reason for thinking that the weights prescribed by the method of least squares, namely $s\sqrt{R}$, $s'\sqrt{R'}$, are better than the weights prescribed by the approximate* inverse-proper solution; namely sR, $s'R'$?

(ϵ) Last is the general case in which it is here contended that the method of least squares is neither theoretically correct nor practically useful.

abcdēfgh̄. In case of asymmetry the method of least squares, though equally *practicable* in virtue of Poisson's extension of the *law of error*, becomes more *insecure*—as Poisson† himself seems to indicate.

abcdēf̄ḡ. When it is not given that the weight of all the observations is the same, then the analogy of probability-curves negatives Airy's‡ principle that we should treat the observations as all of the same weight and confirms the general principle, though not the particular practice, of De Morgan§, Peirce‡, and Glaisher‖.

Here for the present may be terminated our discussion of class *d*. It will have been observed that in the discussion of these problems we have tacitly assumed that the quæsitum is the *most probable* value of the unknown quantity. Given a set of observations x_1, x_2, &c., we have shown that in the long run the real point, from which a set of observations at those relative distances results, occurs at different points with a frequency indicated by the facility-curve $Y = f(x - x_1) \times f(x - x_2) \ldots$. And we have taken for granted that the function of x_1, x_2, &c. which corresponds to the maximum ordinate of that curve is the right method of reduction. This point of view is doubtless natural and obvious and recommended by high authorities. But the more philosophical view, theoretically more correct if not practically different, is that we should select not that value in terms of x_1, x_2, &c. which is most frequently the true one¶ but that one which is *most advantageous, account being taken not only of the frequency but also the seriousness of error*, that value which minimises the detriment of error. Now this quæsitum is of the nature of a *subjective mean*. Its discussion had better be deferred till we come to $\bar{a}$.

abc̄. The case of facility-curves, other than probability-curves, extending to infinity appears to be rather of theoretical interest than practical importance. The general type of reproductive facility-curves *abc̄e*,

$$\frac{k}{\pi}\int_{-\infty}^{\infty} e^{\alpha^t} \cos \alpha k x \, d\alpha,$$

presents the remarkable property that, when t is less than unity, the average of s observations of the same weight ranges under a facility-curve not less, but more, divergent than the primary curve. If the weight (the constant k) is different for different curves, the weighted mean $g_1x_1 + g_2x_2 + \&c. \div sg$, where $g \propto$ the inverse mean $\frac{t}{t-1}$th power, is according

* Above, p. 151.

† *Connoissance des Temps*, 1832, §§ 8, 9. In order to render the method applicable, it would have to be granted that each curve of error has the point of no error as its centre of gravity. For the case where this is not granted see below, p. 168.

‡ *Astronomical Journal*, Cambridge (America), Vol. II. p. 161.

§ *Encycl. Metrop.*, § 136.

‖ *Memoirs Astron. Soc.*, XL. p. 103.

¶ Cp. Glaisher, *Mem. Astronom. Journal*, XL. p. 101.

Digitized by Google

to the method of "assumed inversion" the arithmetical mean of, not maximum, but *minimum*, accuracy, when t is <1. It would be interesting to compare this conclusion of assumed inversion with that of inversion proper. The comparison is easy in the case where $t=1$ (the bounding case between convergent and divergent averages). In this case according to the assumed inversion there is neither gain nor loss in proceeding to an average*. According to inversion proper for certain relative situations of the observations the average is a *most*, for other data it is a *least*, probable reduction. If there be two observations whose law of facility is $\frac{1}{\pi}\frac{1}{1+x^2}$ the most probable value is $\frac{x_1+x_2}{2}$, provided that $\frac{x_1-x_2}{2}<1$. But for higher values of the distance between the observations the average is a position of minimum probability. The most probable value is

$$\frac{x_1+x_2}{2} \pm \sqrt{\frac{(x_1-x_2)^2}{4}-1}\dagger.$$

The curve $\frac{1}{2}ke^{-kx}$ may also be instanced as presenting a contrast between proper and assumed inversion. The proper solution is that value of x which makes $sk(x-x_1)$ (all these differences taken positively) a minimum—in fact Laplace's Method of Situation. The "assumed" solution is very different.

$a\bar{b}$. The difficulties introduced by plural means, as they are of a purely mathematical character, so they seem to have been completely overcome by the powerful mathematicians who have attacked them.

$a\bar{b}c\bar{d}e\bar{f}$. A passing notice is demanded by one case—very important both in physical observations and social statistics—which might be referred to this class. Given two groups (Fig. 7) apparently ranging respectively about two means, to determine the probability that

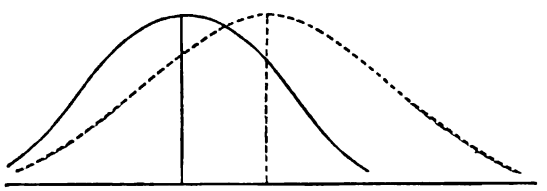


FIG. 7.

this apparent difference corresponds to a real difference of any assigned extent. The analogy of previous reasonings shows that the error arising from treating the apparent as the real difference ranges not under a probability-curve, as usually reasoned, but under‡ a certain non-exponential curve—that is assuming (as is usual) that the two groups of data may be regarded as having diverged each according to a probability-curve.

$\bar{a}b$. We come now to the *subjective Mean*. This mean is selected, not because it most truly represents one real thing outside the data§, but because it most advantageously

* Poisson notices this case (*Connoissance des Temps*, 1827).

† It may be observed that this property of *discontinuity* in the most probable method of reduction may extend to finite curves. This property modifies the remark made by Laplace and followed by De Morgan and Mr Glaisher that the most probable solution for *two* observations is their average.

‡ See above, p. 149.

§ Cp. Article on Probabilities, Sect. v. *Encyclopæd. Britann.* Ed. 9th.

Digitized by Google

represents a whole set of things, the data themselves. Different means may of course be selected for different purposes. The same mode of meaning need not be adopted by the *entrepreneur* basing his estimate upon average prices, and the actuary calculating mean lives, so that on the one hand his office in the long run may profit, and on the other hand it may not be cut out by some rival office.

We are to suppose a variety of values ranging under a facility-curve to occur on different occasions; and we have to determine that one value which may with least detriment be put for the whole set of fluctuating values. For general practical purposes it may perhaps be postulated that the detriment incurred by the assumed mean proving too small on any occasion is equal to the detriment which is incurred when it proves too large to the same extent. The problems thus arising may be arranged under the same headings as those relating to real means.

ābcde. First is the case of a set of statistics so complete as to present by a valid induction the law of their recurrence: namely, arrangement under a probability-curve. (The subdivision into f and $\bar{f}$ is not, as was remarked with reference to *ad*, of much importance.)

ābcdefgh. Put l for the sought subjective mean. And let the detriment incurred by putting on any occasion l for a thing whose real measure is x be $F(x-l)$; where F is a symmetrical function (agreeably to the hypothesis lately made), $F(0)=0$, F' is continually positive, but the form of the function F is otherwise unknown. The difference between x and l is to be taken always as positive. Then, if y be the probability-curve under which the statistics range, the detriment incurred* in the long run by putting l on each occasion for the true value is

$$\int_l^\infty F(x-l)\,y\,dx + \int_{-\infty}^l F(l-x)\,y\,dx = \text{say } E.$$

To find the value of l for which this detriment is a minimum, we have to equate $\frac{dE}{dl}$ to zero and observe the sign of $\frac{d_2E}{dl^2}$.

$$\frac{dE}{dl} = -\int_l^\infty F'(x-l)\,y\,dx + \int_{-\infty}^l F'(l-x)\,y\,dx$$

+ terms† outside the sign of integration which vanish.

y being symmetrical about the point $x=0$, $\frac{dE}{dl}$ vanishes when $l=0$.

To find the second term of variation, we may either differentiate at once with regard to l, or first transform $\frac{dE}{dl}$ as thus (by partial integration as to x):

$$\frac{dE}{dl} = -\Big[F(x-l)\,y\Big]_l^\infty + \int_l^\infty F(x-l)\frac{dy}{dx}\,dx$$
$$\qquad - \Big[F(l-x)\,y\Big]_{-\infty}^l + \int_{-\infty}^l F(l-x)\frac{dy}{dx}\,dx,$$

* This theory is illustrated by Laplace's reasoning at p. 365 (national ed.), p. 333 (third ed.), *Théorie Analytique;* though his reasoning there relates, not to *ābcd*, but, as will be pointed out presently, to *abcd̄*.

† Cp. Laplace, *loc. cit.*

Digitized by Google

(which vanishes as it ought when $l = 0$, since $F \times y = 0$ for $x = 0$ or $\pm\infty$*, and $\frac{dy}{dx}$ has opposite signs for x above and below zero). Hence

$$\frac{d_2E}{dl^2} = {}_l^{\infty}\Big[F'(x-l)\,y\Big] - {}_{-\infty}^{l}\Big[F'(l-x)\,y\Big]$$
$$-\int_l^{\infty} F'(x-l)\frac{dy}{dx}\,dx + \int_{-\infty}^{l} F'(l-x)\frac{dy}{dx}\,dx.$$

The portion without the integral sign vanishes, and the remainder is positive (since every element under the sign of integration is positive). Therefore E is a minimum when $l = 0$. Therefore the central point of the curve is the sought Mean.

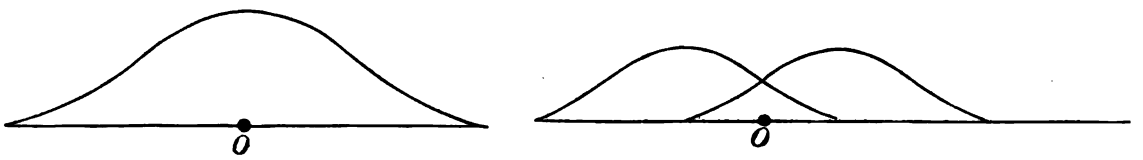
$\bar{a}bcdefg\bar{h}$ (the case of asymmetry) considered as a variation from the preceding case is seen to require a deflection from the centre in the direction of the preponderating side of the curve.

$\bar{a}bcde\bar{g}$ hardly occurs. There may be two groups of statistics, each of which has a different modulus. But, if they have the same mean, no new problem arises; and, if they have different means the case rather belongs to class $\bar{e}$; to which we proceed.

$\bar{a}bcd\bar{e}$: under this heading four cases deserve special notice:

(α) The case of *any* symmetrical disposition of the statistics. By reasoning similar to that which has just been employed, the centre is a *critical* point. It is one of minimum detriment if the arrangement is a simple continuous facility-curve (zero at the centre and extremities) Fig. 8 (1). It is very likely to be one of maximum detriment, when it lies in a trough between two elevations as in Fig. 8 (2).

(1) FIG. 8. (2)



(β) Where the statistics may be arranged under two probability curves, or more generally any simple continuous facility-curves of type $y = \frac{1}{c} f\left(\frac{x}{c}\right)$, not having the same centre nor modulus; and practical exigencies require that one and the same Mean should be put for the whole system. Let us first consider the case in which the centres of the two curves are very close together, as in Fig. 9.

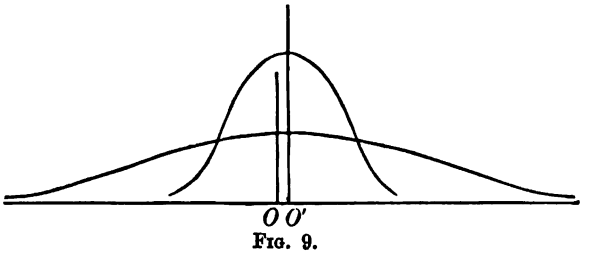


FIG. 9.

* For, if $F(\infty) \times y_\infty$ were finite, the detriment incident to any subjective mean would be infinite: *cadit quæstio.*

Digitized by Google

If we suppose the quæsitum to be for a moment at the point equidistant from the two centres it will tend to move in that direction in which the increment of detriment is least. We can determine then in the direction of which centre it will move by comparing the values of the quantity $\frac{d_2E}{dl^2}(l=0)$ (see p. 162) for each curve.

The relative magnitude will depend upon the nature of the function of detriment (as well as upon the moduli of the curves). If the curves be of the form $\frac{1}{\sqrt{\pi}}he^{-h^2x^2}$ it may be shown that, if the function of error is a power (or sum of powers) less than the second, then $\frac{d}{dh}\frac{d_2E}{dl^2}$ is continually +. Therefore $\frac{d_2E}{dl^2}$ is largest for the largest h and the mean gravitates towards the centre of the weightier curve.

If the distance between the centres is finite, we have to consider for which curve $\frac{dE}{dl}$ is greatest, l being the half distance between the centres.

(γ) A third subcase is that of statistics diverging by a very small range, a very contracted facility-curve. In that case we may be allowed to write for the detriment $F(l)$, $F'(0)\times e$ the error taken positively and multiplied by a constant. The typical mean is then such that $\Sigma(x-x_1)$, the quantity within the bracket always positive, should be a minimum—Laplace's *method of situation;* of which this reasoning is submitted as the rationale. It is submitted also that this reasoning throws light on the fundamental assumptions made by Gauss and Laplace on the function of detriment. If we suppose the function F to involve e in the first power (against which there seems no valid objection) then Laplace's assumption* of the *mean error* tends to be approximatively correct; but Gauss' *mean square of error*, if F is a function of l^2, is preferable.

(δ) Another general remark is that the Mean describable as the Greatest Ordinate appears entitled to great consideration; especially in the case of amputated curves such as the anthropometrical statistics of an army (men below a certain height or strength having been rejected); or† where the Arithmetical mean and even the Median are liable to be displaced by the admixture of returns foreign to the type with which we are concerned, yet not capable of being specified and set apart.

Next in order comes the subjective mean pursued by the way of inverse probability. But at this point it seems advisable to return to a problem in Observations which we had to postpone‡, because it involved as a proximate end, a secondary quæsitum, a fictitious or subjective Mean. Let us look back then from our new point of view at the problem *abcdf*. Let us begin with the case of a *single* observation§, partly because it is the simplest, and partly because in this case only (as above contended) is Laplace's "à priori" method of least squares perfectly satisfactory‖. In this case only then will

* See Gauss, *Theoria Combinat.*, Part I., § 6.

† The use of the Greatest Ordinate Mean is well exemplified in the statistical writings of Dr Charles Roberts.

‡ See p. 158, above.

§ See the present writer's remarks in *Phil. Mag.*, Nov. 1883.

‖ See above, p. 156.

Digitized by Google

our exposition of the most advantageous value coincide in principle with the 'most advantageous' first considered by him; namely in that second definition* of the quæsitum (in the case of "observations non faites encore") which has been a stumbling-block† to many.

Suppose then (as above) that we had to content ourselves with a single observation made with an instrument. Suppose we had our choice between two instruments; and that the accuracy of each could be tested by long previous experience. Let that experience be expressed by the two facility-curves in Figure 9, which instrument is to be

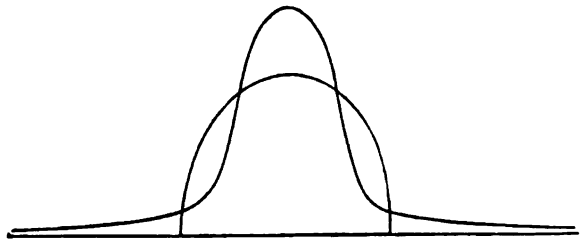


FIG. 9.

preferred. If we take as our end the *most probable* value in the sense so clearly defined by Mr Glaisher, undoubtedly the instrument whose curve is more contracted at the centre (though more expanded at the extremities). But it may well be that the other instrument would be in the long run *more advantageous*, account being taken of the extent, as well as frequency, of error. In the notation above adopted, that instrument is to be preferred for which‡ $2\int_0^\infty F(x)\,y$ is the least. This agrees with Laplace's second view of the quæsitum in his method of least squares; except that for the general function F he puts the particular function x (the mean error)—an often legitimate approximation, as we have just seen.

When the number of observations is plural, we must part company with the method of least squares; which can more clearly be contemplated from the present point of view. The principle of its second step as explained§ above is to test a proposed *method of reduction* $\theta(x_1, x_2 \ldots)$, as we would test an instrument which presented a series of errors respectively equal to

$$\theta(x_1, x_2 \ldots), \qquad \theta(x_1', x_2' \ldots), \text{ \&c.}$$

(where $x_1, x_2 \ldots\ldots, x_1', x_2' \ldots\ldots$, &c. are random sets of values taken from under the facility-curves of the observations). Accordingly this generic principle has two species according as we prefer that measure which affords the *most probable* value, to which our attention under a former heading was confined, or, what now comes into view, the *most advantageous* value. It is here contended that, if the generic principle could be accepted, then the *most advantageous* species would be the philosophically preferable. Mr Glaisher's strictures on that‖ species do not seem deserved.

* *Théorie Analyt.*, ch. IV. § 20, p. 318, third edition.

† Cp. Glaisher, *op. cit.*, p. 101.

‡ Assuming that the function of detriment is symmetrical.

§ P. 152.

‖ *Memoirs of the Astronomical Society*, XL. *loc. cit.*

Digitized by Google

But the generic principle cannot be accepted. And therefore we pass on to inversion proper; which also is divided into two species, according as the quæsitum is the *most probable* or the *most advantageous* value. It is remarkable that Laplace in his general account of the *à posteriori*, or proper inverse method seems to have confined himself to the* *most advantageous* species. The procedure there adopted by Laplace is submitted here as the most philosophical treatment of the inverse problem—with this modification that, whereas Laplace employs a particular function namely x (mean error) to express the detriment of error, the general undetermined symbol F is here recommended. The inverse investigation of the *most advantageous* mean is exactly parallel to that which has been given under† *ābcde* except that Y (Laplace's y') now denotes, not the facility-curve (obtained by simple induction from the very large set of statistics x_1, x_2 .....) expressing the frequency with which members of a group of real things have each value x; but the facility-curve (obtained by inverse probability from the not very large set of observations $x_1 x_2$ .....) expressing the frequency with which the real thing behind the observations has each value x. We find by inverse probability statistics for the occurrence of the cause of the given observations; and we take the subjective mean of those statistics.

The theoretical superiority of the *most advantageous* method is accompanied with a practical difference in the determination of the modulus. We have seen that the modulus for a set of observations (supposed to have diverged according to a single unknown law of probability) has different values with a frequency expressed by the curve

$$y = \frac{1}{\pi^{\frac{n}{2}} c^n} e^{-\frac{S\epsilon^2}{c^2}}$$

where $S\epsilon^2$ is the sum of squares of apparent errors. Whence it follows that the *most probable* value of c is $\frac{2S\epsilon^2}{n}$. But it is probably not the *most advantageous* value. For *from the asymmetry*‡ of the curve y it is probable that the most advantageous is greater than the most probable value. Thus the received solution $\frac{2S\epsilon^2}{n-1}$, though it has no pretensions to precision, yet differs in the right direction from $\frac{2S\epsilon^2}{n}$. The ordinary formula is to that extent correct, but not for the ordinary reason.

Having now discussed the application of the simply inductive subjective Mean (*ād*), to the inverse real Mean (*ad*), we may return to the point§ at which our exposition of pure statistics was interrupted and consider the inverse statistical mean.

ābcdefgh. The simplest case is that of a not very large set of statistics considered as random specimens of a whole series which ranges under a probability curve of known modulus

* *Théorie Analytique*, p. 333, third edition, p. 365, national edition.

† Above, p. 160.

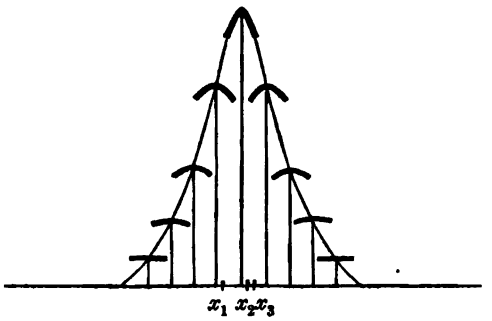
‡ Above, p. 161.

§ See above, p. 162.

Digitized by Google

but unknown centre. One way of treating the problem is—knowing that the centre of the real series of which the given statistics are specimens is the most advantageous Mean—to

Fig. 10.



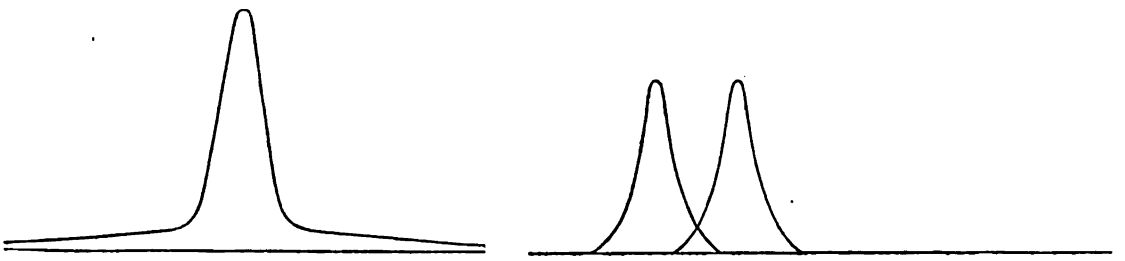
determine by inverse probability the frequency with which the centre of the real series has each value x, and to take that value of the centre which most frequently, *most probably*, occurs. But, it is submitted, the more philosophical procedure—theoretically preferable, though not practically different—is to regard the given statistics x_1, x_2, x_3 ... as indicative of a whole series of groups of statistics whose size and centre is indicated by the height and position of the ordinates of the probability-curve*

$$\frac{1}{\sqrt{\pi s c}}\, e^{\left(x-\frac{x_1+x_2+x_3+\&c.}{s}\right)^2 \div c^2}.$$

We have then to take the subjective mean of this series of series upon principles similar to those already explained. This distinction becomes clearer when the two points of view are no longer coincident, as in the next case.

ābcdēfg. To exhibit the difference just indicated suppose the facility-curve under which the (complete) series from which our samples are selected ranges is changed from the form of Fig. 11 (1) to that of Fig. 11 (2), while the quantity $\int f(z)\, z^2$ (between extreme limits) is kept

(1) Fig. 11. (2)



constant. Then, if we are employing the arithmetic mean, the worth of the mean in the case of observations is unaltered; but in the case of statistics is impaired. For the central

* The area of the curve whose central point has the abscissa x is $erf.\left(\frac{\sqrt{s}}{c}x\right)$. The length of each curve's central ordinate is the area divided by $\sqrt{\pi}c$.

Digitized by Google

point of a series such as that indicated by Fig. (2) has a representative character in a less degree than the central point of Fig. (1). Hence in the course of a series of series*, if we continue taking the arithmetical mean, though we shall as often be as near the real central point as before the supposed change, yet our Mean will no longer have the same worth.

It appears then that the excellence of the Mean under consideration depends upon two qualities: the *precision* with which our Mean of the given samples approximates to that point which, if the series from which the samples are taken were indefinitely prolonged, would constitute a certain assigned Mean of that series, a certain function of its terms; and the representative character of that point. As to the second quality there is nothing to be added under this heading to what has been said† under the heading of Induction. The first quality is an excellence common to Statistics and Observations. And indeed at this point it is very difficult to determine how far an element of Observations enters into inquiries commonly called statistical; how far the sought Mean is purely "subjective," or also an "index"‡ according to the just distinction drawn by§ Dr Bertillon; from whom the former term has been adopted into this paper. Accordingly the following remarks on the worth of Means in respect of precision cover certain cases of Observations transferred here from their proper place.

It follows from the discussion at pp. 153—8 that it is in general hopeless to enquire what is the *best* Mean in respect of precision. All that we can say is that certain Means are *better* than certain others.

For the purpose of comparison Means might be classified as (1) The Arithmetical Mean, (2) Two Means which compete with the Arithmetical in respect of convenience, (3) All other Means.

The third class may in general in the absence of special reasons to the contrary be rejected on the ground of convenience.

The battle of the Means becomes triangular. It is to be decided by comparing the first with each of the others, and these with one another, on the ground both of precision and convenience||.

(α) The prerogative of the Arithmetical Mean, that it is the more accurate value in the case of probability-curves, extends of course to curves in the neighbourhood of that family, and probably far beyond. Take for example the facility-curve $y = \frac{1}{h^2}(h - z)$ (z positive both ways), Fig. 12. The arithmetic mean of s observations ranges under a probability-curve whose modulus squared is $\frac{1}{3}h^2 \div s$. The Median of s observations ranges under a probability-

* Above, p. 165.

† Above, pp. 161, 162.

‡ The subject of the writer's paper on Methods of Statistics is: "What is the worth in respect of precision of an assigned index-mean"?

§ *Dict. Encycl. des Sciences Médicales.* Cp. Mr Venn on the Law of Error in *Logic of Chance*.

|| Saving of labour in the working; a consideration which of course enters into every practical approximative operation.

Digitized by Google

curve whose modulus squared* is $\frac{1}{2}h^2 \div s$. The former is therefore more accurate than the latter. There is however one case in which the Median has the advantage in respect of

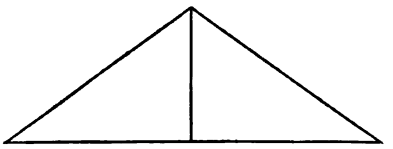


FIG. 12.

precision: namely, when the apex of the curve is very high and its extremities very much extended, as in Fig. 13.

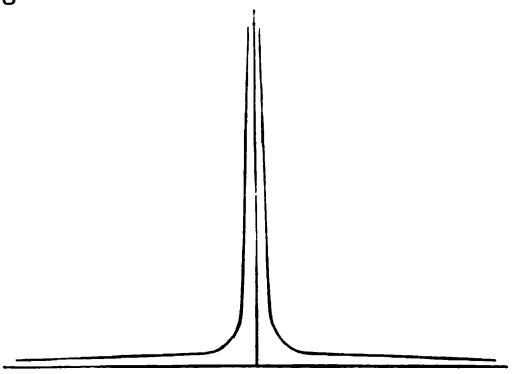


FIG. 13.

For in this case the mean square of error, which is proportional to the modulus-squared which measures the precision of the Arithmetic Mean, is likely to be very great. At the same time the inverse-square of the greatest ordinate, which is proportional to the modulus-squared† appertaining to the Median will be very small.

On another ground the Median has the advantage in respect either of precision or of convenience, or perhaps both. Suppose that the data are measurements made with a measure not minutely graduated, e.g. the numbers of men whose height lies between each two consecutive inches. To obtain the real arithmetic mean (of statistics taken from under a continuous facility-curve) it would be necessary to adjust each of the given figures by smoothing out our data into a continuous curve. But to obtain the real Median it would be necessary to operate upon only one figure by somewhat hypothetically dividing the central compartment.

Where no reduction is thought necessary, the Median has an obvious and not‡ unnoticed advantage in respect of convenience. It has also an unsuspected advantage in that the *weight* of the Arithmetic Mean, the inverse (double) mean square of error, is ascertained by an often very tedious process; whereas the weight of the Median the (double) central ordinate squared is very easily ascertained. Considering that it is frequently, if not generally, more important

* Laplace, *Théorie Analytique*, Supplement 2, sect. 2.

† *Ibid.*

‡ Galton in *Phil. Mag.*, 1875. Fechner in *Abhandlungen Saxische Wissenschaftliche Gesellschaft*, Bd. 18. Fechner, complaining that Laplace had neglected the Median, seems to ignore that Laplace not only deduced the characteristic property of this Mean—that it minimises the Mean Error—but also employed it: not only by way of simple induction (our *d*) in the Method of Situation (*Mécanique Céleste*, III. 39), but also by way of Inverse Probability in the important passage already referred to: on the reduction of observations *déjà faites* (above, p. 160).

Digitized by Google

to know what the precision of a mean is, than that it is more precise than another, this last advantage in favour of the Median seems very important.

(β) Comparing the Arithmetical Mean with the Greatest Ordinate, we are met with the difficulty that the greatest ordinate has *no weight;* or at least that it has never been weighed. For the operation which Laplace has performed in the second supplement for the Median has not, as far as I know, been performed for the Greatest Ordinate. In view of the remark at the end of the preceding paragraph, the Greatest Ordinate appears in this respect to labour under a considerable disadvantage. In respect of convenience it has a considerable advantage over the Arithmetical Mean; and

(γ) a less marked advantage over the Median. In respect of the necessity of adjustment the Median and greatest ordinate may seem on an equal footing. As possessing a definite weight the Median has the same advantage as the Arithmetic Mean over the Greatest Ordinate.

$\bar{a}bcd\bar{e}fg\bar{h}$. The remarks made under the preceding head may be transferred here *mutatis mutandis.* The following are special to this head.

Here come in (by courtesy) observations about which, though we know the (unsymmetrical) facility-curve under which they range, we do not know at what point thereof* occurs the real thing indicated.

The difficulties connected with the process of adjustment are greater here for the Median and Greatest Ordinate; but so are they also for the Arithmetic Mean.

The *weight* above assigned for the Median requires a slight modification. By an extension of Laplace's analysis† it may be found that the Medians of several sets of s observations taken at random from an unsymmetrical facility-curve range under two probability-curves whose centres are distant from the Median of the parent facility-curve by a quantity of the order $\frac{1}{s}$, and therefore neglectible; and that the weight of these coincident curves (inverse modulus squared) is still in Laplace's notation $2s\overline{\phi(0)}|^2$; only that $\phi(0)$ means now the Median ordinate not the Greatest ordinate‡.

To exemplify this heading, let us take the following problem. The ages at death of a number, s, of men of and above the age of 66, distinguished from the general male population of the country by no other attribute§ than one which is the subject of investigation, have been registered. And it is required to determine‖ whether the difference between the mean age of the observed class differs from the mean age of the general population to an extent which is more than accidental, which is indicative that the observed class belong to a different category in respect of healthiness from the general population. To facilitate calculation let us suppose a stationary population.

* Above, p. 158.

† *Théorie Anal.*, Supplement 2. The p's of Laplace are to be put each equal to 1; his k equal to $\frac{1}{2}$.

‡ As Laplace says, speaking of symmetrical curves.

§ E.g. abstinence from liquor.

‖ See Introduction to a paper communicated by the writer to the British Association, 1885.

Digitized by Google

(α) If we employ the Arithmetic Mean, we must find the mean age at death of the general male population at or above 66 from the Life Tables, say for England 76 (more accurately 75·8). Then we must multiply each figure in the column d_x, by the square of the difference between the age corresponding to that figure and the mean age 76; and divide the sum of squares of errors thus found by the number of persons alive at the age of 66. The quotient 90 (approximately and without adjustment) multiplied by two and divided by s is the modulus squared for the probability-curve under which the arithmetical means of random selection of s lives would range. Hence if the arithmetic mean of the observed class differ from 75·8 by more than two (or three) times the quantity $\sqrt{\frac{90}{s}}$, it is practically certain that the observed class has a more than accidental difference in longevity from the general population from which it is selected.

(β) If we employ the Median, we must find the median age of the general male population, i.e. about 75·5 years. The proportion of men dying at this age out of the number living at 66 is about $\frac{75}{1438}$. Hence* for the modulus squared of the probability-curve under which the median of s random selections from the general population would range $\frac{1}{2s \times \left(\frac{75}{1438}\right)^2}$ or $\frac{184}{s}$. The weight of the Median is thus about half that of the Arithmetic Mean.

(γ) If we employ the greatest ordinate we must compare the age at which the greatest ordinate of the set of s observations occurs with 72 the corresponding age for the general male population. But we seem unable in the present state of the calculus of probabilities to determine the weight which should be attached to any assigned difference between the two greatest ordinates: how far such a difference is significant and not accidental.

$\bar{a}bcd\bar{e}f\bar{g}$. The subject special to this head is the relative weight of data emanating from different facility-curves. Weight is of two kinds; that which depends on volume only, and specific weight. To the former category belong statistical returns which are the means respectively of n_1, n_2, &c. constituents; all the constitutents emanating from one and the same facility-curve. Where n_1, n_2, &c. are each large, or where the whole number of given returns is large, this case is easily treated according to the methods applicable to probability-curves and the analogy of the two preceding heads.

As to specific weight, it follows from the discussion at pp. 152—7, that the system of weights which consists of the inverse-mean-squares, though a good, is not necessarily the best system. In general there seems possible no more precise determination of weight than the following. In the combination of data more weight is to be assigned to that which has emanated from a higher source: a higher source being one which in past experience has yielded results less wide of the mark. This principle is equally applicable to Physics and the Moral Sciences.

* Above, p. 167.

Digitized by Google

VIII. *On a new method of obtaining interference-fringes, and on their application to determine whether a displacement-current of electricity involves a motion of translation of the electromagnetic medium.* By L. R. Wilberforce, B.A., Scholar of Trinity College, Cambridge. (Plates III. IV. V.)

CONTENTS.

PART I.

General sketch of the experiments which have been made to decide whether an electric current involves a motion of the electromagnetic medium in the same direction, and of the consequences of such a connection.

PART II.

Summary of the methods which have been adopted for producing interference-fringes suitable for experiments of this nature.

PART III.

Theoretical and practical details of a new method of producing such interference-fringes.

PART IV.

Description of some experiments similar to the above which were made upon displacement-currents of electricity, and in which this new method of producing fringes was employed.

PART I.

According to the view of the electromagnetic field taken by Maxwell, there is a certain "ether" or "medium" pervading the whole of space, with which the molecules of ordinary matter are in some way associated or connected, and which is the seat of all electric and electromagnetic forces.

It is necessary carefully to distinguish between a portion of matter and that part of the medium which is at any given time associated with it.

The equations of electromotive force given by Maxwell:

$$\left.\begin{aligned} P &= c\dot{y} - b\dot{z} - \frac{dF}{dt} - \frac{d\Psi}{dx} \\ Q &= a\dot{z} - c\dot{x} - \frac{dG}{dt} - \frac{d\Psi}{dy} \\ R &= b\dot{x} - a\dot{y} - \frac{dH}{dt} - \frac{d\Psi}{dz} \end{aligned}\right\}$$

have for their experimental basis observations upon moving conducting circuits, and in the case first considered $\dot{x}$, $\dot{y}$, $\dot{z}$ are velocities of portions of conducting matter. It is next assumed that the equations are true generally, but it is not definitely stated whether $\dot{x}$, $\dot{y}$, $\dot{z}$ still refer to the velocities of the portion of matter at the point (xyz) or to those of the associated medium.

If we adopt the second of these alternatives we must assume at the same time that if any portion of conducting matter is set in motion the medium associated with it will move with the same velocity.

Digitized by Google

That the second alternative is the one which agrees with experiment is evident from the following considerations.

It has been proved by J. J. Thomson (*Phil. Mag.* April, 1880) that, on the assumption that light is an electromagnetic disturbance, its velocity in a substance which is moving in the direction of propagation of the light is increased by half the velocity of the substance if the first alternative be taken, by half the velocity of the medium associated with the substance if the second be taken. The distinction is not formally made in his paper, but it is obvious on considering the different meanings that the fundamental equations have in the two cases.

Now, if air be the substance, it has been established by Fizeau (*Ann. de Chim. et de Phys.* 3me serie, t. LVII., 1859, also *Comptes Rendus*, XXXIII.) that the change of velocity, if it exists at all, is very much smaller than this.

Thus the first hypothesis is rendered untenable.

Fizeau further found that if water be the substance the change of velocity is about half the velocity of the water.

The conclusion is that the velocity of the medium associated with moving air is small compared with the velocity of the air, while in the case of water the substance and the medium move with sensibly the same velocity.

A repetition of Fizeau's experiments with the employment of as many different fluids as possible would probably throw some light on the way in which the refractive index of a substance depends upon the intimacy of connection between the substance and the associated medium.

The important questions of what changes of properties in the medium give rise to electric polarization and to electric currents have not yet been answered.

The first experiments made for the purpose of deciding whether an electric current consisted of a motion of translation of the medium were undertaken by Roiti (*Pogg. Ann.* Vol. 150, p. 164, 1873). He was compelled to employ an ordinary Arago's interferential refractometer, which, as will be seen, is not a very suitable instrument for the purpose, and he could find no change in the velocity of light passing through a solution of sulphate of zinc due to a current of electricity flowing in the same direction. Further experiments on the same lines were made by Lecher (*Rep. de Phys.* 20, p. 151, 1884). He also seems to have employed a refractometer identical with Arago's in principle and general arrangement. The substance that he used was a solution of silver nitrate through which he passed a current of about six ampères, but his results were also negative.

In these experiments the current was electrolytically conducted, and it is clear that the case of ordinary conduction cannot be treated by this method, since substances of good conductivity are opaque to light.

There is however according to Maxwell another kind of current which consists of variation of electric "displacement" in a dielectric, and it seemed antecedently probable that such a current should be accompanied by, if not equivalent to, a motion of the

Digitized by Google

medium. It must be remembered that by the term "displacement" Maxwell meant nothing more definite than a certain change at each point of the medium about which no more is asserted than that it is a vector one. The simplest vector change is a change of position, and the experiments that I have undertaken have been made with the object of deciding whether this "displacement" was an actual shifting of the medium, and if so, of what magnitude.

A certain probability is given to this view of the question by the fact that electric "displacement" obeys the same law of distribution as the motion of an incompressible fluid (the motion of the medium being generally assumed to possess the same characteristic), but it cannot at the same time be denied that Roiti and Lecher have by their experiments considerably discredited it, for, as we consider all electric currents as moving in closed circuits, in the case of a circuit of which part is a conductor and part an electrolyte, or in the case in which we have the plates of a condenser connected by means of an electrolyte, we can hardly imagine that the medium should have a permanent or temporary motion along a part of the circuit, and none along the remainder.

There is however one consequence of this hypothesis of a connection between currents and motion of the medium which I am surprised to find has not before been noticed.

If an electric current in a conductor is accompanied by a movement of the medium in the same direction, remembering that the velocities in Maxwell's equations of electromotive force are the velocities of the medium, we see that an electromotive force will be produced in the conductor if it be placed in a magnetic field, which force will be proportional to the strength of the field within the conductor and to the velocity of the medium, and will be alike perpendicular to the lines of magnetic force and to those of flow of the current. This electromotive force has been observed by Hall (*Phil. Mag.* 5, Vols. IX. and X. 1880) in various metallic conductors, but his results cannot be uniformly explained by the hypothesis of a moving medium, since in that case the substance of which the conductor was made would have no influence upon the phenomenon except through its magnetic permeability, which was far from being the case.

Again, if a "displacement current" in a dielectric be accompanied by a movement of the medium, we deduce at once the equations which Rowland (*Phil. Mag.* 5, Vol. XI. 1881) by assuming Hall's effect obtained for the propagation of an electromagnetic disturbance in a dielectric along lines of magnetic force, and arrive in the same way at the fact of the magnetic rotation of the plane of polarization of a ray.

Consider a current (u) flowing along the axis of x, and let the magnetic induction (c) be along the axis of z, then, if the current (u) be accompanied by a motion of the medium with velocity ($\varpi . u$) in the same direction, we have, for the electromotive force,

$$P = 0,$$
$$Q = -c . \varpi . u,$$
$$R = 0.$$

According to Hall $\frac{Q}{cu}$ for gold is in C.G.S. units 6.6×10^{-18}.

Digitized by Google

Hence $$\varpi = -6.6 \times 10^{-18}.$$

Thus if the current used were one ampére for each square centimetre of cross section of the conductor the velocity of the medium deduced from this would be

$$6.6 \times 10^{-14} \text{ centimetres per second}$$

in the direction opposite to the current. The change in the velocity of light due to adding or subtracting half this velocity is too small to be experimentally recognizable.

From the numerical results given by Rowland we see that if we deduce the value of (ϖ) from the magnetic rotation in heavy glass we obtain a result not widely differing from the above. There are doubtless various causes that combine to produce Hall's effect, as has been pointed out by Shelford Bidwell, and no importance can be attached to the value of (ϖ) deduced from it, but the fact that the existence of such an effect follows from supposing a motion of the medium to accompany an electric current appears to me to deserve notice.

A calculation of the "velocity of electricity" has been given by Boltzmann (*Kaiserliche Akad. der Wissenschaften in Wien, math-naturw. Classe*, Jan. 15, 1880, pp. 11—13, the note is translated in *Phil. Mag.* 5, Vol. IX. p. 307), in which he assumes that a conductor carrying a current is charged with electricity of a certain density, that the electromagnetic force it experiences is the result of Hall's electromotive force acting on this electricity, and that the current is its motion along the conductor. It will be seen that Boltzmann's method has nothing to do with that given above, and his assumptions have been well shown by Hall to involve the supposition of an enormous mechanical force on any conductor carrying a current which acts in the direction of the current even when there is no external magnetic force, and therefore to be utterly untenable.

PART II.

From the foregoing considerations it will appear that the experimental method to be followed is to ascertain whether the velocity of propagation of light through a dielectric is altered by the passing of a displacement-current of electricity in the same direction. If such a change did manifest itself it would probably be a small one, and would only last for the short time in which the displacement was changing and thus the experimental difficulties are two-fold.

A small change made in the velocity of light by any influence is best detected and measured by allowing two pencils of light to interfere, and, while observing the fringes, to cause the influence to act upon one of the pencils and to measure the consequent shifting of the fringes. Any form of apparatus for producing interference fringes for this purpose is called an interferential refractometer.

The various methods of producing interference-fringes, the methods of measuring the shifting of the fringes, or of bringing them back to their original position by compensators of various forms and calculating the shifting from the change of adjustment of the compensator, and the experimental details of the setting up of apparatus for these

Digitized by Google

purposes have all been so fully and ably discussed by Quincke (*Pogg. Ann.* 132, p. 29, 1867) that there is nothing to be added on the subject. Different experimenters have from time to time employed in their refractometers fringes produced by almost all the methods mentioned in the above paper, but the only instruments of which it is necessary, on account of their connection with experiments of a nature similar to those that I have undertaken, to give a detailed description are those of Arago, Jamin, and Fizeau.

In Arago's refractometer (*Œuvres,* Vol. x. p. 313, 1858) there is a slit placed in front of the source of light and a convex lens is employed which is placed so that the slit is in its principal focus. Two tubes are arranged so that half the light which has been refracted through the lens passes through each. After emerging from these tubes the two beams pass through two slits a millimetre wide and two millimetres apart which are separated by a small prismatic obstacle whose vertex is towards the tubes. They next traverse two plates of glass of equal thickness which can be adjusted so as to make any angle with the beams and which serve as a compensator, they are then rendered convergent and intersecting by the object-glass of a telescope, and the resulting interference-fringes viewed through its eye-glass.

The arrangement is represented in plan in fig. 1. It was used for determining the difference of refractive index of moist and dry air.

It will be seen that the method of producing these fringes is in reality Young's, a very primitive one, that the illumination is feeble, as a narrow slit has to be used to admit the light, and that the interfering rays are only separated by a distance of two millimetres, a circumstance which, though indifferent in Arago's experiment, might be often inconvenient. There is a further disadvantage in that the fringes produced by the method are generally small and close together. They can, however, be made rather larger by inclining the plates of the compensator so that the angles of incidence of the rays upon them are large.

The interference arrangement employed by Jamin (*Ann. de Chim. et de Phys.* 3 Series, Vol. 52, p. 163) is one which could be very advantageously used in all researches on variation of refractive index, for in it all the defects mentioned in connection with Arago's instrument are overcome. The fringes used by Jamin are those which were observed by Brewster and explained by Herschel. Two glass plates with plane parallel surfaces and exactly of the same thickness are required. These are silvered at the back and placed almost parallel to each other, and light is allowed to fall upon one of them so as to be reflected thence in the direction of the other. If the light after reflection at the second mirror be received by the eye of the observer either directly or through a telescope, curved interference bands are seen which become larger and further apart as the angle between the mirrors is made smaller. The form of the bands has been determined by Ketteler (*Beobachtungen über die Farbenzerstreuung der Gase,* Bonn, 1865) and the subject will be more fully treated at a subsequent period. The general explanation of the formation of the fringes is very simple. If we consider any ray incident on the first mirror, we see that it is divided into two, one which is reflected at the first surface and one which is reflected at the second. The sizes of the mirrors are so arranged that rays which are reflected more than once at the hinder surfaces are cut off. A part of the ray reflected at the second surface

Digitized by Google

of the first mirror emerges from the glass and is then parallel to the ray which had been reflected at the first surface. These two being reflected at the second mirror we have finally four rays, one reflected at two front surfaces, one at two back surfaces, one first at a front surface and then at a back surface, and one first at a back surface and then at a front surface. The first two are cut off by means of suitable diaphragms. The last two will evidently be parallel, and nearly equal in intensity since the mirrors are nearly parallel, and will have traversed nearly the same length of path. Thus they will give well-marked interference phenomena if viewed by the eye.

The course of the rays and the position of the diaphragms are shown in fig. 4.

The more obvious advantages of this arrangement are that the rays between the mirrors corresponding to any particular incident ray are parallel, and that the distance between them may be made reasonably large by employing thick mirrors and a suitable angle of incidence, both of which circumstances make it easy to pass the rays through double tubes like those employed by Arago. We may further notice that the size of the fringes will evidently depend upon the angle between the mirrors and can thus be varied at pleasure, and that no slit need be used between the source of light and the first mirror, and thus the illumination will be increased and observation made more easy.

We now come to the instrument employed by Fizeau in his experiment upon the velocity of light in moving media to which reference has already been made. The disposition of its parts will be seen from fig. 2.

Sunlight falls on a cylindrical lens A, and converges to f, thence it falls on a plate of glass B and is reflected as if it came from g. The light next passes through a lens C so placed that g is in its principal focus. A part of it next traverses a thick glass plate D, then the two parts pass through tubes containing the fluid to be experimented upon, next they fall upon the lens E whose centre is in the line joining g to the centre of C, and in whose principal focus is a plane mirror F perpendicular to this line. From F to g each beam travels over the path that had been taken by the other, they interfere at g and are observed with the eye-piece G. By using the glass D we allow of a considerable separation (9 millimetres in the actual experiment) of the rays within the tubes without making the fringes exceedingly small. Their size may be varied by altering the inclination of D to the rays.

The arrangement may be exhibited in another way by taking the reflection of the whole apparatus in the mirror F, as in fig. 3. (The arrowheads show the direction of motion of the fluid.)

We obviously may consider a luminous line at g' to be the source of light. Looked at in this way Fizeau's instrument becomes a kind of double Arago's refractometer, but the combination of the plate D and D' now can only be used to enlarge the fringes and will not serve as a compensator.

The peculiarity of Fizeau's method is that he makes each ray retrace the path traversed by the other. One practical outcome of this is that the effective length of his

Digitized by Google

tubes is doubled, as may be seen from fig. 3. Thus, by further making the fluid in the tubes move with equal velocities in opposite directions the effect on the velocity of the light is four times as great as would be observed if an ordinary refractometer of the same dimensions were used in one of whose tubes the fluid was at rest and in the other in motion.

There is however another and a far more important advantage of the two rays travelling over the same path in opposite directions. In this case if the refractive index of any part of the substance employed be altered by any accidental cause, such as change of pressure or temperature, there will clearly be no effect upon the fringes.

The only change in the velocity of light which will shift them will be one which is reversed when the light travels in the opposite direction.

The importance of thus being able completely to eliminate the effects of all causes except the one with which we are concerned cannot be overrated. We may indeed be able readily to distinguish between the effects of change of temperature or pressure and those of the motion of the medium, but if the latter are small they may be completely masked by the former, and at least cannot be observed with nearly the same accuracy as if the former did not exist.

This difficulty was one under which Roiti laboured to a considerable extent, as will be seen from his account of his experiments, and it is principally due to this that the superior limit which he was able to assign to the velocity of the medium was such a very large one.

It may be observed that if Fizeau's method, or any other in which the rays traverse the same path, be adopted, the size of the fringes and the distance that they shift must be measured and the change in the velocity of light which causes the shifting calculated; for the circumstance which renders their position independent of change of refractive index of any substance they may pass through will in general preclude the use of a compensator.

PART III.

It seemed to me to be necessary to the successful prosecution of a research on the effect of a displacement current of electricity upon the velocity of light that a form of apparatus should be employed for exhibiting interference effects between two rays of light which should allow (i) the rays to traverse the same path (consisting in part of two parallel straight lines) in opposite directions, (ii) the fringes to be large and the field of view bright, and (iii) the parallel parts of the path to be as far apart as possible.

The reasons for the first requisite have been already fully discussed, the second is necessary if a small shifting of the fringes lasting but for a short time is to be observed, and the better the third condition is complied with, the more possible it is to arrange condensers in the path of the rays so that a displacement current may be made

Digitized by Google

to travel with one ray at a part of its path without travelling against it at another part.

The first of these desiderata has been pointed out to be the peculiar advantage of Fizeau's method, and the second and third are seen to be to a great extent attained by Jamin's arrangement. What must be sought after is a form of apparatus which will combine these advantages, and it is further desirable that it should be simpler in its parts and easier to set up than Fizeau's, in which the cylindrical lens has to be very accurately focussed, and not only the position but also the direction of the small mirror F (fig. 2) most carefully adjusted.

It occurred to me that suitable interference-fringes might be obtained by the following method.

Two plane reflecting surfaces are fixed so as to be accurately at right angles, and a thick mirror, the surfaces of which are plane and parallel, is arranged so that they are nearly parallel to the line of intersection of the two perpendicular mirrors. Any ray of light incident on the thick mirror has a portion reflected at the front surface, and a portion reflected once at the back surface and then emergent into the air. These are reflected by each of the two mirrors, but in reverse order, and return to the thick mirror where they are again reflected.

By properly placing diaphragms we intercept all rays except the two which have undergone a reflection at each of the two surfaces of the thick mirror, and these are in a condition to interfere.

The course of the rays and the position of the stop may be seen from fig. 4, where the rays travel over nearly the same path, and it is clear that we can by shifting the position of the perpendicular mirrors make the two portions into which any particular ray is divided traverse paths which would be exactly identical if the thick mirror were accurately parallel to the line of intersection of the two.

In fig. 4 M is the thick mirror, A and B the mirrors at right angles, OF the course of the incident ray, and X the stop.

It will be seen that by means of this apparatus all the conditions which were above stated as desirable are attained.

The form of the fringes obtained when monochromatic light is allowed to fall in all directions upon the thick mirror can be most easily investigated by employing a preliminary artifice. We know that when we form an image of any object by successive reflections at two perpendicular mirrors, on whichever of them the light is first incident, the position of the image is the same.

Let (fig. 4) M', O', F', G', H' be the images formed by the mirrors A and B of M, O, F, G, H. Then we see that the rays return from A and B to M as if they had had their rise in a ray $O'F'$ reflected by a mirror M', where it is clear that the angle between M and its image M' is twice the angle that M makes with the line of intersection of A and B. Thus our fringes will be of exactly the same nature as those obtained by means of Jamin's mirrors.

Digitized by Google

It has been already stated that a discussion of the form of these fringes has been given by Ketteler, but the methods that he has employed for obtaining some of his formulae seem to me to be so needlessly lengthy that it would not be out of place to show how the question may be more simply treated.

Let D be the thickness of the glass plate M (fig. 5), μ its index of refraction, ϕ, ϕ' the angles of incidence and refraction of a ray $O'F'$ falling on M', ψ, ψ' the angles of incidence and refraction of the rays $F'h$ and $H'k$ falling on M.

Draw a plane U through H' perpendicular to $H'k$ meeting $F'h$ in n, and draw any plane V perpendicular to the three rays leaving M at k, f, and h, and meeting them in p, q, and r.

The difference of path between the two rays from O' to U is clearly

$$2\mu . F'G' - F'n$$

$$= 2\mu \frac{D}{\cos\phi'} - 2D \tan\phi' \sin\phi$$

$$= 2\mu D \cos\phi'.$$

The path from H' to p is known to be the same as that from n to r. Thus the difference of the paths from H' to p and from n to q is the same at that between the paths from n to q and from n to r, that is

$$2\mu D \cos\psi'.$$

Hence the whole difference of path of the two rays from O' to the plane V is

$$2\mu D (\cos\phi' - \cos\psi').$$

If now these rays together undergo any number of reflections and refractions and finally meet, the paths from V to the point of intersection will be equal.

It will be seen that the difference of path between two rays depends only on the direction and not on the position of the ray which gave rise to them.

Thus if all the rays which leave the mirror M are allowed to pass through a lens in the focal plane of which a screen is placed, each point of the screen will be illuminated by a set of pairs of rays, the difference of path between the two rays of each pair in the same set being the same, and on the screen we shall see a series of bands alternately dark and bright, if the light be monochromatic, or of bands of white light separated by dark spaces with coloured edges, if ordinary light be employed.

In order to find the form of the bands on the screen we must remember that the line joining the centre of the lens to the point on the screen illuminated by any set of rays is parallel to the path of the rays before they reach the lens.

Let us for the sake of simplicity consider the two mirrors at right angles to be vertical and the plane of the paper horizontal, and let M (fig. 6) be a section of the thick mirror by the plane of the paper. Let SZ be perpendicular to the sections of the surfaces of M by the plane of the paper, ρ the angle a ray QS incident on M at S makes with SZ, and θ the angle that the plane QSZ makes with the plane of the paper.

Digitized by Google

Let ν be the angle that the thick mirror makes with the vertical.

Then we see at once, from fig. 6, that

$$\left.\begin{aligned}\cos\psi &= \cos\nu\cos\rho - \sin\nu\sin\rho\sin\theta\\ \cos\phi &= \cos\nu\cos\rho + \sin\nu\sin\rho\sin\theta\end{aligned}\right\}.$$

Now, since
$$\begin{aligned}\mu\cos\phi' &= \sqrt{\mu^2 - \sin^2\phi}\\ &= \sqrt{\mu^2 - 1 + \cos^2\phi},\end{aligned}$$

if Δ be the difference of path, we have

$$\frac{\Delta}{2D} = \sqrt{\mu^2 - 1 + (\cos\nu\cos\rho + \sin\nu\sin\rho\sin\theta)^2} - \sqrt{\mu^2 - 1 + (\cos\nu\cos\rho - \sin\nu\sin\rho\sin\theta)^2}.$$

Put $\frac{\Delta}{2D} = c$. The above equation can be most easily rendered rational by putting it in the form

$$c = \sqrt{\alpha + \beta} - \sqrt{\alpha - \beta},$$

whence
$$\frac{2\beta}{c} = \sqrt{\alpha + \beta} + \sqrt{\alpha - \beta},$$

and
$$\left(c + \frac{2\beta}{c}\right)^2 = 4(\alpha + \beta).$$

Thus the equation as rationalized is

$$(c^2 + 4\cos\nu\cos\rho\sin\nu\sin\rho\sin\theta)^2 = 4c^2[\mu^2 - 1 + (\cos\nu\cos\rho + \sin\nu\sin\rho\sin\theta)^2] \ldots\ldots\text{(i)}.$$

Now since, neglecting signs, ρ and θ are the same for the incident and the reflected rays, by substituting in this

$$\left.\begin{aligned}\tan^2\rho &= \frac{x^2 + y^2}{f^2}\\ \tan\theta &= \frac{y}{x}\end{aligned}\right\},$$

where f is the focal length of the lens, and considering c as a parameter, we get the equations in rectangular co-ordinates to the curves of equal illumination on the screen given by light of any particular refractive index.

These equations will however only hold for small values of x and y, since the focal plane is not the locus of the foci of parallel pencils falling on the lens in different directions, but is only a tangent plane to it.

Let us imagine conical surfaces with the centre of the lens for vertex and (i) for the equations to their generating lines to be drawn, and consider their intersections with a sphere of radius (a) having the centre of the lens as centre.

These curves will not be unlike those formed on the retina (as seen from behind) if the light were received directly into the eye.

Neglecting powers of ν above the second, equation (i) becomes

$$c^4 + 16\sin^2\rho\cos^2\rho\sin^2\theta\,.\,\nu^2 = 4c^2(\mu^2 - 1 + \cos^2\rho) + 4c^2\nu^2(\sin^2\rho\sin^2\theta - \cos^2\rho),$$

Digitized by Google

or, as a further approximation, since both c and ν are small,

$$4 \sin^2\rho \cos^2\rho \sin^2\theta \,.\, \nu^2 = c^2 (\mu^2 - \sin^2\rho) \quad \text{(ii)}.$$

The equation to the projections of these curves on a plane parallel to the lens is got by putting

$$\left.\begin{aligned} \sin^2\rho &= \frac{x^2+y^2}{a^2} \\ \tan\theta &= \frac{y}{x} \end{aligned}\right\}.$$

Hence we get

$$c^2 [a^4\mu^2 - a^2 (x^2+y^2)] = \nu^2 y^2 (a^2 - x^2 - y^2) \quad \text{(iii)}.$$

The tracings of these curves for different values of c are given in fig. 7.

If the axis of the eye is inclined to the line SZ (fig. 6), as will be in general the case, we shall see only a part of the above system of fringes. For example, if the eye be placed as in fig. 4, remembering that the image on the retina must be inverted to give what is actually seen, the bands will be something like those shown in fig. 8.

The distance apart of the rays $F'h$, $H'k$ (fig. 4), which it is advisable to have as large as possible, is with our notation evidently

$$2D \tan\phi' \cos\phi.$$

If this is a maximum it is easily shown that

$$\tan^4\phi = \frac{\mu^2}{\mu^2 - 1}.$$

For glass $\mu = \frac{3}{2}$, therefore ϕ is about 50°.

The practical details for the setting up of the apparatus for these fringes are as follows:

An ordinary lime-light was used as the illuminating source. Since the rays finally leaving the thick mirror follow the same path as those incident upon it, it was necessary to reflect the light upon it by means of a piece of glass. In order to avoid the trouble caused by having light reflected at each of two surfaces incident upon the mirror, the front surface of a prism of small angle (about 18°) was used as a reflector. The thick mirror was one of a pair furnished by Elliot Bros. for producing Jamin's fringes. Its thickness was about 20 mm. and its surfaces on being tested proved to be very accurately plane and parallel. With the two perpendicular mirrors I had some little trouble. I first tried to use a right-angled prism, but could not obtain one whose faces were accurately perpendicular. Ordinary mirrors would be very annoying to use, as the reflections from the front surfaces would render the field of view confused. I next tried mirrors of black glass such as are often used for Fresnel's interference experiment, but found the intensity of the light reflected from them to be exceedingly small. Finally I used a pair which were silvered on the front surface and which reflected a very large proportion (almost 90 per cent.) of the light that fell on them. The glasses were carefully selected before being silvered and were very accurately plane.

I had at first expected that interference-fringes would be produced if the angle beween the mirrors were approximately a right angle, but I found that unless it was accurately

Digitized by Google

so none were to be seen. That this is in accordance with the theory given above is clear, for if the mirrors are not at right angles the two rays to which any ray of light gives rise will be no longer parallel but will include an angle which varies with the angle of incidence of the original ray, and their difference of path will depend upon the position as well as upon the direction of the original ray. Thus the screen cannot be placed so that all the pairs of rays intersect on it, and if at any point of it a pair of rays do happen to intersect the brightness there will not be sensibly altered, since it is illuminated by a large number of rays not in a condition to interfere.

The interference-fringes were viewed through a telescope focussed for an infinitely distant object. The most convenient method of supporting at once the telescope and the small prism was to employ a spectrometer from which the collimator had been removed.

The thick mirror was mounted on a plate so as to be movable in the direction of its surface, the adjustment being made with a screw. The plate was fastened to a large block of lead with three short legs. This was placed on a board which was supported by three levelling screws. In the block of lead three screws were driven, and a plate of glass rested upon them. The thick mirror being adjusted so as to be accurately vertical, the screws were arranged so that the glass plate was horizontal. Thus by observing a spirit level placed on the glass plate it was easy at any time to bring the mirror back to the vertical position by means of the levelling screws that supported the board. A drawing of the arrangement is given at A in Plate IV.

The thick mirror was in the first instance adjusted to be at right angles to the line of collimation of a good surveyor's level by focussing it to infinity, illuminating its cross wires, and getting their reflection in the mirror coincident with them.

The two mirrors which were to be at right angles were supported in the same way as the mirrors used in Fresnel's interference experiment described in Glazebrook's *Physical Optics*, p. 119, except that the second mirror was fastened in a position approximately perpendicular to the one that it usually occupies. The support of the mirrors was then fastened, so that the fixed mirror was vertical, to the upper part of an alt-azimuth stand, whose base was levelled so that the mirrors as a whole could be turned about a vertical axis.

A drawing of the mirrors and stand is given at B in Plate IV. The movable mirror was then by means of successive small motions of the screws against which it was pressed rendered vertical and at right angles to the fixed one. The criterion of the accuracy of this last adjustment was that when a telescope focussed to infinity and its cross wires illuminated was pointed at their line of intersection the two images of the wires coincided.

In order that no rays of light should enter the telescope but those that give the interference effects the apparatus is arranged as follows.

Between the lime-light and the small prism a lens is placed so that the rays from any point of the lime should emerge as an approximately parallel pencil. By this means the illumination of the field of view will be rendered more intense. A piece of black

Digitized by Google

cardboard with a vertical slit in it about 8 mm. in breadth is placed in front of the lens. The breadth is so chosen that if we look at the thick mirror from a little distance the rays reflected at its first and at its second surface are separated by a narrow interval of darkness. The perpendicular mirrors are then placed so that their line of intersection is in this dark interval.

If we now move the thick mirror very slightly from the vertical and look through the prism towards it with the naked eye, we shall see a very bright portion, a fairly bright portion with interference curves, and a much fainter portion. The first is due to light which is twice reflected at the silvered surface of the thick mirror, and can be got rid of by moving the mirror parallel to itself by means of the screw. The third is due to light reflected twice at the first surface of the thick mirror, and can be cut off by placing a diaphragm between the prism and the mirror and moving it until it just encroaches upon the fringed part, which is the portion we require.

The course of the rays and the methods of cutting off the superfluous ones can be readily seen from fig. 9. It may seem that, as the diaphragm X is sufficient to keep the light from falling on the further part of the front surface of the thick mirror, the slit S could be dispensed with, but, by illuminating only as much of the prism as is necessary, we keep out a great deal of scattered light which would otherwise pass into the telescope. With the naked eye I found it quite easy to make out the form of the fringes given in fig. 8.

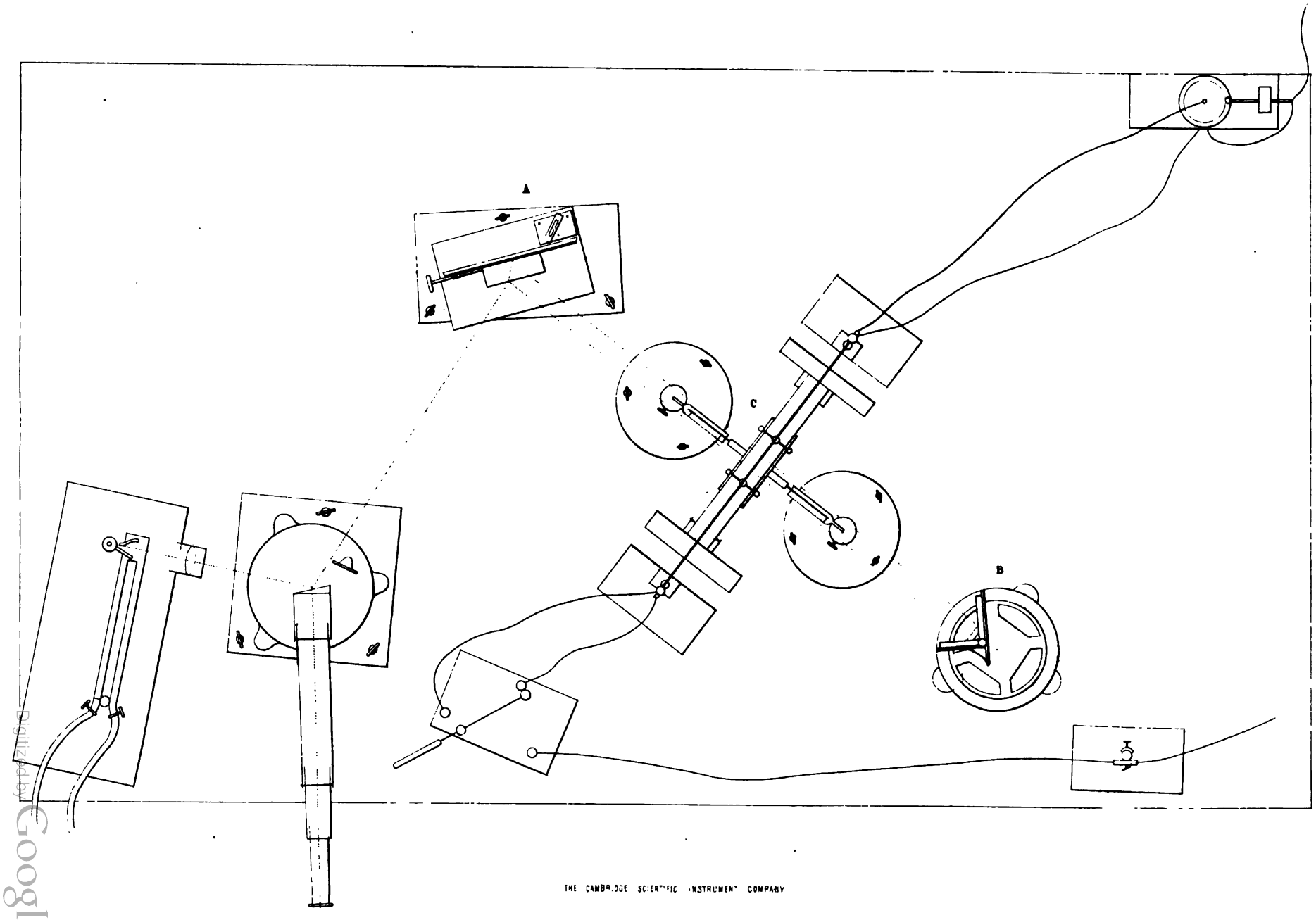
The only rays which are to be employed in the experiments on the velocity of light being those that are almost horizontal in their course between the thick mirror and the perpendicular mirrors, the others were excluded by placing a diaphragm with two small equal holes at a suitable distance apart between the mirrors. When this was adjusted so that the light could be seen through the holes the telescope was pointed in the direction in which they were seen through the prism. On looking into it the field of view was observed to be very bright and crossed by fine large dark bands which were horizontal and had rather narrow fringes of colour at their edges.

That the bands should be horizontal is clear since what we see is that part of fig. 7 enclosed in the dotted circle.

By altering the inclination of the thick mirror to the vertical we can make the bands move up or down, at the same time becoming larger or smaller, but if we make the mirror so nearly vertical that the bands get very large indeed before vanishing they are no longer suitable to work with, as they become indistinct and much distorted, the latter being due according to Quincke to imperfections in the workmanship of the glass.

In order to make it evident that the fringes would not be shifted by any change in the refractive index of the substances through which the rays passed, plates of glass of various thickness were placed before one of the holes in the diaphragm and it was found that the fringes were unchanged if the plate was placed perpendicularly to the rays passing through the hole, but if the plate was inclined the fringes were shifted, as can evidently be explained by the theory of Jamin's compensator.

Digitized by Google



Digitized by Google

Digitized by Google

It will be noticed that if an inclined plate be put before one of the holes the two rays no longer traverse the same path, the one being raised and the other depressed by passing through the plate if it be inclined to the vertical.

The permanence of the fringes in the first experiment is a sufficient guarantee that they would not be affected by any change of refractive index which it is possible could occur, having regard to the symmetrical form which the apparatus will take. I thought that, the arrangement being set up, it might be worth while to show that the fringes can be exhibited on a screen in the focal plane of a lens, for Jamin (*Cours de Phys. Opt. Phys.* Chap. II.) gives an explanation which makes their formation depend upon the light having passed through some small hole such as the pupil of the eye, and which seems certainly unsound.

Placing a screen of ground glass in the focus of the object-glass of the telescope and examining it with the eye-piece I found that the fringes were distinctly visible on it and that therefore Jamin's explanation breaks down.

PART IV.

The question as to the most convenient arrangement for allowing a "displacement-current" of electricity to pass along the paths of the interfering rays next arises for decision.

First we must notice that the time-variation of the line-integral of the electric "displacement" along the path of either ray will constitute the whole displacement current along that ray.

Now as the ray returns to the point from which it starts, the line-integral of the electric displacement taken all along its path is always zero if the path is all the time through a substance of constant specific inductive capacity. Thus we shall have to use a condenser containing a dielectric whose specific inductive capacity is different from that of air.

The simplest arrangement that can be made is represented diagrammatically in fig. 10.

Two plates A and B are placed side by side and two others C and D are placed side by side and opposite to them, being separated from them by a medium of specific inductive capacity K_2, that of air being K_1. A and D are connected and raised to potential V, and B and C are connected and put to earth. Small holes in the plates allow the rays of light to pass through.

Let us suppose that the mirrors are so far away from the condenser that the potential at each of them is zero. Then, if P be the electromotive force at any point of the path of a ray in the direction of the ray, the line-integral of the displacement will be

$$2\int_M^B \frac{K}{4\pi} P ds,$$

which may be written $$2\int_M^B \frac{K_1}{4\pi} P ds + 2\int_A^C \frac{K_2 - K_1}{4\pi} . P ds.$$

Digitized by Google

Now
$$\left.\begin{aligned}\int_M^E Pds &= 0,\\ \int_A^C Pds &= V.\end{aligned}\right\}$$

Hence if τ be the time of charging the condenser the line-integral of the mean displacement current will be
$$\frac{K_2 - K_1}{2\pi} \cdot \frac{V}{\tau}.$$

For the other ray this will of course be the same but with the opposite sign.

We see that the result is independent of the thickness of the dielectric between the plates of the condenser.

Let the velocity of light in air be (v) and the velocity of the "medium" for a displacement-current $\dot{f}$ be $\varpi\dot{f}$.

Thus the velocity of light in the moving medium is
$$v + \frac{\varpi\dot{f}}{2},$$
and the time of traversing the space ds is
$$\frac{ds}{v + \frac{\varpi\dot{f}}{2}} = \frac{ds}{v}\left(1 - \frac{\varpi\dot{f}}{2v}\right),$$
since $\varpi\dot{f}$ is small compared with v.

Thus the whole time by which one ray is accelerated or retarded by the motion of the "medium" is
$$\frac{\varpi}{2v^2}\int\dot{f}ds = \frac{\varpi}{2v^2} \cdot \frac{K_2 - K_1}{2\pi} \cdot \frac{V}{\tau}.$$

The difference in time taken by the two rays to traverse their course will be twice this, or
$$\frac{\varpi}{v^2} \cdot \frac{K_2 - K_1}{2\pi} \cdot \frac{V}{\tau}.$$

Thus, their difference of path in air will be
$$\frac{\varpi}{v} \cdot \frac{K_2 - K_1}{2\pi} \cdot \frac{V}{\tau}.$$

A piece of plate glass was employed as the dielectric of the condenser. Although its thickness does not appear in the formula just investigated, yet it is clear that it must be large compared with the diameter of the holes through which the light passes if the potential over each of these is to be sensibly the same as that of the conducting plate in which they are made. The plate of glass that I used had a thickness of about 26 mm. and its surfaces were very nearly plane and parallel, so that when it was placed in the path of the rays and moved about the fringes were not at all displaced.

The conducting plates of the condenser were made of pieces of tin-foil. These might have been fastened directly to the thick plate, but it was thought to be more convenient

Digitized by Google

if different parts of it could from time to time be used if necessary, and another disadvantage of fixing them was the extreme difficulty that had been found in some preliminary experiments in getting the holes in the tin-foil exactly opposite to each other, so the pieces were fastened with shellac varnish on two thinner glass plates of thickness 3 mm. each which were brought up so as to be in contact with the thick plate.

The holes in the tin-foil might have been at first made at any convenient distance apart, so long as it was less than the maximum distance between the two parts of a ray, and the angle of incidence of the light on the thick mirror adjusted so that the rays should pass through them if the plates were properly placed. This would however have been very troublesome, and it was found easier, having set up the interference arrangement in a permanent manner, to make the holes at the suitable distance apart.

This distance could be most accurately found by placing between the thick mirror and the perpendicular mirrors a screen in which a narrow horizontal slit had been cut so that the height of the slit was the same as that of the middle of the thick mirror, and sliding up and down before it another screen in which two narrow slits inclined at equal small angles to the vertical were cut, until the rays passed through the holes thus formed.

The pieces of tin-foil on the glass plates were of the shape shown in fig. 11. The diameter of each of the holes was 5 mm. and their centres were 15 mm. apart. In order that sparks should not pass from one piece to the other a slip of glass was fastened half way between them and at right angles to the plate by means of electrical cement. Directly the condenser was employed, however, the cement was perforated by a spark. A block of paraffin was then melted on to the plate, the surface of the glass being first carefully cleaned and then rendered very slightly greasy with turpentine, and by means of this the pieces of tin-foil were completely insulated from each other. The same was of course done with the other plate. The shape of the paraffin block is shown by the dotted lines in fig. 11.

After the paraffin was put on, the rest of the front surface of the glass was given several coats of shellac varnish, so that the insulation of the pieces of tin-foil was rendered very good.

The readiest way of setting up the condenser in the path of the rays was found to be to take one of the thin plates and to support it vertically and perpendicular to the light incident upon it, so that the rays went through the two holes in the tin-foil. This adjustment could be made with very little trouble by placing the plate with the holes on the same level and nearly in the right place, and then moving the perpendicular mirrors slightly by turning round the upper part of the alt-azimuth stand. The other plate was then put in contact with it back to back, and its support adjusted so that the holes in the opposite pieces of tin-foil exactly faced each other, which, on account of the small thickness of the plates, was also easy.

The second plate was then carefully moved away from the first in such a manner that the rays still passed through the holes. The thick plate mounted on a frame was

Digitized by Google

then placed between them and parallel to them, and the two plates moved up to it one after the other, taking care as before that the rays had a path open to them. The plates can be adjusted perpendicular to the rays incident upon them by noticing the direction of the light reflected at their surfaces.

The contacts with the tin-foil plates were somewhat difficult to make satisfactorily, as on account of the high potential to which they had to be raised brush discharges into the air were liable to occur. An instrument was designed and made for the purpose which is shown at *C* in Plate IV., in which the contacts are made by curved rods which descend from two horizontal cross bars and press by their own weight against the plates.

In order that any effect that the displacement-current may have on the fringes should be observable the time during which the current passes must not be less than $\frac{1}{5}$ second.

This condition was satisfied by connecting one of the cross bars of the contact-maker with the inner coating of a Leyden jar of which the outer coating was put to earth, as also the other cross bar.

The former bar could be connected by a sort of large switch mounted on a block of paraffin either to the outer coating of the jar or to one electrode of a Holtz machine of which the other was to earth.

An earth-connected knob was placed near the knob of the jar so that its potential should not rise above a certain value, depending on the distance of the knobs, which could be adjusted at pleasure. The potential of the jar had to be limited so that the insulation of the condenser should not be broken down.

The whole arrangement is shown in plan in Plate III. and in elevation in Plate IV.

Various sizes of Leyden jar were tried which, together with the condenser, took from $\frac{1}{2}$ to $\frac{1}{5}$ of a second to charge.

On connecting the upper bar of the contact-maker with the electrical machine by means of the switch, the potential of the parts connected with it rose rapidly from zero to a certain limit when they were discharged by a spark passing to the safety knob, and the same variation of potential recurred. A considerable number of experiments was made with this apparatus, but no shifting at all of the fringes could be observed.

To be able very accurately to decide upon the question of whether there was any shifting or not, a thin wire was placed in the focus of the object-glass of the telescope so as to be parallel to the fringes as seen through the eye-piece, and the telescope pointed so that there was a very narrow line of light between the wire and one of the black bands which was very well marked. A widening or narrowing of this line corresponding to a shifting of the fringes over $\frac{1}{50}$ of their breadth could have been detected.

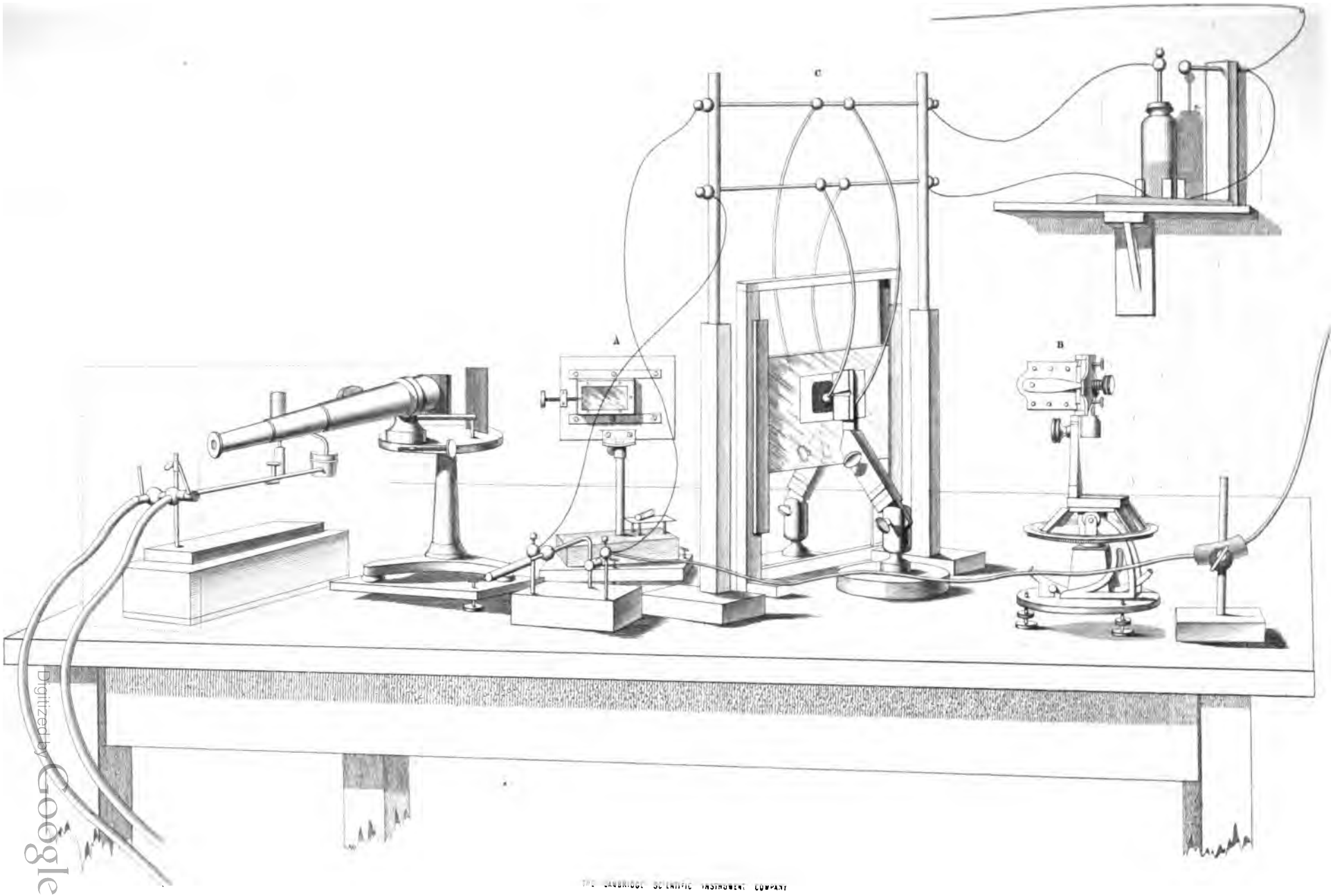
Hence the difference of path of the rays is less than $\frac{1}{100}$ of the mean wave-length of light, that is, less than

$$6 \times 10^{-7} \text{ centimetres.}$$

Hence referring to the value obtained for the difference of path

$$\frac{\varpi}{v} \frac{K_2 - K_1}{2\pi} \frac{V}{\tau} < 6 \times 10^{-7} \text{ centimetres.}$$

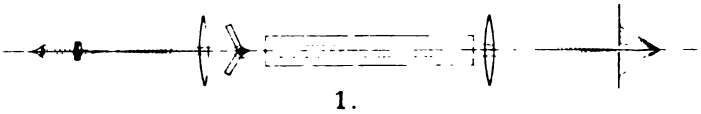
Digitized by Google



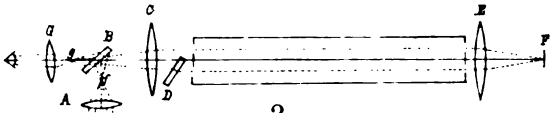
THE CAMBRIDGE SCIENTIFIC INSTRUMENT COMPANY

Digitized by Google

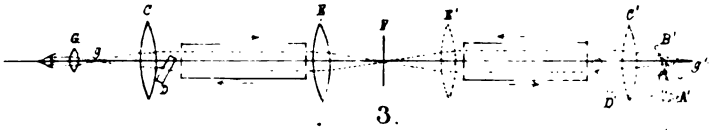
Digitized by Google



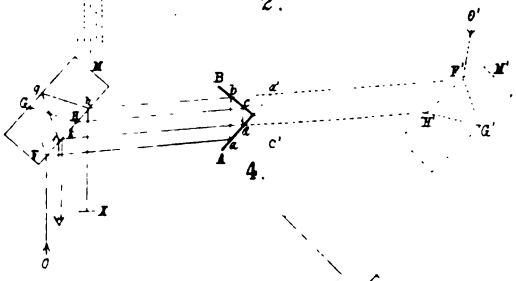
1.



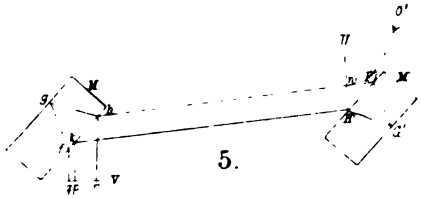
2.



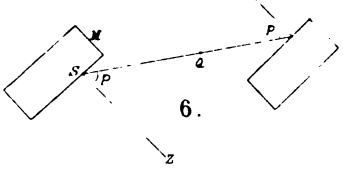
3.



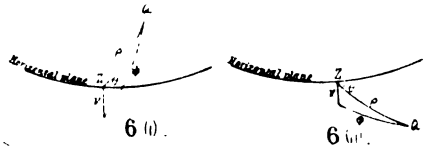
4.



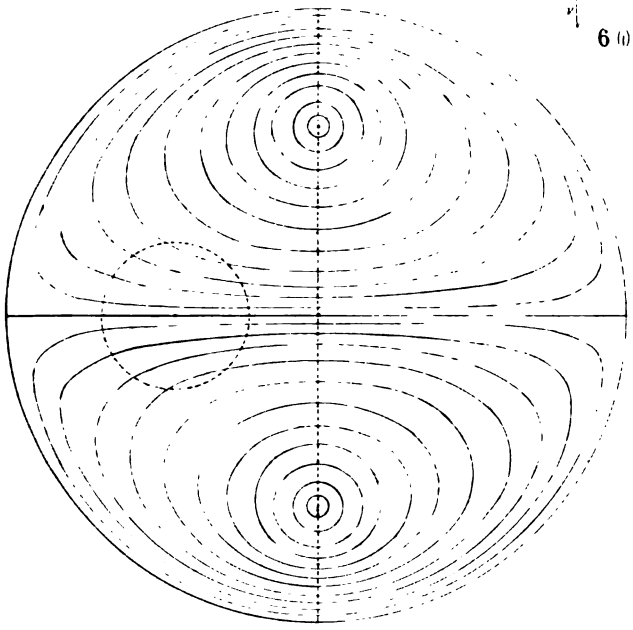
5.



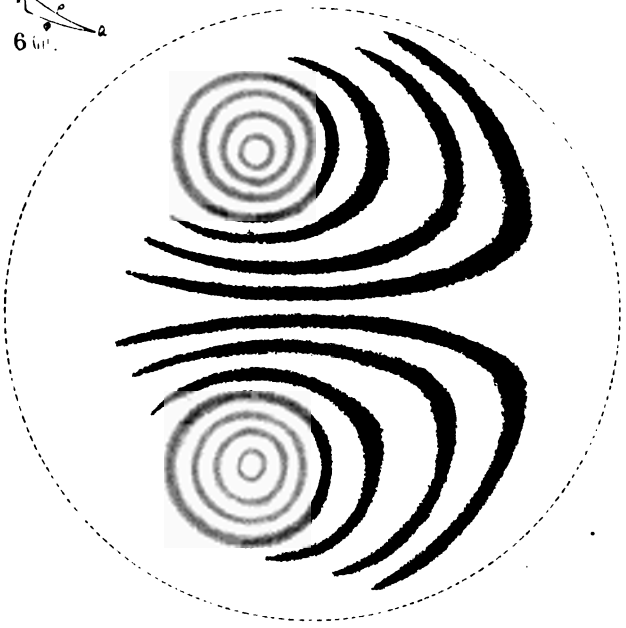
6.



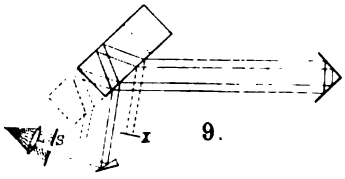
6 (i). 6 (ii).



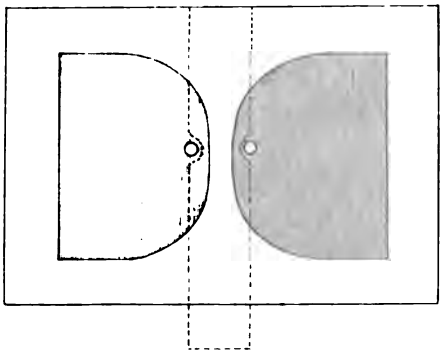
7.



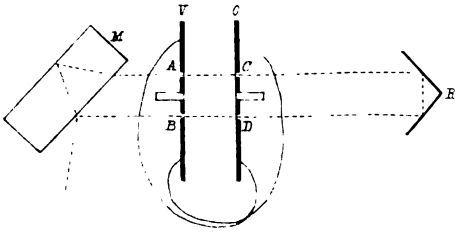
8.



9.



11.



10.

THE CAMBRIDGE SCIENTIFIC INSTRUMENT COMPANY

Digitized by Google

Digitized by Google

Now if C be the mean value of the displacement-current through the glass while the condenser is being charged, and if δ be the thickness of the glass,

$$C = \frac{K_2}{4\pi} \cdot \frac{V}{\tau\delta}.$$

Hence $$2\frac{\varpi}{v} \cdot \frac{K_2 - K_1}{K_2} \cdot \delta \cdot C < 6 \times 10^{-7} \text{ centimetres,}$$

or, if δ be measured in centimetres,

$$\varpi \cdot C < 3\frac{K_2}{K_2 - K_1} \cdot \frac{v}{\delta} \cdot 10^{-7},$$

and $\varpi \cdot C$ is clearly the velocity of the medium corresponding to a displacement-current C.

Now $$K_2 = 3{\cdot}2 \times K_1,$$
$$\delta = 2{\cdot}6 + {\cdot}6 = 3{\cdot}2,$$
$$v = 3 \times 10^{10} \text{ in C.G.S. measure;}$$

therefore $$\varpi \cdot C < \frac{9}{2{\cdot}2} \times 10^3 \text{ in C.G.S. units,}$$
$$< 41 \text{ metres per second.}$$

If the superior limit thus obtained be compared with that given by Roiti, namely 200 metres per second, the increased accuracy attainable by the use of the interference apparatus which I have employed will be apparent, more especially if it be borne in mind that the vessels containing the electrolyte in his experiment were 45 centimetres long, as against 3·2 centimetres of glass used by me.

The great disadvantage of making observations such as those that I have described upon a displacement-current is that the current is of necessity exceedingly small compared with those which it is possible to have conducted through an electrolyte, on account of the duration which it must necessarily have in order that its effects should be visible, and that therefore, though the velocity of the "medium" ϖC can be shown to lie within very narrow limits, yet C is so small that we derive little information about the value of ϖ.

Thus, in the above experiments,

$$V = \text{about } 500{,}000 \text{ volts.,}$$
$$= \tfrac{1}{2}10^{14} \text{ in C.G.S.,}$$
$$\tau = \tfrac{1}{5},$$
$$\delta = 3{\cdot}2,$$
$$K_2 = \frac{3{\cdot}2}{9} \cdot 10^{-20}.$$

Whence we obtain:

$$C = \frac{K_2}{4\pi}\frac{V}{\tau \cdot \delta} = \frac{5}{9 \times 8\pi} \cdot 10^{-6}$$
$$= 2{\cdot}2 \times 10^{-6} \text{ in C.G.S. units per square centimetre,}$$
$$= 2{\cdot}2 \times 10^{-7} \text{ ampères per square centimetre.}$$

Digitized by Google

IX. *On the mutual action of oscillatory twists in an elastic medium, as applied to a vibratory theory of electricity.* By A. H. Leahy, M.A., Pembroke College.

[*Read, Nov.* 23, 1885.]

The oscillations discussed in this paper are such as may be produced in an elastic medium by the tangential displacement of the surfaces of tubes of small sectional area: the tubes either forming closed curves or extending indefinitely in both directions. The direction and circumstances of the motion are in general analogous to ordinary vortex motions in an incompressible fluid, and it is shewn that, if the period of the oscillation be such that the waves produced are "long"; so that the inverse square of the length of the wave can be neglected compared with the corresponding power of ordinary finite distances; then the displacement due to these tangential displacements is the same as the velocity due to a vortex ring of the same form as the tubular surfaces. There are however sufficient differences between the two theories to prevent the name "vortex" being given to this species of motion, and we shall for the future call the tubular surfaces "oscillatory twists" or "twists" for the sake of brevity. It is found in the subsequent work, by making use of the ordinary equations of an elastic medium, that two such twists will act upon each other so as to give rise to a force which depends upon the product of the angular displacements of the two surfaces, and which is of course of the second order in these displacements. Now the equations of elasticity have been established on the assumption that terms of the second order may be neglected, and hence an objection may be raised at the outset that, if we are to take account of forces depending upon the second order terms, we must investigate the equations of elasticity to the second order, and until this is done that the whole theory is unsound. In answer to this objection it may be remarked that, if the medium in which the motions are supposed to take place is "perfectly elastic" or non-dissipative, as the luminiferous ether must nearly be, if it be capable of transmitting vibrations from ordinary stellar distances as it is supposed to do in the undulatory theory of light, then it will appear to be at least probable that in such a medium the energy due to a given extension is equal to that due to an equal and opposite compression. If this be so the terms of the second and all even orders will disappear from the equations of elasticity, and consequently if we keep below the third order of the displacement our equations will be correct and the results arrived at can be relied on.

It has been shewn* that if W is the total energy of an elastic body in any state of strain, W is a function of $s_1\, s_2\, s_3\, \alpha\, \beta\, \gamma$ where

$$s_1 = \frac{du}{dx}, \qquad s_2 = \frac{dv}{dy}, \qquad s_3 = \frac{dw}{dz},$$

$$\alpha = \frac{dw}{dy} + \frac{dv}{dz}, \qquad \beta = \frac{du}{dz} + \frac{dw}{dx}, \qquad \gamma = \frac{dv}{dx} + \frac{du}{dy},$$

* See Thomson and Tait, *Nat. Philosophy*, Art. 678 and Green's *Mathematical Papers*, p. 249.

Digitized by Google

and since s_1, s_2, etc. are small W can be expanded in a very convergent series of the form

$$W = W_0 + W_1 + W_2 + \ldots\ldots$$

where W_n is a homogeneous function into which these six quantities enter in the n'th order: u, v, w being the displacements parallel to the co-ordinate axes.

In the ordinary theory W_0 and W_1 are shewn to be zero, and the equations of an elastic medium depend upon W_2. Following Green notice that the energy must remain unchanged if $-z$ and $-w$ are put for z and w; i.e. as much energy is spent in stretching the medium up as in stretching it down. This substitution would change α and β into $-\alpha$ and $-\beta$, and we see that the energy must not contain any odd powers of α and β. Similarly it must contain no odd powers of γ. Thus the 56 terms in W_3 reduce to 19, namely:

$$s_1^3,\ s_2^3,\ s_3^3;\ s_1^2 s_2,\ s_1^2 s_3,\ s_1 s_2^2,\ s_1 s_3^2,\ s_2^2 s_3,\ s_2 s_3^2,\ s_1 s_2 s_3;$$
$$\alpha^2 s_1,\ \beta^2 s_1,\ \gamma^2 s_1;\ \alpha^2 s_2,\ \beta^2 s_2,\ \gamma^2 s_2;\ \alpha^2 s_3,\ \beta^2 s_3,\ \gamma^2 s_3.$$

But, if the energy of compression is equal to that of extension, W must remain unchanged by putting $-u$ for u, $-v$ for v and $-w$ for w. Thus we see that in this case all the terms in W_3 vanish and W becomes $W_2 + W_4$ to the third approximation, whence it follows that the equations, which were established by putting W equal to W_2, will continue to give correct results if terms of the third order are neglected.

As practically bearing on the question whether the equations of elasticity can be relied on in the case of the ether I may mention that Professor Pearson of University College, has, in a recently published* paper, investigated the equations of elasticity to a second and higher orders; and that he finds among several other curious results that, if the second powers of the strain are taken into account, the index of refraction of light will depend upon the intensity. As no trace of such a phenomenon has, I believe, been observed, this fact appears to strengthen the supposition that results obtained from the equations on the ordinary hypothesis will be reliable up to the second order of approximation at least.

2. The results obtained in this paper may be briefly summed up as follows. In the first place a twist along any curve will give a displacement equal in magnitude and direction to the magnetic induction due to a current of strength $\frac{\mu b^2 \omega \sin pt}{2}$ where b is the radius of the twist, $\omega \sin pt$ its angular displacement, and μ the coefficient of magnetic permeability of the medium: the waves being supposed to be "long" as in § 1. This is shewn to be the case if the axis of the twist is either rectilinear or circular, and might be inferred from the corresponding theorems in fluid motion on kinematical considerations. The second result, which does not appear to bear any analogy to the theorems of fluid motion, is that, if the field of vibration is explored by a rectilineal twist of the same period as that of the vibration, the twist will experience a force at right angles to the plane containing the twist and the direction of the displacement which would exist if the twist were removed.

This force is in such a direction that a person standing in the twist so that the rotation of the surface of the twist is in the opposite direction to that of the hands of

*. *Proceedings of the Cambridge Philosophical Society*, Vol. v. Part iv. page 296.

Digitized by Google

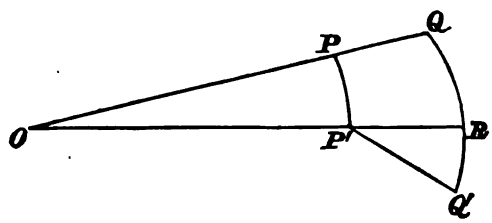
a watch, and having the direction of the displacement acting from his right hand to his left will be urged forwards by the force.

Displacement produced by a rectilineal twist.

3. Taking cylindrical co-ordinates it is clear that the displacement at any point will be wholly perpendicular to the axis and to the shortest distance from the given point to the axis. Denoting this by v we shall have (by variation of angular momentum)

$$\rho \,.\, 2\pi r^2 \frac{d^2 v}{dt^2} = \frac{d}{dr}(\tau \,.\, 2\pi r^2),$$

if ρ is the density of the medium and τ the tension exerted by the consecutive layers of the medium.



But if O is the projection of the axis and if two points PQ are displaced to P' and Q',

$$\tau = \mu \times \text{angle}\, RP'Q' = \mu\left(\frac{dv}{dr} - \frac{v}{r}\right);$$

therefore we get
$$\frac{d^2 v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2} = \frac{\rho}{\mu}\frac{d^2 v}{dt^2}.$$

Therefore
$$v = Ae^{ipt} I_1(hr) + A'e^{-ipt} I_1(hr)$$

is the complete periodic value of v of period $\frac{2\pi}{p}$ if $h^2 = \frac{\rho p^2}{\mu}$ and I_1 is the solution of Bessel's equation of the first order, applicable to space outside a cylinder.

This is known* to be

$$I_1(hr) = \frac{1}{hr}\left\{1 - \frac{h^2r^2}{2^2} + \frac{h^4r^4}{2^2 \,.\, 4^2}\cdots\right\} + \left\{hrS_1 - \frac{h^3r^3}{2^2 \,.\, 4}\, S_2\cdots\right\} - \left(\Gamma + \tfrac{1}{2}\pi r + \log\frac{hr}{2}\right)\left(\frac{hr}{2} - \frac{h^3r^3}{2^2 \,.\, 4}\cdots\right),$$

if Γ is Euler's constant and $S_m = 1 + \frac{1}{2} + \ldots \frac{1}{m}$.

If squares of hr are neglected, we have, as an approximation,

$$v = \frac{A \sin pt}{r},$$

which is the approximate expression for the displacement given by a single oscillatory twist if the wave produced is long.

* See Rayleigh, *Theory of Sound*, Art. 341.

Digitized by Google

Supposing the angular displacement of the twist at time t to be $\omega \sin pt$ and the radius to be b, we have

$$v = \frac{b^2 \omega \sin pt}{r} \quad \text{.....(1).}$$

The magnetic induction due to a straight current of strength c is $\frac{2c}{\mu r}$ if μ is the "magnetic permeability."

Therefore the displacement due to a rectilineal twist is equal to the magnetic induction due to a current whose strength is equal to

$$\tfrac{1}{2}\mu b^2 \omega \sin pt.$$

4. From the result obtained for the case of two dimensions, it appears to be probable that a displacement due to any twist along the axis of a closed curve will be the same as the magnetic induction due to a current along the curve if second powers of rh, where h is the inverse wave length, are neglected. The integration of the equations of elasticity is however very difficult when the boundary is not symmetrical; but the required result can be obtained for the case of a twist along the circular axis of a ring of small section.

Using "toroidal co-ordinates," take as orthogonal surfaces a series of rings generated by the revolution of coaxial non-intersecting circles about their common radical axis, and a series of spheres cutting these rings orthogonally, the third set of surfaces being planes through the axis of revolution of the rings. These co-ordinates have been investigated* by Mr Hicks, who has applied them to find solutions of Laplace's equation at the surface of an anchor ring or tore.

If the rings are defined by $u = \text{const.}$ and the spheres by $v = \text{const.}$ it has been shewn that

$$\left.\begin{aligned} r \sin\theta &= a \frac{\sin hu}{\cos hu - \cos v} \\ r \cos\theta &= a \frac{\sin v}{\cos hu - \cos v} \end{aligned}\right\} \quad \text{.....(2),}$$

r and θ being polar co-ordinates and θ the co-latitude.

Also, if b is the radius of the cross section of a ring, c the radius of circular axis of a ring, R radius of a sphere and a of the critical circle,

$$\cos hu = \frac{c}{b}, \qquad \sin hu = \frac{a}{b}, \qquad \sin v = \frac{a}{R} \quad \text{.....(3).}$$

The ordinary equations of elasticity can be transformed to these co-ordinates and an approximate solution obtained for the case of a long wave by neglecting the square of the inverse power of the wave length.

If ξ be the displacement perpendicular to the surface of one of the rings and η that perpendicular to one of the spheres; ξ being reckoned positive when measured from the

* *Transactions of the Royal Society*, Vol. 172, p. 614.

Digitized by Google

centre of a ring; and η being reckoned positive if measured from the axis when the point considered is above the plane of the critical circle; we shall have for the equations of elasticity

$$\left.\begin{aligned}(\lambda+2\mu)\sin hu\frac{d\epsilon}{du}+\mu(\cos hu-\cos v)\frac{d\omega}{dv}&=-\rho a\frac{d^2\xi}{dt^2}\frac{\sin hu}{\cos hu-\cos v}\\(\lambda+2\mu)\sin hu\frac{d\epsilon}{dv}-\mu(\cos hu-\cos v)\frac{d\omega}{du}&=-\rho a\frac{d^2\eta}{dt^2}\frac{\sin hu}{\cos hu-\cos v}\end{aligned}\right\}\text{.....(4)},$$

where λ and μ are the ordinary coefficients of elasticity, ρ is the density, and*

$$\epsilon=-\frac{1}{a}\left\{\left(\frac{d\xi}{du}+\frac{d\eta}{dv}\right)(\cos hu-\cos v)+\frac{\xi(1-\cos hu\cos v)-\eta\sin hu\sin v}{\sin hu}-(\xi\sin hu+\eta\sin v)\right\},$$

$$\omega=\frac{1}{a}\left\{\left(\frac{d\eta}{du}-\frac{d\xi}{dv}\right)\sin hu+\frac{\xi\sin v\sin hu-\eta\sin^2 hu}{\cos hu-\cos v}\right\}.$$

These equations give, by differentiation and reduction,

$$\frac{d^2\epsilon}{du^2}+\frac{d^2\epsilon}{dv^2}+\frac{1-\cos hu\cos v}{\sin hu(\cos hu-\cos v)}\frac{d\epsilon}{du}-\frac{\sin v}{\cos hu-\cos v}\frac{d\epsilon}{dv}=\frac{a^2\rho\dfrac{d^2\epsilon}{dt^2}}{(\lambda+2\mu)(\cos hu-\cos v)^2}\text{...(5)},$$

$$\frac{d^2\omega}{du^2}+\frac{d^2\omega}{dv^2}-\frac{1-\cos hu\cos v}{\sin hu(\cos hu-\cos v)}\frac{d\omega}{du}+\frac{\sin v}{\cos hu-\cos v}\frac{d\omega}{dv}=\frac{a^2\rho\dfrac{d^2\omega}{dt^2}}{\mu(\cos hu-\cos v)^2}\text{.....(6)}.$$

If these equations are solved we shall be able to obtain general values of ξ and η which will satisfy any specified surface conditions. I have been unable to obtain solutions of these equations beyond a particular integral, it will however be noticed that when the corresponding equations are obtained in solving for a general displacement in polar or cylindrical co-ordinates, if symmetrical about an axis; the complete solution will, the squares of the inverse wave length being neglected, be the solutions of the equations got by neglecting the effective force. These will not be the general solutions, nor shall we be able to express an arbitrary displacement by means of them; but we shall find values for the displacement that will satisfy several definite surface conditions, and among these will be the conditions resulting from a uniform twist if the ring is of small section. Now equation (5), if the right-hand side be neglected, is Laplace's ordinary equation, as can be shewn by transformation. This is indeed obvious since it is manifestly the transformed equation of the dilatation

$$\nabla^2\epsilon=\delta^2\frac{d^2\epsilon}{dt^2}.$$

The solution of Laplace's equation in toroidal co-ordinates which is applicable to space without a tore has been shewn by Mr Hicks to be

$$\sqrt{\cos hu-\cos v}\,\Sigma A_n P_n\cos(nv-a_n),$$

* ϵ and ω are connected with the ordinary dilatation and twist by the equations

$$\epsilon=\frac{1}{r^2}\frac{d(r^2u')}{dr}+\frac{1}{r^2\sin\theta}\frac{d(rv'\sin\theta)}{d\theta}\qquad\dot{\omega}=\frac{\sin\theta}{a}\left(\frac{du'}{d\theta}-\frac{d(rv')}{dr}\right),$$

where u', v' are the displacements along the radius vector and the tangent to the meridian. See Lamé, *Elasticity*, Art. 84.

Digitized by Google

where P_n is one of the solutions of the differential equation

$$\frac{d^2P_n}{du^2} + \cot hu \frac{dP_n}{du} - (n^2 - \tfrac{1}{4}) P_n = 0.$$

Hence we have to this approximation

$$\left.\begin{aligned} \epsilon &= \sqrt{\cos hu - \cos v}\, \Sigma A_n P_n \cos(nv - \alpha_n), \\ \text{and it can be shewn that} \quad \omega &= \frac{\sin hu}{\sqrt{\cos hu - \cos v}} \Sigma B_n \frac{dP_n}{du} \cos(nv - \beta_n) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

It might have been expected that any boundary condition could thus be satisfied; but this is not the case: for, when the values thus obtained for ϵ and ω are substituted in (4), the constants combine so that A_n and B_n give only one condition at the boundary.

Substituting the value of μ and putting

$$\begin{aligned} -\tfrac{1}{2}B_0 - \tfrac{3}{4}B_1 \cos\beta_1 &= C_0, & -\tfrac{3}{4}B_1 \sin\beta_1 &= D_0, \\ \tfrac{1}{4}B_1 \cos\beta_1 - \tfrac{5}{4}B_2 \cos\beta_2 &= C_1, & \tfrac{1}{4}B_1 \sin\beta_1 - \tfrac{5}{4}B_2 \sin\beta_2 &= D_1, \\ \ldots\ldots\ldots\ldots & & \ldots\ldots\ldots\ldots & \\ \frac{2n-3}{4} B_{n-1} \cos\beta_{n-1} - \frac{2n+1}{4} B_n \cos\beta_n &= C_{n-1}, & \frac{2n-3}{4} B_{n-1} \sin\beta_{n-1} - \frac{2n+1}{4} B_n \sin\beta_n &= D_{n-1}, \end{aligned}$$

and making use of the following relations, which are easily proved to hold from the sequence equations between the P_n functions, given in the paper previously referred to,

$$\left.\begin{aligned} \sin hu P_n &= \frac{1}{2n}\left(\frac{dP_{n+1}}{du} - \frac{dP_{n-1}}{du}\right), \\ \cos hu \frac{dP_n}{du} &= \frac{1}{4n}\left\{(2n-1)\frac{dP_{n+1}}{du} + (2n+1)\frac{dP_{n-1}}{du}\right\}, \\ \cos hu P_n &= \frac{1}{4n}\{(2n+1)P_{n+1} + (2n-1)P_{n-1}\}, \\ \sin hu \frac{dP_n}{du} &= \frac{4n^2-1}{8n}(P_{n+1} - P_{n-1}), \\ \text{with} \quad \sin hu \frac{dP_0}{du} + \tfrac{1}{2}\cos hu P_0 &= \tfrac{1}{2}P_1, \\ \cos hu \frac{dP_0}{du} + \tfrac{1}{2}\sin hu P_0 &= \frac{dP_1}{du}, \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (8);$$

we finally arrive at

$$\left.\begin{aligned} \frac{\rho a p^2}{\mu}\xi = \sqrt{\cos hu - \cos v}\, \Big\{ & D_0 \frac{dP_1}{du} + \left(C_0 \frac{dP_0}{du} - C_1 \frac{dP_2}{du}\right)\sin v - \left(D_0 \frac{dP_0}{du} - D_1 \frac{dP_2}{du}\right)\cos v \\ & + \Sigma\left(C_{n-1}\frac{dP_{n-1}}{du} - C_n \frac{dP_{n+1}}{du}\right)\sin nv - \left(D_{n-1}\frac{dP_{n-1}}{du} - D_n \frac{dP_{n+1}}{du}\right)\cos nv \\ \frac{\rho a p^2}{\mu}\eta = \frac{\sqrt{\cos hu - \cos v}}{2}\, [& -C_0 P_1 + (C_0 P_0 - 3C_1 P_2)\cos v + (D_0 P_0 - 3D_1 P_2)\sin v \\ & + \Sigma\{(2n-1)C_{n-1}P_{n-1} - (2n+1)C_n P_{n+1}\}\cos nv \\ & + \{(2n-1)D_{n-1}P_{n-1} - (2n+1)D_n P_{n+1}\}\sin nv] \end{aligned}\right\} (9).$$

Digitized by Google

If we had made use of the value of e we should have simply altered the arbitrary constants: but in the case of a twist there is no dilatation at the boundary, no wave of dilatation is propagated from the boundary, and e is zero throughout, so far as the disturbance caused by a twist affects the surrounding medium.

5. These are the general values for the displacement at any point; it is necessary to see whether it is possible to satisfy the surface conditions over a ring of small sectional area.

Taking all the constants to be zero, except C_0, we have

$$\xi = 2C_0 \sin v \frac{dP_0}{du} \sqrt{\cos hu - \cos v},$$

$$\eta = C_0 (P_0 \cos v - P_1) \sqrt{\cos hu - \cos v}.$$

We shall shew that these values will satisfy the surface conditions over a ring whose axis is along the critical circle.

Resolving along and parallel to the axis of the system, we have

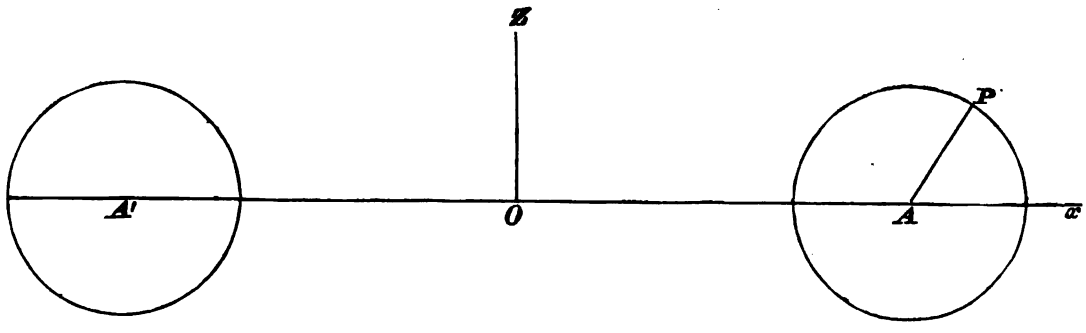
displacement parallel to the axis $= \dfrac{\xi \sin hu \sin v + \eta (1 - \cos hu \cos v)}{\cos hu - \cos v}$

$$= C_0 (P_1 \cos v - P_0) \sqrt{\cos hu - \cos v},$$

displacement perpendicular to the axis $= \dfrac{-\xi (1 - \cos hu \cos v) + \eta \sin hu \sin v}{\cos hu - \cos v}$

$$= -2C_0 \sin v \frac{dP_1}{du} \sqrt{\cos hu - \cos v},$$

after reduction by relations (8).



But if AA' is a section of the critical circle

displacement along AP $= C_0 \left(2 \dfrac{dP_1}{du} - P_1\right) \dfrac{a \sin v}{b} \sqrt{\cos hu - \cos v},$

displacement perpendicular to $AP = -C_0 \left\{ \cos v \left(2 \dfrac{dP_1}{du} - P_1\right) + 2 \dfrac{dP_0}{du} + P_0 \right\} \dfrac{a}{b} \sqrt{\cos hu - \cos v},$

b being the radius of the circle with centre at A.

Digitized by Google

*But in the neighbourhood of A,

$$P_0 = 2e^{-\frac{1}{2}u} \log (4e^u),$$

$$P_1 = \pi e^{\frac{1}{2}u};$$

therefore, if b is small,

displacement along AP = zero,

displacement perpendicular to $AP = -4C_0 e^{-\frac{1}{2}u} . \frac{a}{b} \sqrt{\cos hu}$

$$= -2\sqrt{2} C_0 \frac{a}{b}.$$

Hence a displacement $b\omega \sin pt$ at every point of the surface of the ring of small section whose axis lies along the critical circle is represented by

$$\left.\begin{aligned} \xi &= 2C_0 \sin v \frac{dP_0}{du} \sqrt{\cos hu - \cos v}, \\ \eta &= C_0 (P_0 \cos v - P_1) \sqrt{\cos hu - \cos v}, \end{aligned}\right\} \text{.....(10)},$$

if $$-2\sqrt{2}\, C_0 a = b^2 \omega \sin pt.$$

The displacement along and perpendicular to the axis of the system has been given to be

$$\left.\begin{aligned} w' &= -\frac{b^2\omega \sin pt}{2\sqrt{2}\, a} (P_1 \cos v - P_0) \sqrt{\cos hu - \cos v} \\ u' &= +\frac{b^2\omega \sin pt}{\sqrt{2}\, a} \frac{dP_1}{du} \sin v \sqrt{\cos hu - \cos v} \end{aligned}\right\} \text{.....(11)}.$$

Putting $$P_1 = \int_0^\pi \frac{d\phi}{\{\cos hu - \sin hu \cos \phi\}^{\frac{1}{2}}} = \int_0^\pi \sqrt{\cos hu - \sin hu \cos \phi} . d\phi,$$

$$P_0 = \int_0^\pi \frac{d\phi}{\sqrt{\cos hu - \sin hu \cos \phi}},$$

and, reducing by the relations,

$$\left.\begin{aligned} \cos hu &= \frac{r^2 + a^2}{L}, & \cos v &= \frac{r^2 - a^2}{L} \\ \sin hu &= \frac{2ar \sin \theta}{L}, & \sin v &= \frac{2ar \cos \theta}{L} \end{aligned}\right\} \text{.....(12)},$$

if $L^2 = (r^2 + a^2)^2 - 4a^2 r^2 \sin^2 \theta$, r being the distance from the origin and θ the co-latitude,

* For $P_0 = \int_0^\pi \frac{d\theta}{\sqrt{\cos hu - \sin hu \cos \theta}}$

$$= 2\sqrt{k'} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = 2\sqrt{k'} \log \frac{4}{k'},$$

if $k' = e^{-u}$,

$$P_1 = \int_0^\pi \sqrt{\cos hu - \sin hu \cos \theta}\, d\theta$$

$$= \frac{2}{\sqrt{k'}} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} . d\theta = \frac{2}{\sqrt{k'}} . \frac{\pi}{2}.$$

Digitized by Google

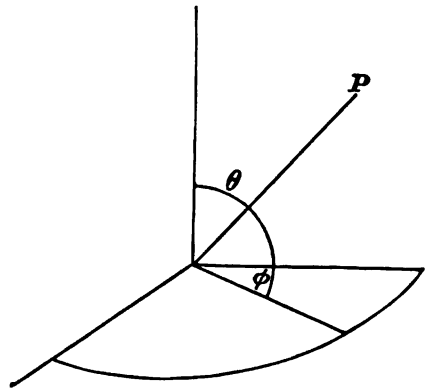
we get, in polar co-ordinates,

$$\left.\begin{aligned} w' &= -b^2 a\omega \sin pt \int_0^\pi \frac{(r\cos\phi\sin\theta - a)\,d\phi}{(r^2+a^2-2ar\sin\theta\cos\phi)^{\frac{3}{2}}} \\ {}^*u' &= +b^2 a\omega \sin pt \int_0^\pi \frac{r\cos\phi\cos\theta\, d\phi}{(r^2+a^2-2ar\sin\theta\cos\phi)^{\frac{3}{2}}} \end{aligned}\right\} \quad \text{.....(13).}$$

These are the components of the magnetic induction due to a circular current of strength $\frac{b^2\mu\omega\sin pt}{2}$, where μ is the coefficient of magnetic permeability.

For if C be the strength of the current, F, G and H the components of the vector potential at any point P,

$$F=0, \qquad H=0, \qquad G=2\frac{C}{\mu}\int_0^\pi \frac{\cos\phi\, a\, d\phi}{\sqrt{r^2+a^2-2ar\sin\theta\cos\phi}};$$



therefore the components of the magnetic induction are

$$a = -2\frac{C}{\mu}\int_0^\pi \frac{r\cos\theta\cos\phi\, a\, d\phi}{(r^2+a^2-2ar\sin\theta\cos\phi)^{\frac{3}{2}}},$$

$$b=0,$$

$$c = -2\frac{C}{\mu}\int_0^\pi \frac{r\sin\theta\cos\phi - a}{(r^2+a^2-2ar\sin\theta\cos\phi)^{\frac{3}{2}}}.$$

See Maxwell's *Electricity*, equations A, Art. 591.

As a particular case let us suppose the radius of the circular axis of the ring to be small. In this case we shall get the simplest components by resolving along and perpendicular to the radius vector from the centre of the circular axis. We get

$$\text{displacement along the radius vector} = b^2a^2\omega\sin pt\int_0^\pi \frac{\cos\theta\, d\phi}{(r^2+a^2-2ar\sin\theta\cos\phi)^{\frac{3}{2}}}$$

$$= \frac{\pi b^2 a^2 \omega \sin pt}{r^2}\cos\theta;$$

* For $\int_0^\pi \sqrt{\cos hu - \sin hu\cos\phi}\, d\phi$

$$= \int_0^\pi \frac{d\phi}{\{\cos hu - \sin hu\cos\phi\}^{\frac{3}{2}}};$$

$$\therefore \int_0^\pi \frac{\sin hu(1+\cos^2\phi) - 2\cos hu\cos\phi}{\{\cos hu - \sin hu\cos\phi\}^{\frac{3}{2}}}\, d\phi = 0,$$

whence, taking $P_1 = \int_0^\pi \sqrt{(\cos hu - \sin hu\cos\phi}\, d\phi$, and reducing by above relation, we get required result.

Digitized by Google

$$\left.\begin{array}{l}\text{displacement perpendicular to the radius vector and tending from the axis of the system}\\ \qquad\qquad\qquad\

form, since the displacements given in equations (9) will give the strain at any point produced by several other displacements of the boundary besides a simple twist: in fact all which can be deduced from a dilatation and twist given by

$$\epsilon = \sqrt{\cos hu - \cos v}\,\Sigma(A_n P_n \cos nv + A_n' P_n' \sin nv),$$

$$\omega = \frac{\sin hu}{\sqrt{\cos hu - \cos v}}\,\Sigma\left(B_n \frac{dP_n}{du}\cos nv + B_n' \frac{dP_n'}{du}\sin nv\right).$$

The general values for the displacement can also by the help of the foregoing be deduced for the case of a long wave, if the displacement is symmetrical about an axis.

For in this case the dilatation is known* to be

$$\epsilon = \Sigma L_n r^{-(n+1)} Z_n \left\{1 + \frac{k^2 r^2}{(2n-1)}\right\},$$

to a second approximation if Z_n is the zonal harmonic of the order n.

Also we have

$$\Sigma L_n r^{-(n+1)} Z_n = \sqrt{\cos hu - \cos v}\,\Sigma(A_n P_n \cos nv + A_n' P_n \sin nv),$$

since each side of the above equation is the general solution of Laplace's equation in the case of symmetry about an axis.

Hence $$\Sigma \frac{L_n r^{-(n+1)} Z_n}{2(2n-1)} = \sqrt{\cos hu - \cos v}\,\Sigma(C_n P_n \cos nv + C_n' P_n \sin nv),$$

if the C's satisfy the equations

$$\begin{aligned}
A_0 &= -4C_0 - C_1, & & \\
A_1 &= 2C_0 - 4C_1 - 3C_2, & A_1' &= -4C_1' - 3C_2', \\
A_2 &= 3C_1 - 4C_2 - 5C_3, & A_2' &= 3C_1' - 4C_2' - 5C_3', \\
A_3 &= 5C_2 - 4C_3 - 7C_4, & A_3' &= 5C_2' - 4C_3' - 7C_4', \\
A_n &= (2n-1)C_{n-1} - 4C_n - (2n+1)C_{n+1}, & A_n' &= (2n-1)C_{n-1}' - 4C_n' - (2n+1)C_{n+1}',
\end{aligned}$$

as can be verified by differentiation with respect to r.

Whence we get as the value of ϵ, if fourth powers of hr be neglected,

$$\epsilon = \sqrt{\cos hu - \cos v}\,\Sigma(A_n P_n \cos nv + A_n' P_n' \sin nv)$$
$$+ k^2 a^2 \frac{\cos hu + \cos v}{\sqrt{\cos hu - \cos v}}\,\Sigma(C_n P_n \cos nv + C_n' P_n' \sin nv) \ldots\ldots\ldots\ldots\ldots (15),$$

if the C's and A's are connected by the equations just given.

The complete value for ω to a second approximation might also be determined and values for ξ and η obtained which would satisfy any given conditions at the surface of a ring the u of which is given; but the complicated character of the relations connecting the constants would make the determination of the displacement in any special case a matter of considerable difficulty.

* *Transactions*, Vol. XIV. page 7, equations (3).

Digitized by Google

Leaving this part of the subject let us now see how a rectilineal twist will be acted upon if placed in a field of vibration. For this purpose it will be necessary to find the general expression for a displacement in cylindrical co-ordinates in order to get suitable surface conditions. The case of two parallel rectilineal twists will first be investigated and the general result deduced from this simple case.

7. To find expressions for a steady periodic disturbance in an elastic medium unlimited in extent: if the displacement is given for all points which are on the surface of an infinite straight cylinder when in their undisturbed positions, the motion being supposed to take place in planes perpendicular to axis of cylinder. The disturbance will reduce to a case of motion in two dimensions. We shall have, by the ordinary theory of elasticity, the disturbance compounded of u and v along and perpendicular to radius vector from the axis where u and v are given by the equations*

$$\left.\begin{aligned}(\lambda+2\mu)\frac{d\epsilon}{dr}+\frac{\mu}{r}\frac{d\gamma}{d\theta}&=\rho\frac{d^2u}{dt^2}\\ \frac{\lambda+2\mu}{r}\frac{d\epsilon}{d\theta}-\mu\frac{d\gamma}{dr}&=\rho\frac{d^2v}{dt^2}\end{aligned}\right\}\quad\text{.....(16),}$$

where
$$\epsilon=\frac{1}{r}\frac{d(ru)}{dr}+\frac{1}{r}\frac{dv}{d\theta},$$
$$\gamma=\frac{1}{r}\frac{du}{d\theta}-\frac{1}{r}\frac{d(rv)}{dr},$$

r and θ being co-ordinates of undisturbed positions.

Equations (1) reduce to

$$\left.\begin{aligned}\frac{d}{dr}\left(r\frac{d\epsilon}{dr}\right)+\frac{1}{r}\frac{d^2\epsilon}{d\theta^2}&=\frac{\rho r}{\lambda+2\mu}\frac{d^2\epsilon}{dt^2}\\ \frac{d}{dr}\left(r\frac{d\gamma}{dr}\right)-\frac{1}{r}\frac{d^2\gamma}{d\theta^2}&=\frac{\rho r}{\mu}\frac{d^2\gamma}{dt^2}\end{aligned}\right\}\quad\text{.....(17).}$$

Thus ϵ and γ are independent, as can be otherwise shewn; and ϵ is the same arbitrary function of $\lambda+2\mu$ as γ is of μ.

To solve equation for ϵ in a periodic form, put

$$\epsilon=we^{ipt},$$

whence w must satisfy the equation

$$\frac{d}{dr}\left(r\frac{dw}{dr}\right)+\frac{1}{r}\frac{d^2w}{d\theta^2}+k^2rw=0,$$

if
$$\rho p^2=(\lambda+2\mu)k^2.$$

By Fourier's theorem w is of the form

$$\Sigma w_n\sin(n\theta+\alpha_n),$$

where w_n satisfies
$$\frac{d^2w_n}{dr}+\frac{1}{r}\frac{dw_n}{dr}+\left(k^2-\frac{n^2}{r^2}\right)w_n=0\quad\text{.....(18).}$$

* Lamé, *Elasticity*, Art. 78.

Digitized by Google

It is known* that the solution of this equation, which is applicable to space outside the cylinder, is

$$w_n = A_n (kr)^n \left\{\frac{d}{d(kr)^2}\right\}^n w_0 \quad\text{.....(19)},$$

if $$w_0 = \left(\Gamma + \tfrac{1}{2}\pi\iota + \log\frac{kr}{2}\right)\left\{1 - \frac{k^2r^2}{2^2} + \frac{k^4r^4}{2^2.4^2}\ldots\right\} + \left\{\frac{k^2r^2}{2^2} - \frac{k^4r^4}{2^2.4^2}S_2 + \frac{k^6r^6}{2^2.4^2.6^2}S_3\ldots\right\},$$

where $$S_m = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots \frac{1}{m},$$

and Γ is Euler's constant ($= \cdot 5772156\ldots$).

The solution applicable to space within the cylinder is the ordinary Bessel's function of order n.

Hence w_0 being of the form given above

$$\epsilon = \Sigma A_n \sin(n\theta + \alpha_n)\, e^{\iota pt} (kr)^n \left\{\frac{d}{d(kr)^2}\right\}^n w_0,$$

with another term obtained by changing the sign of ι.

Changing the constants and substituting the trigonometrical form of solution for the exponential one, we have

$$\epsilon = \Sigma A_n \sin(n\theta + \alpha_n)\,(kr)^n \left\{\frac{d}{d(kr)^2}\right\}^n y_0,$$

if $$y_0 = \sin(pt + \zeta_1)\left\{\left(\Gamma + \log\frac{kr}{2}\right)\left(1 - \frac{k^2r^2}{2^2} + \frac{k^4r^4}{2^2.4^2} + \ldots\right) + \left(\frac{k^2r^2}{2^2} - \frac{k^4r^4}{2^2.4^2}S_2\ldots\right)\right\}$$
$$+ \frac{\pi}{2}\cos(pt + \zeta_1)\left\{1 - \frac{k^2r^2}{2^2} + \frac{k^4r^4}{2^2.4^2}S_2\ldots\right\} \quad\text{.....(20)},$$

and similarly,

$$\gamma = \Sigma B_n \cos(n\theta + \beta_n)\,(hr)^n \left\{\frac{d}{d(hr)^2}\right\}^n z_0,$$

where $\rho p^2 = \mu h^2$ and z_0 is the same function of hr as y_0 is of kr.

To get u and v we have by equations (16)

$$\left.\begin{aligned} u &= -\frac{1}{k^2}\frac{d\epsilon}{dr} - \frac{1}{h^2 r}\frac{d\gamma}{d\theta} \\ v &= -\frac{1}{k^2 r}\frac{d\epsilon}{d\theta} + \frac{1}{h^2}\frac{d\gamma}{dr} \end{aligned}\right\} \quad\text{.....(21)},$$

which will give general values for the displacement at any point in the medium.

We shall require the terms in ϵ, γ, u and v corresponding to $n = 0$ and $n = 1$.

* See Rayleigh, *Theory of Sound*, Vol. II. Art. 341.

Digitized by Google

They are

$$\left.\begin{aligned}
\epsilon = {} & k^2 A_0 \sin(pt+\zeta_0)\left\{\left(\Gamma+\log\frac{kr}{2}\right)\left(1-\frac{k^2r^2}{2^2}+\frac{k^4r^4}{2^2.4^2}\dots\right)+\frac{k^2r^2}{2^2}S_1-\frac{k^4r^4}{2^2.4^2}S_2\dots\right. \\
& + k^2 A_0 \cos(pt+\zeta_0)\frac{\pi}{2}\left\{1-\frac{k^2r^2}{2^2}+\frac{k^4r^4}{2^2.4^2}\dots\right\} \\
& + k^2 A_1 \sin(pt+\zeta_1).\frac{\sin(\theta+\alpha_1)}{r}\left\{1-\frac{k^2r^2}{2^2}+\frac{k^4r^4}{2^2.4^2}\dots\dots\right\} \\
& - k^2 A_1 \sin(pt+\zeta_1).\frac{\sin(\theta+\alpha_1)}{r}\left\{\left(\Gamma+\log\frac{kr}{2}\right)\left(\frac{k^2r^2}{2}-\frac{k^4r^4}{2^2.4}+\frac{k^6r^6}{2^2.4^2.6}\dots\right)-\left(\frac{k^2r^2}{2}S_1-\frac{k^4r^4}{2^2.4}S_2\dots\right)\right\} \\
& - k^2 A_1 \cos(pt+\zeta)\frac{\sin(\theta+\alpha_1)}{r}.\frac{\pi}{2}\left\{\frac{k^2r^2}{2}-\frac{k^4r^4}{2^2.4}+\frac{k^6r^6}{2^2.4^2.6}\dots\right\}
\end{aligned}\right\}\ (22),$$

γ is the same function of hr as ϵ is of kr.

Also for $n=0$,

$$\left.\begin{aligned}
u_0 = {} & -\frac{A_0}{r}\sin(pt+\zeta_0)\left\{1-\frac{k^2r^2}{2^2}+\frac{k^4r^4}{2^2.4^2}-\frac{k^6r^6}{2^2.4^2.6^2}\dots\right\} \\
& + \frac{A_0}{r}\sin(pt+\zeta_0)\left\{\left(\Gamma+\log\frac{kr}{2}\right)\left(\frac{k^2r^2}{2}-\frac{k^4r^4}{2^2.4}\dots\right)-\left(\frac{k^2r^2}{2}S_1-\frac{k^4r^4}{2^2.4}S_2\dots\right)\right\} \\
& + \frac{A_0}{r}\cos(pt+\zeta_0)\frac{\pi}{2}\left\{\frac{k^2r^2}{2}-\frac{k^4r^4}{2^2.4}+\frac{k^6r^6}{2^2.4^2.6}\dots\right\}, \\
v_0 = {} & \frac{B_0}{r}\sin(pt+\zeta_0')\left\{1-\frac{h^2r^2}{2^2}+\frac{h^4r^4}{2^2.4^2}-\frac{h^6r^6}{2^2.4^2.6^2}\dots\right\} \\
& - \frac{B_0}{r}\sin(pt+\zeta_0')\left\{\left(\Gamma+\log\frac{hr}{2}\right)\left(\frac{h^2r^2}{2}-\frac{h^4r^4}{2^2.4}\dots\right)-\left(\frac{h^2r^2}{2}S_1-\frac{h^4r^4}{2^2.4}S_2\dots\right)\right\} \\
& - \frac{B_0}{r}\cos(pt+\zeta_0')\frac{\pi}{2}\left\{\frac{h^2r^2}{2}-\frac{h^4r^4}{2^2.4}+\frac{h^6r^6}{2^2.4^2.6}\dots\right\}
\end{aligned}\right\}\ \dots\dots(23),$$

ζ_0' being the epoch (in γ) corresponding to ζ_0 in ϵ.

And for $n=1$,

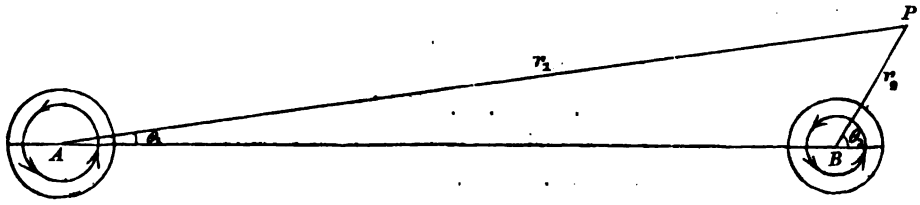
$$\left.\begin{aligned}
u_1 = {} & \frac{A_1}{r^2}\sin(pt+\zeta_1)\sin(\theta+\alpha_1)\left\{1+\frac{3k^2r^2}{2^2}-\frac{7k^4r^4}{2^2.4^2}+\dots\right\} \\
& + \frac{A_1}{r^2}\sin(pt+\zeta_1)\sin(\theta+\alpha_1)\left\{\left(\Gamma+\log\frac{kr}{2}\right)\left(\frac{k^2r^2}{2}-\frac{3k^4r^4}{2^2.4}+\dots\right)-\left(\frac{k^2r^2}{2}-\frac{3k^4r^4}{2^2.4}S_2+\dots\right)\right\} \\
& + \frac{A_1}{r^2}\cos(pt+\zeta_1)\sin(\theta+\alpha_1).\frac{\pi}{2}\left(\frac{k^2r^2}{2}-\frac{3k^4r^4}{2^2.4}+\dots\right\} \\
& + \frac{B_1}{r^2}\sin(pt+\zeta_1')\sin(\theta+\beta_1)\left\{1-\frac{h^2r^2}{2^2}+\frac{h^4r^4}{2^2.4^2}-\dots\right\} \\
& - \frac{B_1}{r^2}\sin(pt+\zeta_1')\sin(\theta+\beta_1)\left\{\left(\Gamma+\log\frac{hr}{2}\right)\left(\frac{h^2r^2}{2}-\frac{h^4r^4}{2^2.4}+\dots\right)-\left(\frac{h^2r^2}{2}-\frac{h^4r^4}{2^2.4}S_2+\dots\right)\right\} \\
& - \frac{B_1}{r^2}\cos(pt+\zeta_1')\sin(\theta+\beta_1).\frac{\pi}{2}\left(\frac{h^2r^2}{2}-\frac{h^4r^4}{2^2.4}+\dots\right)
\end{aligned}\right|$$

Digitized by Google

$$\left.\begin{aligned}
v_1 = &-\frac{A_1}{r^2}\sin(pt+\zeta_1)\cos(\theta+\alpha_1)\left\{1-\frac{k^2r^2}{2^2}+\frac{k^4r^4}{2^2.4^2}\dots\right\}\\
&+\frac{A_1}{r^2}\sin(pt+\zeta_1)\cos(\theta+\alpha_1)\left\{\left(\Gamma+\log\frac{kr}{2}\right)\left(\frac{k^2r^2}{2}-\frac{k^4r^4}{2^2.4}+\dots\right)-\left(\frac{k^2r^2}{2}-\frac{k^4r^4}{2^2.4}S_2+\dots\right)\right\}\\
&+\frac{A_1}{r^2}\cos(pt+\zeta_1)\cos(\theta+\alpha_1)\frac{\pi}{2}\left(\frac{k^2r^2}{2}-\frac{k^4r^4}{2^2.4}+\dots\right)\\
&-\frac{B_1}{r^2}\sin(pt+\zeta_1')\cos(\theta+\beta_1)\left\{1+\frac{3h^2r^2}{2^2}-\frac{7h^4r^4}{2^2.4^2}+\dots\right\}\\
&-\frac{B_1}{r^2}\sin(pt+\zeta_1')\cos(\theta+\beta_1)\left\{\left(\Gamma+\log\frac{hr}{2}\right)\left(\frac{h^2r^2}{2}-\frac{3h^4r^4}{2^2.4}+\dots\right)-\left(\frac{h^2r^2}{2}-\frac{3h^4r^4}{2^2.4}S_2+\dots\right)\right\}\\
&-\frac{B_1}{r^2}\cos(pt+\zeta_1')\cos(\theta+\beta_1).\frac{\pi}{2}\left(\frac{h^2r^2}{2}-\frac{3h^4r^4}{2^2.4}+\dots\right)
\end{aligned}\right\}\quad(24).$$

Mutual action of two rectilineal twists.

8. Let us suppose two cylindrical surfaces to have their centres fixed and to be



moving with oscillatory twists about their centres; the angular displacements at any time t being given by $\omega_1 \sin pt$, $\omega_2 \sin pt$.

We shall suppose the displacements to be so small that there is no slipping at the boundary; also that the waves are long, so that the distance c between their centres is small compared with the length of the wave of displacement. The radii are supposed to be small compared with the distance c.

For all points in space, the distance of which from either cylinder is not comparable with the wave-length, we have by equation (1) a displacement compounded of

$$\frac{\omega a^2 \sin pt}{r_1} \text{ perpendicular to } AP$$

and

$$\frac{\omega b^2 \sin pt}{r_2} \text{ perpendicular to } BP,$$

a and b being the radii of the cylinders.

These are equivalent to a component

$$\frac{\omega_1 a^2}{r_1^2}\, c \sin\theta_2 \sin pt \text{ along } BP,$$

and a component $\dfrac{\omega_1 a^2}{r_1^2}\, c\left(\cos\theta_2 - \dfrac{r_2}{c}\right)\sin pt + \dfrac{\omega_2 b^2}{r_2}\sin pt$ perpendicular to BP.

Digitized by Google

In the immediate neighbourhood of the surface of B these reduce to

$$\frac{\omega_1 a^2}{c}\left(\sin\theta_2 - \frac{r_2}{c}\sin 2\theta_2 + \frac{r_2^2}{c^2}\sin 3\theta_2 - \frac{r_2^3}{c^3}\sin 4\theta_2 \ldots\right)\sin pt,$$

and
$$\frac{\omega_1 a^2}{c}\left(\cos\theta_2 - \frac{r_2}{c}\cos 2\theta_2 + \frac{r_2^2}{c^2}\cos 3\theta_2 - \frac{r_2^3}{c^3}\cos 4\theta_2 \ldots\right)\sin pt + \frac{\omega_2 b^2}{r_2}\sin pt.$$

Hence if the terms due to the reflection at the surface of B are represented by u_2, v_2, we have

$$\left.\begin{aligned}
&\text{displacement along } BP &&= \frac{\omega_1 a^2}{c}\left(\sin\theta_2 - \frac{r_2}{c}\sin 2\theta_2 \ldots\right)\sin pt + u_2\\
&\text{displacement perpendicular to } BP &&= \frac{\omega_1 a^2}{c}\left(\cos\theta_2 - \frac{r_2}{c}\cos 2\theta_2 \ldots\right)\sin pt + v_2 + \frac{\omega_2 b^2}{r_2}\sin pt
\end{aligned}\right\}\ldots(25);$$

u_2 and v_2 must be of the forms given in equations (24), the constants being determined by the boundary conditions.

Suppose for example that there is no slipping at the boundary.

The component along the radius from the centre must be zero and the component perpendicular to the radius equal to $b\omega\sin pt$.

Taking the terms dependent on $\sin\theta_2$, $\cos\theta_2$ we get for the constants in equation (24)

$$\zeta_1 = \zeta_1' \text{ and } \tan\zeta_1 = -\frac{\pi(h^2+k^2)}{2\left\{k^2\log\left(\frac{kb}{2}+\Gamma\right) + h^2\log\left(\frac{hb}{2}+\Gamma\right)\right\}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26),$$

$$A_1 = \frac{4a^2\omega_1}{\pi c(h^2+k^2)}\sin\zeta_1\left(1+\frac{h^2b^2}{4}\right),$$

$$B_1 = -\frac{4a^2\omega_1}{\pi c(h^2+k^2)}\sin\zeta_1\left(1+\frac{k^2b^2}{4}\right),$$

whence

$$u_2 = \frac{\sin(pt+\zeta_1)}{r^2}\sin\theta_2\left[A_1\left\{1+\left(\log\frac{kr}{2}+\Gamma+\frac{1}{2}\right)\frac{k^2r^2}{2}\right\} + B_1\left\{1-\left(\log\frac{hr}{2}+\Gamma-\frac{1}{2}\right)\frac{h^2r^2}{2}\right\}\right]$$

$$+\cos(pt+\zeta_1)\sin\theta_2(k^2A_1 - h^2B_1)\frac{\pi}{4},$$

$$v_2 = -\frac{\sin(pt+\zeta_1)}{r^2}\cos\theta_2\left[A_1\left\{1-\left(\log\frac{kr}{2}+\Gamma-\frac{1}{2}\right)\frac{k^2r^2}{2}\right\} + B_1\left\{1+\left(\log\frac{hr}{2}+\Gamma+\frac{1}{2}\right)\frac{h^2r^2}{2}\right\}\right]$$

$$+\cos(pt+\zeta_1)\cos\theta_2(k^2A_1 - h^2B_1)\frac{\pi}{4},$$

Digitized by Google

or

$$\left.\begin{aligned}
u_2 = {} & \frac{a^2\omega_1}{c}\left\{\frac{h^2-k^2}{\pi(h^2+k^2)}\left(\frac{b^2}{r^2}-1\right)\sin\zeta_1 - \frac{k^2\left(\log\frac{kr}{2}+\Gamma\right)+h^2\left(\log\frac{hr}{2}+\Gamma\right)}{k^2\left(\log\frac{kb}{2}+\Gamma\right)+h^2\left(\log\frac{hb}{2}+\Gamma\right)}\cos\zeta_1\right\}\sin(pt+\zeta_1)\sin\theta_2 \\
& + \frac{a^2\omega_1}{c}\sin\zeta_1\cos(pt+\zeta_1)\sin\theta_2, \\
v_2 = {} & -\frac{a^2\omega_1}{c}\left\{\frac{h^2-k^2}{\pi(h^2+k^2)}\left(\frac{b^2}{r^2}-1\right)\sin\zeta_1 + \frac{k^2\left(\log\frac{kr}{2}+\Gamma\right)+h^2\left(\log\frac{hr}{2}+\Gamma\right)}{k^2\left(\log\frac{kb}{2}+\Gamma\right)+h^2\left(\log\frac{hb}{2}+\Gamma\right)}\cos\zeta_1\right\}\sin(pt+\zeta_1)\cos\theta_2 \\
& + \frac{a^2\omega_1}{c}\sin\zeta_1\cos(pt+\zeta_1)\cos\theta_2.
\end{aligned}\right\} \quad \text{.....(27).}$$

Since the series in equations (24) are convergent it is clear that, at any distance from the twists which is comparable with the length of the wave, the reflected displacement will be negligible as compared with the direct twisting action. This will not be the case if we are dealing with distances which are small compared with the length of the wave, but it can be shewn that the effect is of the same kind as that already given, or is of higher orders in $\frac{b}{c}$ and $\frac{a}{c}$. Omitting for the present the terms due to this "induced twist*", we shall proceed to find the force between the twists—the displacement being taken to be that given in equations (25): u_2, v_2 being given in equations (27).

To find the force on B we must first calculate the normal and tangential forces

which act at any point of its surface, resolve and then integrate for the time $\frac{2\pi}{p}$ which is a complete period.

* If $m = \frac{h^2-k^2}{\pi(h^2+k^2)}\sin\zeta_1$, a double reflection will give a resolved displacement at the surface of B composed of

$$u_2' = \frac{a^2\omega_1}{c}\sin\theta_1\{\sin pt - 2m\sin(pt+\zeta_1) + m^2\sin(pt+2\zeta_1)\}.$$

$$v_2' = \frac{a^2\omega_1}{c}\cos\theta_2\{\sin pt - 2m\sin(pt+\zeta_1) + m^2\sin(pt+2\zeta_1)\}.$$

This can be treated like the terms $\frac{a^2\omega_1}{c}\sin\theta_2$ and $\frac{a^2\omega_1}{c}\cos\theta_2$ in equation (25), and will by result (29) give a force

$$-\frac{8\pi\rho pa^2\omega_1\omega_2}{c(h^2+k^2)}\sin\zeta_1\cos\zeta_1\{1-2m\cos\zeta_1+m^2\cos 2\zeta_1\}.$$

A subsequent reflection will give a force

$$-\frac{8\pi\rho pa^2\omega_1\omega_2}{c(h^2+k^2)}\sin\zeta_1\cos\zeta_1$$
$$\{1-4m\cos\zeta_1+6m^2\cos 2\zeta_1-4m^3\cos 3\zeta_1+m^6\cos 4\zeta_1\},$$

the series being continued for r reflections at each surface if $(2r+1)c$ is comparable with the wave length.

All these terms will give the same effect as the one investigated in the text since m is small.

Digitized by Google

The force acting along the normal at any point of the surface is

$$N_r = (\lambda + 2\mu)\frac{du}{dr} + \lambda\left(\frac{u}{r} + \frac{1}{r}\frac{du}{d\theta}\right),$$

and the force along the tangent to the surface is

$$T = \mu\left(\frac{dv}{dr} + \frac{1}{r}\frac{du}{d\theta} - \frac{v}{r}\right).$$

The only terms to be taken into account are those involving $\cos\theta_2$, $\sin\theta_2$, as all the others disappear on integration over the surface.

At the surface $\quad u = 0, \qquad v = b\omega_2 \sin pt,$

$$\frac{du}{d\theta} = 0, \qquad \frac{dv}{d\theta} = 0;$$

$$\frac{du}{dr} = \frac{4a^2\omega_1}{\pi bc}\frac{k^2}{h^2+k^2}\sin\zeta\sin(pt+\zeta)\sin\theta_2,$$

$$\frac{dv}{dr} = \frac{4a^2\omega_1}{\pi bc}\frac{h^2}{h^2+k^2}\sin\zeta\sin(pt+\zeta)\cos\theta_2;$$

therefore, since $\quad (\lambda + 2\mu)k^2 = \mu h^2 = \rho p^2,$

$$\left.\begin{aligned} N_r &= \frac{4a^2\rho p^2\omega_1}{\pi bc(h^2+k^2)}\sin\zeta\sin(pt+\zeta)\sin\theta_2 \\ T &= \frac{4a^2\rho p^2\omega_1}{\pi bc(h^2+k^2)}\sin\zeta\sin(pt+\zeta)\cos\theta_2 \end{aligned}\right\} \quad \text{.....(28).}$$

In calculating the force on B it must be remembered that P is in its displaced position: hence the angle which the normal to the surface makes with AB is no longer θ_2 but $\theta_2 + \omega_2 \sin pt$.

Therefore, $\quad$ Force on AB revolving along AB in the direction AB

$$= \int_0^{2\pi}\{N_r\cos(\theta_2 + \omega_2\sin pt) - T\sin(\theta_2 + \omega_2\sin pt)\}\, b\,d\theta$$

$$= -\frac{4a^2\rho p^2\omega_1}{\pi c(h^2+k^2)}\sin\zeta\sin(pt+\zeta)\int_0^{2\pi}\sin(\omega_2\sin pt)\,d\theta$$

$$= -\frac{8a^2\rho p^2\omega_1\omega_2}{c(h^2+k^2)}\sin\zeta\sin pt\,(pt+\zeta),$$

since ω_2 is small;

Force perpendicular to AB

$$= \frac{8a^2\rho p^2\omega_1}{c(h^2+k^2)}\cos(\omega_2\sin pt)\sin(pt+\zeta)\sin\zeta.$$

The only admissible part of this is

$$\frac{8\rho p^2 a^2\omega_1}{c(h^2+k^2)}\sin(pt+\zeta)\sin\zeta,$$

the term $\omega_1\omega_2^2$ being negligible as being of a higher order of small quantities.

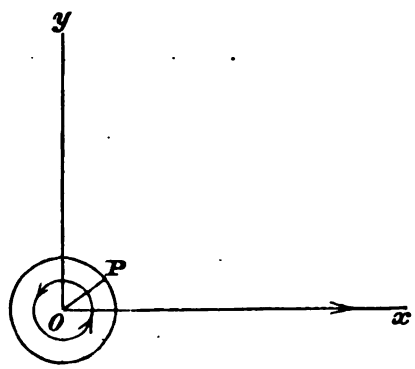
Digitized by Google

To find the whole force on B we must integrate for the time $\frac{2\pi}{p}$, and we get the force perpendicular to AB to be zero; and the force in direction AB during the period $\frac{2\pi}{p}$

$$= -\frac{8\pi\rho p a^2 \omega_1 \omega_2}{c(h^2+k^2)} \sin\zeta \cos\zeta \text{.....(29).}$$

This is an attraction if the twists are in the same sense, and a repulsion if they are in opposite senses at the same time.

9. More generally, let a tube of twist be placed in an elastic medium which is undergoing any periodic disturbance $F \sin pt$, the angular displacement of the surface of



the tube at time t being $\omega \sin pt$, and reckoned to be positive in the direction indicated by the arrow heads in the figure.

Let Ox be the direction of the component of F resolved perpendicularly to the axis of the tube, the trace of which on the paper is O.

The component which is parallel to the axis will clearly have no action on the surface during a complete oscillation, for its action during that period will be $2\pi b \int_0^{\frac{2\pi}{p}} Z \sin pt\, dt$, which is zero, Z being the component parallel to the axis. The section of the tube being small, X may be taken to have the same value all over the surface: and even if the small variations of X are taken into account it can be seen that they will disappear on integration over the surface like the higher terms in equation (25) above.

As before we shall have to add terms similar to those given in equation (27), namely, along OP

$$X\left\{\frac{h^2-k^2}{\pi(h^2+k^2)}\left(\frac{b^2}{r^2}-1\right)\sin\zeta_1 - \frac{k^2\left(\log\frac{kr}{2}+\Gamma\right)+h^2\left(\log\frac{hr}{2}+\Gamma\right)}{k^2\left(\log\frac{kb}{2}+\Gamma\right)+h^2\left(\log\frac{hb}{2}+\Gamma\right)}\cos\zeta_1\right\}\sin(pt+\zeta_1)\sin\theta$$
$$+ X \sin\zeta_1 \cos(pt+\zeta_1)\sin\theta;$$

Digitized by Google

and perpendicular to OP

$$-X\left\{\frac{h^2-k^2}{\pi(h^2+k^2)}\left(\frac{b^2}{r^2}-1\right)\sin\zeta_1+\frac{k^2\left(\log\frac{kr}{2}+\Gamma\right)+h^2\left(\log\frac{hr}{2}+\Gamma\right)}{k^2\left(\log\frac{kr}{2}+\Gamma\right)+h^2\left(\log\frac{hr}{2}+\Gamma\right)}\cos\zeta_1\right\}\sin(pt+\zeta_1)\cos\theta$$
$$+X\sin\zeta_1\cos(pt+\zeta_1)\cos\theta,$$

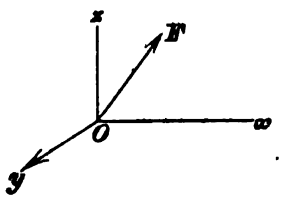
X being the radius and ζ_1 given in equation (26).

Hence we shall, as before, find the force on the twist to be in the direction Oy and of magnitude

$$-\frac{8\pi\rho pX_0\omega}{h^2+k^2}\sin\zeta_1\cos\zeta_1,$$

and we have the following results.

If the field of vibration be explored by a twist which is not supposed to have inductive action on the surfaces giving rise to the displacement, and if its axis is along



Oz, the direction of the twist bearing to the positive direction of OZ the same relation as the thread does to the axis of a right-handed screw, then if the displacement in the medium at any point is represented by $F\sin pt$; a force will be experienced by the twist in the direction Oy, where Oy is at right angles to the plane containing the axis of the twist and the direction of F, the magnitude of the force being equal to

$$kF\omega\sin FOz,$$

where k is a constant, depending on the nature of the medium and the period of the twist.

Also, if Ox is the normal to the plane of yOz drawn in the same direction as F; Ox, Oy, Oz will be in cyclical order. This is the same as the relation between the magnetic induction, the electromotive force, and the current given in Maxwell's *Electricity*, Vol. II. Chap. 8, where it is stated that if the primary circuit is fixed and the primary current kept constant, and if the magnetic field be explored by a linear circuit conveying a current it will experience a force similar in magnitude and direction to that given above if F represents the magnetic induction and ω the current.

10. It remains to add a note on the possible application of these results in the formation of a theory of magnetic action. It has for some time been inferred from

Digitized by Google

several experimental results* that the particles of a magnetic body are polarised, and that magnetization simply means the inclination of their axes in the same direction. If we suppose these particles to be small oscillatory circular twists, we see by result (15) that the displacement produced by each of them will be the same as the magnetic induction due to magnets of strength $\frac{\pi b^2 a^2 \omega \sin pt}{2}$, where ω is the angular displacement at the surface, b the radius of the cross section, and a that of the circular axis of the twist. Also, it has been shewn that a force will act upon a rectilineal twist similar to that which would act on a current in the magnetic field, and it can be proved that the force on another small ring would be that experienced by a small magnet at the point. If magnetization of a body means the inclination of the axes in a particular direction, the summation of the effect of these rings will be the summation of the supposed magnetic effect of the particles of the body. The general condition of a bar magnet on this theory will be, that its surface is vibrating tangentially, so that the direction of vibration is parallel to the axis of the body.

A cylindrical body carrying a current will, on a similar hypothesis, have its surface also vibrating tangentially; but in this case the direction of vibration will be perpendicular to the axis of the cylinder: and, since it has been shewn in a former paper† that pulsating spheres in an elastic medium will act upon each other like electrified spheres, those in like phases repelling and those in unlike phases attracting one another; a statically electrified body may be considered to have its surface vibrating normally.

Thus, if the surface of a body is vibrating in any manner, the components of the vibration will correspond to the magnetic, electrodynamic, and electrostatical condition of the vibrating substance. The chief objection that I see to the theory is that it is difficult to imagine how the alternate polarizations and depolarizations of particles ranged along a wire could give rise to oscillatory twists. Several suggestions might be made as to shape, mode of motion, etc. of the particles, in order to explain this, but they would be mere speculations while they have no experimental basis to rest upon. This is at present an impediment, but does not appear to be a fatal objection to the theory.

That magnetic and electrical action is not propagated through an intervening medium is almost inconceivable after Faraday's experiments and Maxwell's investigations; and that there must be tangential stress: i.e. that the medium is not a fluid appears from the condition of the magnetic field due to a current. Moreover it is difficult to think that the remarkable results given in Chap. XX. of Maxwell's *Electricity* are a mere accident, and that the media through which light and electricity are propagated should not be related. There remain the possibilities of transmission by means of statical stress, finite displacement, and vibration. This last has always been objected to on the ground that the flow of electricity takes place in all directions, and not in a straight line at right angles to the wave front, as is the case with light. But this property of the transmission of light

* See *Proceedings of the Royal Society*, Vol. XXXV. p. 19 and p. 178.

† *Transactions of the Cambridge Philosophical Society*, Vol. XIV. Part 1.

Digitized by Google

in straight lines depends upon the shortness of the wave lengths; and if we consider long waves the transmission in a straight line will no longer follow. It may be noticed that, if the medium of propagation is identical with the luminiferous ether, a long wave will not imply a slow vibration; for example, a wave length of 200 miles will correspond to about 1000 vibrations in a second, taking the velocity of light to be 200,000 miles a second. It may at least be said for a vibratory theory that it gives results which correspond with observed facts, namely, that like normally vibrating bodies repel and unlike attract: also that like twisting bodies attract and unlike repel; and that in the case of simple forms of the bodies the law of force is the same as that of observed electrical and magnetic actions. Moreover, by a small change in our hypothesis, an explanation of gravitation can be deduced from what has been proved in the case of pulsating spheres; which is in accordance with what has always been expected to be the case, if the explanation of electrical actions should be found. In the result worked out for pulsating spheres the radius at any time t is supposed to be $a(1+k_1 \sin pt)$, which is the natural supposition; a being the radius of the sphere in a state of equilibrium. But if we vary this supposition by supposing the radius to be $a(1 + k_1 \sin pt - k_1' \sin^2 pt)$, [which is not very unnatural since the work required to compress a sphere into one of radius $a(1-k)$ is greater than would be required to extend it to a radius $a(1+k)$], then, in addition to our previous result that two spheres repel if the amplitudes k_1, k_2 have the same sign and attract if they have opposite signs, we should get a statical stress, the coefficient of which would be $-\frac{1}{2}k_1'$, and which would give an attraction varying according to the law of the inverse square, as may easily be seen by geometrical considerations.

The possibility of electrical action by means of statical stress or of finite displacement still remains untouched. It may be shewn that in the former case a twisting stress would give rise to a force of the kind required between two cylindrical surfaces, but there would in this case be a much more important term urging them along the lines of magnetic induction, which is not consistent with observation. As several experiments point to the idea that an electric current is accompanied by a twisting action, this looks like an objection to a theory of statical stress. Moreover the mathematical theories of finite displacements in an elastic medium or of stress in it are far more difficult than that of vibrations, and I would venture to submit a remark of Lamé on this question which had better be reproduced in his own words,* "En général, sauf quelques cas simples, les problèmes "relatifs à l'équilibre d'élasticité sont incomparablement plus difficiles à traiter par l'analyse "mathématique que les problèmes relatifs aux vibrations.....Une si grande différence dans "la facilité d'aborder les deux genres de problèmes pourrait être une indication naturelle. "Parmi les questions de Physique mathématique qui résistent aux efforts des géomètres, "ou qu'ils traitent péniblement par des formules longues et compliquées, il en est beaucoup "dont l'importance est fort douteuse. Au contraire, un grand nombre de questions qui se "résolvent par des calculs et des formules simples sont d'une importance incontestable. "Serait-ce donc que l'équilibre d'élasticité joue dans la nature un rôle moins important "que les vibrations?"

* Lamé, *Leçons sur l'élasticité*, Art. 69.

Digitized by Google

Digitized by Google

X. *On a class of Spherical Harmonics of complex degree with application to physical problems.* By E. W. Hobson, M.A.

[*Read* Feb. 28, 1887.]

The object of the present communication is to present an investigation of the properties of Spherical Harmonics of degree $-\frac{1}{2}+p\sqrt{-1}$, and to give some examples of the application of these functions to express the solution of certain questions in the theory of the potential. In the Appendix on Spherical Harmonics in the first volume of Thomson and Tait's *Natural Philosophy*, the importance of these Harmonics in the determination of the steady motion of heat in spaces bounded by concentric spheres and co-axial cones is pointed out, and some general remarks are made about Harmonics of fractional and imaginary degrees. These remarks are the basis of the investigations in the present paper. The functions with which this paper is concerned were introduced as independent functions, defined by certain definite integrals, by Herr Mehler (see *Crelle's Journal*, Vol. LXVIII., and a memoir entitled "Ueber eine mit den Kugel- und Cylinderfunctionen verwandte Function," Elbing, 1870), and are called by him "Kegelfunctionen," which we should render as "Conal Harmonics."

An account of these functions, with Mehler's application of them to potential problems connected with the infinite half-cone and the space bounded by a rotating circular segment, will be found in Heine's *Kugelfunctionen*. I have treated these functions entirely from the point of view of Thomson and Tait, that is to say as a particular case of ordinary Spherical Harmonics. The only writing besides the above-mentioned Appendix of Thomson and Tait, with which I am acquainted, in which the problem of conduction of heat or electricity in the space bounded by two concentric spheres and a cone with the vertex at the centre is touched upon, is a note by Mr W. Burnside in the *Mathematical Messenger* for 1885. In this note two problems are solved, in which the potential function is symmetrical with respect to the axis of the figure.

In the first part of the present communication the zonal and tesseral harmonics of degree $-\frac{1}{2}+p\sqrt{-1}$ are exhibited as series in various forms; in order to make the requisite transformations free use is made of the general theorems for hypergeometric series in Kummer's paper on the subject, in *Crelle's Journal*, Vol. XV.

Digitized by Google

The addition formula for the zonal harmonic of the first kind is then obtained. In the second part these functions are applied to the following problems:

(1) The determination of the permanent temperature at any point in a uniform conductor bounded by portions of two concentric spherical surfaces and by a right circular cone with its vertex at the centre of the spheres, when the temperature is given at every point of the conal boundary and is zero over the spherical boundary.

(2) The same problem for the space bounded by two such spheres and two such cones, the temperature being given over each of the conal boundaries.

(3) The case in which, besides the boundaries in (1), there are also two plane boundaries each passing through the axis of the cones, and which are maintained at temperature zero.

(4) The problems corresponding to the above, when there is only one spherical boundary.

(5) The case, given by Heine (*Kugelfunctionen*), of the space bounded by an infinite half-cone.

(6) The case in which the boundaries are those in (1) but in which the temperature is also given arbitrarily over the spherical boundaries.

(7) The determination of the electricity induced on the surface of an infinite conductor with a hollow of the shape in (1) due to an electrified point inside the hollow.

(8) The corresponding problem for the space in (4).

(9) By inversion, the determination of the potential at any point external to a conductor bounded by two non-intersecting spheres and the surface formed by the revolution round its chord, of a segment of a circle joining the two common inverse points of the two spheres due to a free charge on the insulated conductor. A direct method of solving this problem has been indicated by C. Neumann (*Math. Ann.* Vol. 18, Ueber die Mehlerschen functionen).

(10) The determination of the potential of the electricity induced on a conductor of the shape in (9) by an external electrified point.

(11) The determination of the current function of the motion in an infinite perfect incompressible fluid, due to the motion of a solid body whose surface is of the form generated by the rotation of a circular segment, in the direction of the axis of rotation.

In conclusion a remark is made as to the two dimensional harmonics of imaginary degree.

It should be observed that the arbitrary functions in these problems are supposed to be such that the integrals and series are convergent and that the latter have not in general (i.e. except at particular points or along particular lines) infinitely slow convergency. This will be the case for all functions which need be considered in physical problems.

I hope on a future occasion to consider some approximations made from the results obtained, especially in the problems (9) and (10). These last problems are interesting from

Digitized by Google

the fact that when the rotating segment is only a small portion of a circle the body becomes two spherical conductors connected by a conducting wire. It may be interesting to know how the charge is divided between the spheres and the wire, and thus to know how great the effect of such a connection is upon the distribution of electricity on spheres.

PART I.

Investigation of the form and properties of Spherical Harmonics of degree $-\frac{1}{2}+p\sqrt{-1}$.

1. The equation satisfied by the zonal harmonic $P_n(\mu)$ is

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{du}{d\mu}\right\}+n(n+1)u=0;$$

if for n we put $-\frac{1}{2}+p\sqrt{-1}$, this equation becomes

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{du}{d\mu}\right\}-(p^2+\tfrac{1}{4})u=0 \quad\text{.....}\quad (1).$$

If we assume that the series $\alpha_0+\alpha_1\mu+\alpha_2\mu^2+\ldots\ldots$ satisfies this equation, we obtain by substituting and equating the coefficients of the various powers of μ to zero,

$$\alpha_{2n}=\frac{p^2+(2n-\frac{3}{2})^2}{(2n-1)(2n)}\alpha_{2n-2},$$

$$\alpha_{2n+1}=\frac{p^2+(2n-\frac{1}{2})^2}{(2n+1)(2n)}\alpha_{2n-1};$$

hence, writing A, B for α_0, α_1 respectively, we obtain the solution of (1) in the form

$$v=A\left\{1+\frac{p^2+(\frac{1}{2})^2}{1.2}\mu^2+\frac{(p^2+\overline{\tfrac{1}{2}}|^2)(p^2+\overline{\tfrac{5}{2}}|^2)}{1.2.3.4}\mu^4+\ldots\right\}$$

$$+B\mu\left\{1+\frac{p^2+\overline{\tfrac{3}{2}}|^2}{1.2.3}\mu^2+\frac{(p^2+\overline{\tfrac{3}{2}}|^2)(p^2+\overline{\tfrac{7}{2}}|^2)}{1.2.3.4.5}\mu^4+\ldots\right\}.$$

If we denote by $F(a, b, c, x)$ the hypergeometric series

$$1+\frac{a.b}{1.c}x+\frac{a(a+1).b(b+1)}{1.2.c(c+1)}x^2+\ldots$$

this solution may be written

$$v=AF(\tfrac{1}{4}+\tfrac{1}{2}p\iota,\ \tfrac{1}{4}-\tfrac{1}{2}p\iota,\ \tfrac{1}{2},\ \mu^2)+B\mu F(\tfrac{3}{4}+\tfrac{1}{2}p\iota,\ \tfrac{3}{4}-\tfrac{1}{2}p\iota,\ \tfrac{3}{2},\ \mu^2)\ \text{.....}\ (2).$$

2. Each of the hypergeometrical series in (2) is convergent for values of μ between ± 1, but is divergent when $\mu=\pm 1$, for we find

$$\frac{n+1}{n}\cdot\frac{\alpha_{2n}}{\alpha_{2n-2}}=\frac{4n^3-2n^2+\ldots}{4n^3-2n^2},$$

$$\frac{n+1}{n}\cdot\frac{\alpha_{2n+1}}{\alpha_{2n-1}}=\frac{4n^3+2n^2+\ldots}{4n^3+2n^2};$$

Digitized by Google

which shews that the terms of each series alternately diminishes at a rate comparable with the rate of diminution of the terms of the series $1+\frac{1}{2}+\frac{1}{3}\ldots\ldots$

It is however possible so to choose the ratio of A to B that the value of v in (2) may be finite when $\mu=1$.

The series $F(a, b, c, x)$ may be written

$$1+\frac{\Pi(c-1)}{\Pi(a-1)\,\Pi(b-1)}\left\{\frac{\Pi(a)\,\Pi(b)}{\Pi(1)\,\Pi(c)}x+\ldots+\frac{\Pi(a+r-1)\,\Pi(b+r-1)}{\Pi(r)\,\Pi(c+r-1)}x^r+\ldots\right\};$$

hence, if

$$A\frac{\Pi(-\frac{1}{2})}{\Pi(-\frac{3}{4}+\frac{1}{2}p\iota)\,\Pi(-\frac{3}{4}-\frac{1}{2}p\iota)}-B\frac{\Pi(\frac{1}{2})}{\Pi(-\frac{1}{4}+\frac{1}{2}p\iota)\,\Pi(-\frac{1}{4}-\frac{1}{2}p\iota)}=0,$$

the value of v will be finite if we suppose that an equal number of terms of the two series is always taken. Strictly, we ought to write the terms of the two series alternately, but for convenience we retain the form (2); thus, if

$$\frac{A}{B}=\frac{\Pi(-\frac{3}{4}+\frac{1}{2}p\iota)\,\Pi(-\frac{3}{4}-\frac{1}{2}p\iota)}{2\Pi(-\frac{1}{4}+\frac{1}{2}p\iota)\,\Pi(-\frac{1}{4}-\frac{1}{2}p\iota)},$$

the value of v in (2), under the supposition just mentioned, remains finite when $\mu=1$. If the ratio $\frac{A}{B}$ have the above value with the sign changed, v will remain finite when $\mu=-1$. We can now divide the value of v into two parts, one of which is finite when $\mu=1$ and infinite when $\mu=-1$, whilst the other is finite when $\mu=-1$ and infinite when $\mu=1$.

3. It is convenient to choose absolute values of A and B such that $v=1$ when $\mu=1$. This may be done most conveniently by means of a theorem given by Kummer (*Crelle*, Vol. XV. p. 82),

$$F\left(a, b, \frac{a+b+1}{2}, \frac{1-\sqrt{x}}{2}\right)=\frac{\sqrt{\pi}\,\Pi\left(\frac{a+b-1}{2}\right)}{\Pi\left(\frac{a-1}{2}\right)\Pi\left(\frac{b-1}{2}\right)}F\left(\frac{a}{2}, \frac{b}{2}, \frac{1}{2}, x\right)$$
$$-\frac{2\sqrt{\pi}\,\Pi\left(\frac{a+b-1}{2}\right)}{\Pi\left(\frac{a}{2}-1\right)\Pi\left(\frac{b}{2}-1\right)}F\left(\frac{a+1}{2}, \frac{b+1}{2}, \frac{3}{2}, x\right).$$

When $x=1$ the left-hand side of this equation becomes unity; if we put $a=\frac{1}{2}+p\iota$, $b=\frac{1}{2}-p\iota$, $x=\mu^2=\cos^2\theta$, we obtain the equation

$$F\left(\tfrac{1}{2}+p\iota, \tfrac{1}{2}-p\iota, 1, \sin^2\frac{\theta}{2}\right)=\frac{\sqrt{\pi}\,F(\frac{1}{4}+\frac{1}{2}p\iota, \frac{1}{4}-\frac{1}{2}p\iota, \frac{1}{2}, \mu^2)}{\Pi(-\frac{1}{4}+\frac{1}{2}p\iota)\,\Pi(-\frac{1}{4}-\frac{1}{2}p\iota)}-\frac{2\sqrt{\pi}\,F(\frac{3}{4}+\frac{1}{2}p\iota, \frac{3}{4}-\frac{1}{2}p\iota, \frac{3}{2}, \mu^2)}{\Pi(-\frac{3}{4}+\frac{1}{2}p\iota)\,\Pi(-\frac{3}{4}-\frac{1}{2}p\iota)}.$$

If we denote the right-hand side of this equation by $K_p(\mu)$, the complete solution of the equation (2) is

$$v=CK_p(\mu)+DK_p(-\mu)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

where C and D denote arbitrary constants.

Digitized by Google

The expression $K_p(\mu)$ is equal to unity when $\mu=1$ and is infinite when $\mu=-1$, and $K_p(-\mu)$ is unity when $\mu=-1$ and infinite when $\mu=+1$.

Moreover

$$K_p(\mu)=F\left(\tfrac{1}{2}+p\iota,\ \tfrac{1}{2}-p\iota,\ 1,\ \sin^2\frac{\theta}{2}\right)=1+\frac{4p^2+1}{2^2}\sin^2\frac{\theta}{2}+\frac{(4p^2+1)(4p^2+3^2)}{2^2\,.\,4^2}\sin^4\frac{\theta}{2}+\ldots$$

$$K_p(-\mu)=F\left(\tfrac{1}{2}+p\iota,\ \tfrac{1}{2}-p\iota,\ 1,\ \cos^2\frac{\theta}{2}\right)=1+\frac{4p^2+1}{2^2}\cos^2\frac{\theta}{2}+\frac{(4p^2+1)(4p^2+3^2)}{2^2\,.\,4^2}\cos^4\frac{\theta}{2}+\ldots\ldots(4).$$

The first of these equations shews that $K_p(\mu)=P_{-\frac{1}{2}+p\iota}(\mu)$. Heine has shewn (*Handbuch der Kugelfunctionen*, Vol. II. p. 225) that $K_p(-\mu)=\frac{1}{2}\{Q_{-\frac{1}{2}+p\iota}(\mu)+Q_{-\frac{1}{2}-p\iota}(\mu)\}$, where $Q(\mu)$ denotes the zonal harmonic of the second kind.

4. If in the equation (1) we put $z^2=1-\mu^2$ and change to z as independent variable, the equation will be found to become

$$(1-z^2)\frac{d^2v}{dr^2}-\left(\frac{1-2z^2}{z}\right)\frac{dv}{dr}-(p^2+\tfrac{1}{4})\,v=0.$$

Assuming $v=\Sigma A_n z^n$ as a solution of this equation, we find

$$\frac{A_{n+2}}{A_n}=\frac{(n+\frac{1}{2})^2+p^2}{(n+2)^2}$$

thus
$$v=A_0\left\{1+\frac{p^2+\overline{\tfrac{1}{2}}|^2}{2^2}\sin^2\theta+\frac{(p^2+\overline{\tfrac{1}{2}}|^2)(p^2+\overline{\tfrac{5}{2}}|^2)}{2^2\,.\,4^2}\sin^4\theta+\ldots\right\}$$
$$+A_1\sin\theta\left\{1+\frac{p^2+\overline{\tfrac{3}{2}}|^2}{3^2}\sin^2\theta+\frac{(p^2+\overline{\tfrac{3}{2}}|^2)(p^2+\overline{\tfrac{7}{2}}|^2)}{3^2\,.\,5^2}\sin^4\theta+\ldots\right\}.$$

Since $K_p(\cos\theta)$ does not change sign with θ we have

$$K_p(\cos\theta)=1+\frac{p^2+\overline{\tfrac{1}{2}}|^2}{2^2}\sin^2\theta+\frac{(p^2+\overline{\tfrac{1}{2}}|^2)(p^2+\overline{\tfrac{5}{2}}|^2)}{2^2\,.\,4^2}\sin^4\theta+\ldots\ldots\ldots\ldots\ldots(5),$$

a result which is valid so long as θ lies between $\pm\frac{\pi}{2}$.

5. The function $K_\mu(\cos\theta)$ may, by means of two theorems given by Kummer in his above-mentioned memoir on hypergeometric series, be expanded in cosines of multiples of θ. These theorems are

$$F\left(\frac{a}{2},\frac{b}{2},\frac{1}{2},\cos^2\theta\right)=\frac{\Pi\left(\frac{-a-b}{2}\right)}{\Pi\left(-\frac{a}{2}\right)\Pi\left(-\frac{b}{2}\right)}$$
$$\left\{1+\frac{2ab}{(a-2)(b-2)}\cos 2\theta+\frac{2a(a+2)b(b+2)}{(a-2)(a-4)(b-2)(b-4)}\cos 4\theta+\ldots\right\},$$

$$\cos\theta F\left(\frac{a+1}{2},\frac{b+1}{2},\frac{3}{2},\cos^2\theta\right)=\frac{\Pi\left(\frac{-a-b}{2}\right)}{\Pi\left(\frac{1-a}{2}\right)\Pi\left(\frac{1-b}{2}\right)}$$
$$\left\{\cos\theta+\frac{(a+1)(b+1)}{(a-3)(b-3)}\cos 3\theta+\frac{(a+1)(a+3)(b+1)(b+3)}{(a-3)(a-5)(b-5)(b-5)}\cos 5\theta+\ldots\right\}.$$

Digitized by Google

If we substitute in these the values $a=\frac{1}{2}+p\iota$, $b=\frac{1}{2}-p\iota$, we have, referring to the value of $K_p(\cos\theta)$ in equation (2),

$$K_p(\cos\theta)=\alpha\left[\frac{1}{2}+\frac{4p^2+1}{4p^2+3}\cos 2\theta+\frac{(4p^2+1)(4p^2+5^2)}{(4p^2+3^2)(4p^2+7^2)}\cos 4\theta+\ldots\right]$$
$$-\beta\left[\cos\theta+\frac{4p^2+3^2}{4p^2+5^2}\cos 3\theta+\frac{(4p^2+3^2)(4p^2+7^2)}{(4p^2+5^2)(4p^2+9^2)}\cos 5\theta+\ldots\right]\ldots\ldots\ldots\ldots(6),$$

where
$$\alpha=\frac{2\pi}{\{\Pi(-\frac{1}{4}+\frac{1}{2}p\iota)\,\Pi(-\frac{1}{4}-\frac{1}{2}p\iota)\}^2},$$
$$\beta=\frac{2\pi}{\Pi(-\frac{3}{4}+\frac{1}{2}p\iota)\,\Pi(-\frac{3}{4}-\frac{1}{2}p\iota)\,\Pi(\frac{1}{4}+\frac{1}{2}p\iota)\,\Pi(\frac{1}{4}-\frac{1}{2}p\iota)}.$$

This agrees with the formula obtained by Heine (*Kugelfunctionen*, Vol. II. p. 241) by means of the addition theorem for the K function. The above series for $K_p(\cos\theta)$ may be obtained directly by substituting $z=\mu+\sqrt{\mu^2-1}$ in the equation (1), which then takes the form
$$z^2(z^2-1)\frac{d^2v}{dr^2}+2z^3\frac{dv}{dr}+(p^2+\tfrac{1}{4})(z^2-1)=0.$$

If we assume $v=\Sigma A_n z^n$ as a solution we get $A_n\{p^2+(n-\frac{1}{2})^2\}=A_{n-2}\{p^2+(n-\frac{3}{2})^2\}$, and we thus get a series of the form above, but the determination of the constants is troublesome.

6. In order to express $K_p(\cos\theta)$ as a definite integral we shall require to express $F\left(a, b, \frac{a+b+1}{2}, x\right)$ as a definite integral. The general term of the series is
$$\frac{\Pi\left(\frac{a+b-1}{2}\right)\Pi(a+r-1)\,\Pi(b+r-1)}{\Pi\left(\frac{a+b-1}{2}+r\right)\Pi(a-1)\,\Pi(b-1)}\cdot\frac{x^r}{r!};$$

and since
$$\frac{\Pi(2x-1)}{\Pi(x-1)}=\frac{2^{2x}}{\sqrt{\pi}}\Pi(x-\tfrac{1}{2}),$$

this may be written
$$\frac{2^{a+b}}{\sqrt{\pi}}\frac{\Pi\left(\frac{a+b-1}{2}\right)\Pi\left(\frac{a+b}{2}-1\right)}{\Pi(a-1)\,\Pi(b-1)}\cdot\frac{\Pi(a+k-1)\,\Pi(b+k-1)}{\Pi(a+b+2k-1)}\cdot\frac{\frac{a+b}{2}+k-1!}{k!\,\frac{a+b}{2}-1!}(4x)^k,$$

or
$$\frac{2^{a+b}}{\sqrt{\pi}}\frac{\Pi\left(\frac{a+b-1}{2}\right)\Pi\left(\frac{a+b}{2}-1\right)}{\Pi(a-1)\,\Pi(b-1)}\cdot\frac{\frac{a+b}{2}+k-1!}{k!\,\frac{a+b}{2}-1!}(4x)^k\int_0^1 u^{a+k-1}(1-u)^{b+k-1}du;$$

the sum of the series is therefore
$$\frac{2^{a+b}}{\sqrt{\pi}}\frac{\Pi\left(\frac{a+b-1}{2}\right)\Pi\left(\frac{a+b}{2}-1\right)}{\Pi(a-1)\,\Pi(b-1)}\int_0^1\frac{u^{a-1}(1-u)^{\beta-1}}{\{1-4xu(1-u)\}^{\frac{a+b}{2}}}du\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

which is the required expression for $F\left(a, b, \frac{a+b+1}{2}, x\right)$.

Digitized by Google

If we put $a = \frac{1}{2} + p\iota$, $b = \frac{1}{2} - p\iota$, $x = \sin^2\frac{\theta}{2}$, we get

$$K_p(\cos\theta) = \frac{1}{2\Pi(-\frac{1}{2}+p\iota)\,\Pi(-\frac{1}{2}-p\iota)}\int_0^1 \frac{u^{-\frac{1}{2}+p\iota}(1-u)^{-\frac{1}{2}-p\iota} + u^{-\frac{1}{2}-p\iota}(1-u)^{-\frac{1}{2}+p\iota}}{\{1-4xu(1-u)\}^{\frac{1}{2}}}\,du,$$

now $\Pi(-\frac{1}{2}+p\iota)\,\Pi(-\frac{1}{2}-p\iota) = \frac{\pi}{\cosh p\pi}$; if we make the substitution $u = \frac{1}{e^\alpha+1}$, which gives $u(1-u) = \frac{1}{4}\operatorname{sech}^2\frac{1}{2}\alpha$, this integral becomes

$$\frac{\cosh p\pi}{2\pi}\int_{-\infty}^{\infty} \frac{\dfrac{\cos p\alpha}{\cosh\frac{1}{2}\alpha}}{\left\{1-\dfrac{\sin^2\frac{1}{2}\theta}{\cosh^2\frac{1}{2}\alpha}\right\}^{\frac{1}{2}}}\,d\alpha,$$

or

$$\frac{\sqrt{2}\cosh p\pi}{\pi}\int_0^\infty \frac{\cos p\alpha}{(\cosh\alpha+\cos\theta)^{\frac{1}{2}}}\,d\alpha \quad\text{.....}\quad (8).$$

Also, changing θ into $\pi-\theta$,

$$K_p(-\cos\theta) = \frac{\sqrt{2}\cosh p\pi}{\pi}\int_0^\infty \frac{\cos p\alpha}{(\cosh\alpha-\cos\theta)^{\frac{1}{2}}}\,du.$$

The functions $K_p(\cos\theta)$, $K_p(-\cos\theta)$ introduced by Mehler were defined by means of these definite integrals and their properties were deduced from this definition.

7. I shall now investigate the form of the tesseral harmonics of degree $-\frac{1}{2}+p\iota$. Those of the order s are of the form $\begin{matrix}\sin\\ \cos\end{matrix}\, s\phi\,.\,u_s$, where u_s satisfies the differential equation

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{du_s}{d\mu}\right\} - \left\{p^2+\frac{1}{4}+\frac{s^2}{1-\mu^2}\right\}u_s = 0.$$

If we put $u_s = (1-\mu^2)^{\frac{1}{2}s}\,w_s$, the differential equation for w_s is

$$(1-\mu^2)\frac{d^2w_s}{d\mu^2} - 2(1+s)\,\mu\frac{dw_s}{d\mu} - \{p^2+(s+\tfrac{1}{2})^2\}\,w_s = 0.$$

The value of w_s obtained from this differential equation is found as in the former case ($s=0$) to be

$$A\left[1+\frac{(s+\frac{1}{2})^2+p^2}{1.2}\mu^2+\frac{\{(s+\frac{1}{2})^2+p^2\}\{(s+\frac{5}{2})^2+p^2\}}{1.2.3.4}\mu^4+\dots\right]$$

$$+B\mu\left[1+\frac{(s+\frac{3}{2})^2+p^2}{1.2.3}\mu^2+\frac{\{(s+\frac{3}{2})^2+p^2\}\{(s+\frac{7}{2})^2+p^2\}}{1.2.3.4.5}\mu^4+\dots\right],$$

or

$$AF(\tfrac{1}{2}s+\tfrac{1}{4}+\tfrac{1}{2}\mu\iota,\ \tfrac{1}{2}s+\tfrac{1}{4}-\tfrac{1}{2}\mu\iota,\ \tfrac{1}{2},\ \mu^2)$$

$$+B\mu F(\tfrac{1}{2}s+\tfrac{3}{4}+\tfrac{1}{2}\mu\iota,\ \tfrac{1}{2}s+\tfrac{3}{4}-\tfrac{1}{2}\mu\iota,\ \tfrac{3}{2},\ \mu^2).$$

Employing the theorem of Kummer's used in Art. 3, we find that the expression

Digitized by Google

$$\frac{\sqrt{\pi}\,\Pi(s)}{\Pi(\frac{1}{2}s-\frac{1}{4}+\frac{1}{2}p\iota)\,\Pi(\frac{1}{2}s-\frac{1}{4}-\frac{1}{2}p\iota)}F(\tfrac{1}{2}s+\tfrac{1}{4}+\tfrac{1}{2}p\iota,\ \tfrac{1}{2}s+\tfrac{1}{4}-\tfrac{1}{2}p\iota,\ \tfrac{1}{2},\ \mu^2)$$

$$-\frac{2\sqrt{\pi}\,\Pi(s)}{\Pi(\frac{1}{2}s-\frac{3}{4}+\frac{1}{2}p\iota)\,\Pi(\frac{1}{2}s-\frac{3}{4}-\frac{1}{2}p\iota)}\mu F(\tfrac{1}{2}s+\tfrac{3}{4}+\tfrac{1}{2}p\iota,\ \tfrac{1}{2}s+\tfrac{3}{4}-\tfrac{1}{2}p\iota,\ \tfrac{3}{2},\ \mu^2)\ldots\ldots(9)$$

is convergent when $\mu=1$, and is equal to unity when $\mu=0$; and that it is in fact equal to

$$F(s+\tfrac{1}{2}+p\iota,\ s+\tfrac{1}{2}-p\iota,\ s+1,\ \sin^2\tfrac{1}{2}\theta).$$

If in the above expression the two series are added instead of being subtracted, the resulting expression is convergent for $\mu=-1$, and is equal to

$$F(s+\tfrac{1}{2}+p\iota,\ s+\tfrac{1}{2}-p\iota,\ s+1,\ \cos^2\tfrac{1}{2}\theta).$$

It may be shewn as before that, provided θ lies between $\pm\frac{1}{2}\pi$, the expression considered is equal to

$$F(\tfrac{1}{2}s+\tfrac{1}{4}+\tfrac{1}{2}p\iota,\ \tfrac{1}{2}s+\tfrac{1}{4}-\tfrac{1}{2}p\iota,\ s+1,\ \sin^2\theta).$$

If we denote this expression (9) by $K_p^s(\mu)$, the most general value of u_s is

$$\sin^s\theta\,.\,\{C'K_p^s(\cos\theta)+D'K_p^s(-\cos\theta)\},$$

where $K_p^s(\cos\theta)$ is finite except when $\theta=\pi$, and $K_p^s(\cos\theta)$ is finite except when $\theta=0$.

8. If we apply the theorem

$$F(\alpha,\beta,\gamma,x)=(1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\ \gamma-\beta,\ \gamma,\ x)$$

to the expressions just obtained for $K_p^s(\mu)$, we find that they become

$$\frac{\sqrt{\pi}\,\Pi(s)}{\Pi(\frac{1}{2}s-\frac{1}{4}+\frac{1}{2}p\iota)\,\Pi(\frac{1}{2}s-\frac{1}{4}-\frac{1}{2}p\iota)}(1-\mu^2)^{-s}F(\tfrac{1}{4}-\tfrac{1}{2}s-\tfrac{1}{2}p\iota,\ \tfrac{1}{4}-\tfrac{1}{2}s+\tfrac{1}{2}p\iota,\ \tfrac{1}{2},\ \mu^2)$$

$$-\frac{2\sqrt{\pi}\,\Pi(s)}{\Pi(\frac{1}{2}s-\frac{3}{4}+\frac{1}{2}p\iota)\,\Pi(\frac{1}{2}s-\frac{3}{4}-\frac{1}{2}p\iota)}(1-\mu^2)^{-s}\mu F(\tfrac{3}{4}-\tfrac{1}{2}s-\tfrac{1}{2}p\iota,\ \tfrac{3}{4}-\tfrac{1}{2}s+\tfrac{1}{2}p\iota,\ \tfrac{3}{2},\ \mu^2),$$

$$\cos^{-2s}\tfrac{1}{2}\theta\,.\,F(\tfrac{1}{2}-p\iota,\ \tfrac{1}{2}+p\iota,\ s+1,\ \sin^2\tfrac{1}{2}\theta),$$

and
$$\cos\theta\,F(\tfrac{1}{2}s+\tfrac{3}{4}-\tfrac{1}{2}p\iota,\ \tfrac{1}{2}s+\tfrac{3}{4}+\tfrac{1}{2}p\iota,\ s+1,\ \sin^2\theta),$$
respectively.

9. If in the theorems quoted in Art. 5 we put $a=s+\frac{1}{2}+p\iota$, $b=s+\frac{1}{2}-p\iota$, we obtain for $K_p^s(\pm\cos\theta)$ the expression

$$\alpha_s\left[\tfrac{1}{2}+\frac{(2s+1)^2+p^2}{(2s-3)^2+p^2}\cos 2\theta+\frac{\{(2s+1)^2+p^2\}\{(2s+5)^2+p^2\}}{\{(2s-3)^2+p^2\}\{(2s-7)^2+p^2\}}\cos 4\theta+\ldots\right]$$

$$\pm\beta_s\left[\cos\theta+\frac{(2s+3)^2+p^2}{(2s-5)^2+p^2}\cos 3\theta+\frac{\{(2s+3)^2+p^2\}\{(2s+7)^2+p^2\}}{\{(2s-5)^2+p^2\}\{(2s-9)^2+p^2\}}\cos 5\theta+\ldots\right]\ldots\ldots(10),$$

where
$$\alpha=\frac{\sqrt{\pi}\,.\,\Pi(s)\,\Pi(-s-\frac{1}{2})}{\Pi(\frac{1}{2}s-\frac{1}{4}+\frac{1}{2}p\iota)\,\Pi(\frac{1}{2}s-\frac{1}{4}-\frac{1}{2}p\iota)\,\Pi(-\frac{1}{2}s-\frac{1}{4}+\frac{1}{2}p\iota)\,\Pi(-\frac{1}{2}s-\frac{1}{4}-\frac{1}{2}p\iota)},$$

$$\beta=\frac{2\sqrt{\pi}\,\Pi(s)\,\Pi(-s-\frac{1}{2})}{\Pi(\frac{1}{2}s-\frac{1}{4}+\frac{1}{2}p\iota)\,\Pi(\frac{1}{2}s-\frac{1}{4}-\frac{1}{2}p\iota)\,\Pi(-\frac{1}{2}s+\frac{1}{4}+\frac{1}{2}p\iota)\,\Pi(-\frac{1}{2}s+\frac{1}{4}-\frac{1}{2}p\iota)}.$$

Digitized by Google

10. I shall now investigate the addition theorem for the function $K_p(\mu)$. Let

$$\cos\gamma = \cos\theta\cos\theta_1 + \sin\theta\,.\,\sin\theta_1\cos(\phi-\phi_1).$$

Assume
$$K_p(\cos\gamma) = \sum_{s=0}^{s=\infty} u_s \cos s(\phi-\phi_1)$$

substituting in the equation

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dz}{d\mu}\right\} - (p^2+\tfrac{1}{4})z + \frac{1}{1-\mu^2}\frac{d^2z}{d\phi^2} = 0,$$

which $K_p(\cos\gamma)$ satisfies, we have by equating to zero the coefficient of $\cos s(\phi-\phi_1)$ the equation

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{du_s}{d\mu}\right\} - \left[p^2+\tfrac{1}{4}+\frac{s^2}{1-\mu^2}\right]u_s = 0,$$

the solution of which is

$$u_s = \alpha_s' K_p^s(\mu) + \beta_s' K_p^s(-\mu).$$

In this case we must have $\beta_s' = 0$ for $K_p(\cos\gamma)$ remains finite when $\cos\theta = 1$, whereas $K_p^s(-1)$ is infinite, also since $\cos\gamma$ contains μ and μ_1 symmetrically α_s' must contain $K_p^s(\mu_1)$ as a factor; therefore

$$K_p(\cos\gamma) = \Sigma\gamma_s K_p^s(\mu)\, K_p^s(\mu_1)\cos s(\phi-\phi_1),$$

to determine γ_s put $\theta = \theta_1 = \frac{1}{2}\pi$ and $\phi_1 = 0$; this equation becomes then

$$K_p(\cos\phi) = \Sigma\gamma_s K_p^s(0)\, K_p^s(0)\cos s\phi,$$

comparing this with equation (6), we have

$$\gamma_s\{K_p^s(0)\}^2 = \alpha\,.\,\frac{\{4p^2+1^2\}\ldots\ldots\{4p^2+(2s-3)^2\}}{\{4p^2+3^2\}\ldots\ldots\{4p^2+(2s-1)^2\}},$$

or
$$= -\beta\,\frac{\{4p^2+3^2\}\ldots\ldots\{4p^2+(2s-3)^2\}}{\{4p^2+5^2\}\ldots\ldots\{4p^2+(2s-1)^2\}},$$

according as s is even or odd. Now equation (9) shews that

$$K_p^s(0) = \frac{\sqrt{\pi}\,\Pi(s)}{\Pi(\frac{1}{2}s-\frac{1}{4}+\frac{1}{2}p\iota)\,\Pi(\frac{1}{2}s-\frac{1}{4}-\frac{1}{2}p\iota)}$$

$$= \frac{\sqrt{\pi}\,\Pi(s)}{\left\{\frac{(2s-1)^2+p^2}{2^2}\right\}\left\{\frac{(2s-3)^2+p^2}{2^2}\right\}\ldots\ldots}$$

Also (Art. 5)
$$\alpha = \frac{2\pi}{\left\{\frac{1}{4^2}+\frac{p^2}{2^2}\right\}^2\left\{\left(\frac{5}{4}\right)^2+\frac{p^2}{2^2}\right\}^2\left\{\left(\frac{9}{4}\right)^2+\frac{p^2}{2^2}\right\}^2\ldots\ldots}$$

$$\beta = \frac{2\pi}{\left\{\left(\frac{3}{4}\right)^2+\frac{1}{4}p^2\right\}\left\{\left(\frac{7}{4}\right)^2+\frac{1}{4}p^2\right\}\ldots\ldots\left\{\frac{1}{4^2}+\frac{p^2}{2^2}\right\}\left\{\left(\frac{3}{4}\right)^2+\frac{p^2}{2^2}\right\}\ldots\ldots}$$

$$\therefore\ \gamma_s = (-1)^s\,2\,.\,\frac{(4p^2+1^2)(4p^2+3^2)\ldots\ldots\{4p^2+(2s-1)^2\}}{2^2\,.\,4^2\ldots(2s)^2}.$$

Digitized by Google

Hence we obtain the addition theorem

$$K_p\{\cos\theta\cos\theta_1+\sin\theta\sin\theta_1\cos(\phi-\phi_1)\}=K_p(\cos\theta)\,K_p(\cos\theta_1)$$
$$+\sum_{m=1}^{m=\infty}(-1)^s.\,2\,.\,\frac{(4p^2+1^2)(4p^2+3^2)\ldots\{4p^2+(2s-1)^2\}}{2^2\,.\,4^2\ldots(2s)^2}K_p^m(\cos\theta)K_p^m(\cos\theta_1)\cos m(\phi-\phi_1)\ldots(11).$$

PART II.

Applications to physical problems.

I shall now give some examples of the application of the functions of which the properties are investigated above, to the solution of problems in electricity, the conduction of heat and hydrodynamics.

11. If we determine n so that the harmonic $\left(Ar^n+\frac{B}{r^{n+1}}\right)S_n$ may vanish for both the values a and b of r, we obtain from the equations

$$\left.\begin{aligned}Aa^n+\frac{B}{a^{n+1}}=0\\ Ab^n+\frac{B}{b^{n+1}}=0\end{aligned}\right\}$$

the values $n=-\frac{1}{2}+\frac{k\pi\iota}{\log_e\frac{b}{a}}$, where k denotes any integer; thus spherical harmonics of degree $-\frac{1}{2}+\frac{k\pi\iota}{\log_e\frac{b}{a}}$ are adapted to the solution of potential problems in which the boundary of the space considered consists partially of portions of the two spheres $r=a$ and $r=b$.

12. Suppose it required to determine the permanent temperature at any point in the space bounded by the two concentric spherical surfaces $r=a$, $r=b$ and the right circular cone $\theta=\alpha$, when the two spherical boundaries are maintained at temperature zero and the temperature is given as a function of r and ϕ at every point of the conical boundary; r, θ, ϕ denote the ordinary polar coordinates of any point.

Suppose $b>a$, and let $r=ae^\rho$, $b=ae^\sigma$, then if $S_{\frac{k\pi\iota}{\sigma}-\frac{1}{2}}$ denote a surface harmonic of degree $-\frac{1}{2}+\frac{k\pi\iota}{\sigma}$, the solid harmonic $\frac{1}{\sqrt{r}}\sin\left(\frac{k\pi\rho}{\sigma}\right).S_{\frac{k\pi\iota}{\sigma}-\frac{1}{2}}$ vanishes over each of the spherical boundaries. The surface harmonic must not be infinite when $\theta=0$ and the potential function V must be unaltered when $\phi+2\pi$ is substituted for ϕ, therefore a suitable form for V is

$$V=\sum_{k=1}^{k=\infty}\frac{1}{\sqrt{r}}\sin\left(\frac{k\pi\rho}{\sigma}\right)\sum_{m=0}^{m=\infty}(B_{km}\cos m\phi+C_{km}\sin m\phi)\,\frac{K^m_{\frac{k\pi}{\sigma}}(\cos\theta)}{K^m_{\frac{k\pi}{\sigma}}(\cos\alpha)},$$

Digitized by Google

where B_{km}, C_{km} are determined from the condition that when $\theta = \alpha$, V may be equal to a given function of ρ and ϕ for all values of ρ between 0 and σ. Suppose the value of V over the conical boundary given in the form $\frac{1}{\sqrt{r}} f(\rho, \phi)$; assume

$$f(\rho, \phi) = \phi_0(\rho) + \Sigma \{\phi_m(\rho) \cos m\phi + f_m(\rho) \sin m\phi\},$$

then we have
$$\phi_0(\rho) = \frac{1}{2\pi} \int_0^{2\pi} f(\rho, \phi')\, d\phi'$$

$$\phi_m(\rho) \cos m\phi + \mu_m(\rho) \sin m\phi = \frac{1}{\pi} \int_0^{2\pi} f(\rho, \phi') \cos m(\phi - \phi')\, d\phi',$$

and for values of ρ between 0 and σ we have

$$f(\rho, \phi') = \frac{1}{2\pi} \sum_{k=1}^{k=\infty} \sin \frac{k\pi\rho}{\sigma} \int_0^\sigma \sin \frac{k\pi\rho'}{\sigma} . f(\rho', \phi')\, d\rho',$$

hence
$$f(\rho, \phi) = \frac{1}{4\pi^2} \sum_{k=1}^{k=\infty} \sin \frac{k\pi\rho}{\sigma} \int_0^\sigma \int_0^{2\pi} \sin \frac{k\pi\rho'}{\sigma} f(\rho', \phi')\, d\rho'\, d\phi'$$

$$+ \frac{1}{2\pi^2} \sum_{k=1}^{k=\infty} \sum_{m=1}^{m=\infty} \frac{\sin k\pi\rho}{\sigma} \int_0^\sigma \int_0^{2\pi} \sin \frac{k\pi\rho'}{\sigma} \cos m(\phi - \phi') f(\rho', \phi')\, d\rho'\, d\phi';$$

thus we have

$$V = \frac{1}{2\pi^2 \sqrt{r}} \sum_{k=1} \sum_{m=0} \epsilon \sin\left(\frac{k\pi\rho}{\sigma}\right) . \frac{K^m_{\frac{k\pi}{\sigma}}(\cos\theta)}{K^m_{\frac{k\pi}{\sigma}}(\cos\alpha)} . \int_0^\sigma \int_0^{2\pi} \sin \frac{k\pi\rho'}{\sigma} \cos m(\phi - \phi') f(\rho', \phi')\, d\rho'\, d\phi',$$

when $\epsilon = 1$, except when $m = 0$ and then $\epsilon = \frac{1}{2}$.

This value of V satisfies the potential equation, is finite throughout the space considered, vanishes for $r = a$ and $r = b$ and is equal to $\frac{1}{\sqrt{r}} f(\rho, \phi)$ over the conical boundary; it expresses therefore the temperature at any point within the space considered.

In the particular case in which the temperature over the conical boundary is constant and equal to V_1, the value of V is

$$\frac{V_1}{2\pi\sqrt{r}} \Sigma . \sin \frac{k\pi\rho}{\sigma} . \frac{K_{\frac{k\pi}{\sigma}}(\cos\theta)}{K_{\frac{k\pi}{\sigma}}(\cos\alpha)} \int_0^\sigma e^{\frac{\rho'}{2}} \sin \frac{k\pi\rho'}{\sigma}\, d\rho'$$

or
$$\frac{2V_1\sigma}{\sqrt{r}} \Sigma \frac{k}{4k^2\pi^2 + \sigma^2} \left(1 - \sqrt{\frac{b}{a}} \cos k\pi\right) \frac{K_{\frac{k\pi}{\sigma}}(\cos\theta)}{K_{\frac{k\pi}{\sigma}}(\cos\alpha)}.$$

13. In the case in which the space throughout which V is to be found, is bounded by the two spheres $r = a$, $r = b$ and the two cones $\theta = \alpha$, $\theta = \beta$, we require both the functions $K(\cos\theta)$ and $K(-\cos\theta)$ to express the value of V; these are now both finite throughout the space considered.

Digitized by Google

An expression of the form

$$V = \Sigma\Sigma \frac{1}{\sqrt{r}} \sin\left(\frac{k\pi\rho}{\sigma}\right) \{(A_{mk} \cos m\phi + B_{mk} \sin m\phi) K^m_{\frac{k\pi}{\sigma}}(\cos\theta) + (C_{mk} \cos m\phi + D_{mk} \sin m\phi) K^m_{\frac{k\pi}{\sigma}}(-\cos\theta)\}$$

satisfies the boundary conditions $V=0$ for $r=a$, $r=b$ and is finite and continuous throughout the space, and the four sets of coefficients may be determined so that $V = \frac{1}{\sqrt{r}} f_1(\rho, \phi)$ for $\theta=\alpha$, and $V = \frac{1}{\sqrt{r}} f_2(\rho, \phi)$ for $\theta=\beta$, for values of ρ between 0 and σ.

The above expression becomes for $\theta=\alpha$

$$V = \Sigma\Sigma \frac{1}{\sqrt{r}} \sin\frac{k\pi\rho}{\sigma} [\{A_{mk} K^m_{\frac{k\pi}{\sigma}}(\cos\alpha) + C_{mk} K^m_{\frac{k\pi}{\sigma}}(-\cos\alpha)\} \cos m\phi + B_{mk} K^m_{\frac{k\pi}{\sigma}}(\cos\alpha) + D_{mk} K^m_{\frac{k\pi}{\sigma}}(-\cos\alpha) \sin m\phi],$$

now we have as above

$$\frac{1}{\sqrt{r}} f_1(\rho, \phi) = \frac{\epsilon}{2\pi^2\sqrt{r}} \Sigma\Sigma \sin\frac{k\pi\rho}{\sigma} \int_0^\sigma\int_0^{2\pi} \sin\frac{k\pi\rho'}{\sigma} \cos m(\phi-\phi') f_1(\rho', \phi')\, d\rho'\, d\phi',$$

hence we obtain the equations

$$A_{mk}K^m(\cos\alpha) + C_{mk}K^m(-\cos\alpha) = \frac{\epsilon}{2\pi^2}\int_0^\sigma\int_0^{2\pi} \sin\frac{k\pi\rho'}{\sigma} \cos m\phi' . f_1(\rho', \phi')\, d\rho' d\phi',$$

$$B_{mk}K^m(\cos\alpha) + D_{mk}K^m(-\cos\alpha) = \frac{\epsilon}{2\pi^2}\int_0^\sigma\int_0^{2\pi} \sin\frac{k\pi\rho'}{\sigma} \sin m\phi' . f_1(\rho', \phi')\, d\rho' d\phi',$$

and by putting $\theta=\beta$ in the expression for V, we obtain in like manner

$$A_{mk}K^m(\cos\beta) + C_{mk}K^m(-\cos\beta) = \frac{\epsilon}{2\pi^2}\int_0^\sigma\int_0^{2\pi} \sin\frac{k\pi\rho'}{\sigma} \cos m\phi' . f_2(\rho', \phi')\, d\rho' d\phi'$$

$$B_{mk}K^m(\cos\beta) + D_{mk}K^m(-\cos\beta) = \frac{\epsilon}{2\pi^2}\int_0^\sigma\int_0^{2\pi} \sin\frac{k\pi\rho'}{\sigma} \sin m\phi' . f_2(\rho', \phi')\, d\rho' d\phi'.$$

Determining the values of the constants A, B, C, D from the equations, we obtain the following expression for the temperature

$$V = \frac{1}{2\pi^2\sqrt{r}} \Sigma\Sigma \frac{\epsilon}{K^m_{\frac{k\pi}{\sigma}}(\cos\alpha) K^m_{\frac{k\pi}{\sigma}}(-\cos\beta) - K^m_{\frac{k\pi}{\sigma}}(\cos\beta) K^m_{\frac{k\pi}{\sigma}}(-\cos\alpha)} \sin\frac{k\pi\rho}{\sigma}$$

$$\times \int_0^\sigma\int_0^{2\pi} \sin\frac{k\pi\rho'}{\sigma} \cos m(\phi-\phi') \Big[f_1(\rho', \phi') \Big\{K^m_{\frac{k\pi}{\sigma}}(\cos\theta) K^m_{\frac{k\pi}{\sigma}}(-\cos\beta) - K^m_{\frac{k\pi}{\sigma}}(\cos\theta) K^m_{\frac{k\pi}{\sigma}}(\cos\beta)\Big\}$$

$$+ f_2(\rho', \phi') \Big\{K^m_{\frac{k\pi}{\sigma}}(-\cos\theta) K^m_{\frac{k\pi}{\sigma}}(\cos\alpha) - K^m_{\frac{k\pi}{\sigma}}(\cos\theta) K^m_{\frac{k\pi}{\sigma}}(-\cos\alpha)\Big\}\Big]\, d\rho' d\phi',$$

where $\epsilon=1$, except when $m=0$ and then $\epsilon=\frac{1}{2}$.

Digitized by Google

In the particular case in which V is constant over each conical boundary, say $V = V_1$ for $\theta = \alpha$ and $V = V_2$ for $\theta = \beta$, we must put $f_1(\rho, \phi) = V_1 e^{ik\phi}$, $f_2(\rho, \phi) = V_2 e^{ik\phi}$ in the expression just obtained; it then becomes

$$\frac{1}{2\pi\sqrt{r}} \Sigma \frac{1}{K(\cos\alpha) K(-\cos\beta) - K(\cos\beta) K(-\cos\alpha)} \sin\frac{k\pi\rho}{\sigma} \int_0^\sigma \sin\frac{k\pi\rho'}{\sigma} \cdot e^{ik\phi'} d\rho'$$
$$\times [K(\cos\theta) \{V_1 K(-\cos\beta) - V_2 K(-\cos\alpha)\} + K(-\cos\theta) \{V_2 K(\cos\alpha) - V_1 K(\cos\beta)\},$$

which is equal to

$$\frac{2\sigma V_1}{\sqrt{r}} \sum_{k=1}^{k=\infty} \frac{k}{4k^2\pi^2 + \sigma^2} \sin\frac{k\pi\rho}{\sigma} \cdot \frac{K_{\frac{k\pi}{\sigma}}(\cos\theta) K_{\frac{k\pi}{\sigma}}(-\cos\beta) - K_{\frac{k\pi}{\sigma}}(-\cos\theta) K_{\frac{k\pi}{\sigma}}(\cos\beta)}{K_{\frac{k\pi}{\sigma}}(\cos\alpha) K_{\frac{k\pi}{\sigma}}(-\cos\beta) - K_{\frac{k\pi}{\sigma}}(-\cos\alpha) K_{\frac{k\pi}{\sigma}}(\cos\beta)}$$

$$+ \frac{2\sigma V_2}{\sqrt{r}} \Sigma \frac{k}{4k^2\pi^2 + \sigma^2} \sin\frac{k\pi\rho}{\sigma} \cdot \frac{K_{\frac{k\pi}{\sigma}}(\cos\theta) K_{\frac{k\pi}{\sigma}}(-\cos\alpha) - K_{\frac{k\pi}{\sigma}}(-\cos\theta) K_{\frac{k\pi}{\sigma}}(\cos\alpha)}{K_{\frac{k\pi}{\sigma}}(\cos\alpha) K_{\frac{k\pi}{\sigma}}(-\cos\beta) - K_{\frac{k\pi}{\sigma}}(-\cos\alpha) K_{\frac{k\pi}{\sigma}}(\cos\beta)}.$$

This is the expression for the permanent temperature throughout the space considered when the spherical boundaries are maintained at temperature zero and the conical boundaries at constant temperatures V_1 and V_2.

14. In the case in which there are besides the two spherical and the one or two conal boundaries, also two plane boundaries $\phi = 0$ and $\phi = \gamma$, over which V is to vanish, the value of V will consist of terms of the type

$$\frac{1}{\sqrt{r}} \sin\left(\frac{k\pi\rho}{\sigma}\right) \sin\left(\frac{m\phi\pi}{\gamma}\right) \cdot K_{\frac{k\pi}{\sigma}}^{\frac{m\pi}{\gamma}} (\pm\cos\theta)$$

for this vanishes over the four boundaries $r = a$, $r = b$, $\phi = 0$, $\phi = \gamma$. We find that for values of ρ between 0 and σ and of ϕ between 0 and γ,

$$f(\rho, \phi) = \frac{1}{2\pi\gamma} \sum_{k=1} \sum_{m=1} \sin\left(\frac{k\pi\rho}{\sigma}\right) \int_0^\sigma \int_0^\phi \sin\frac{k\pi\rho'}{\sigma} \sin\frac{m\pi\phi'}{\gamma} \cdot f(\rho', \phi') \, d\rho' d\phi'.$$

Hence for the case in which there is one conal boundary $\theta = \alpha$ maintained at temperature $\frac{1}{\sqrt{r}} f(\rho, \phi)$, the permanent temperature at any point is

$$\frac{1}{2\pi\gamma\sqrt{r}} \sum_{k=1} \sum_{m=1} \sin\frac{k\pi\rho}{\sigma} \cdot \frac{K_{\frac{k\pi}{\sigma}}^{\frac{m\pi}{\gamma}}(\cos\theta)}{K_{\frac{k\pi}{\sigma}}^{\frac{m\pi}{\gamma}}(\cos\alpha)} \sin\frac{m\pi\phi}{\gamma} \int_0^\sigma \int_0^\phi \sin\frac{k\pi\rho'}{\sigma} \sin\frac{m\pi\phi'}{\gamma} f(\rho', \phi') \, d\rho' d\phi'.$$

The case in which there are two conal boundaries $\theta = \alpha$, $\theta = \beta$ may be treated in a similar manner, but it seems unnecessary to write down the result in this case.

15. The solution of the problems corresponding to the foregoing ones, when there is only one spherical boundary, may be deduced from the above solutions by making a

Digitized by Google

indefinitely small and replacing the summations with respect to k by corresponding definite integrals, but the solutions may easily be obtained directly as follows.

Terms of the type

$$\frac{1}{\sqrt{r}} \sin\left(p \log_e \frac{b}{r}\right) \begin{matrix}\sin\\ \cos\end{matrix} m\phi \,.\, K_p^m (\pm \cos\theta),$$

are suitable for this case, since they vanish for $r = b$ and are finite and continuous for all points in the space considered (except infinitely near the origin which is the vertex of the cone, and may be supposed excluded by surrounding it by an indefinitely small sphere). Let $r = be^{-\rho}$, then ρ has values between 0 and ∞ for the space considered. If there is one conal boundary over which

$$V = \frac{1}{\sqrt{r}} f(\rho, \phi),$$

we have
$$f(\rho, \phi) = \frac{1}{\pi} \Sigma\epsilon \int_0^{2\pi} f(\rho, \phi') \cos m(\phi - \phi')\, d\phi',$$

and
$$f(\rho, \phi') = \frac{2}{\pi} \int_0^\infty \int_0^\infty \sin p\rho \sin p\rho' \,.\, f(\rho', \phi')\, dp\, d\rho',$$

for values of ρ between 0 and ∞; hence the value of V is

$$\frac{2}{\pi^2} \cdot \frac{1}{\sqrt{r}} \Sigma\epsilon \int_0^{2\pi} \int_0^\infty \int_0^\infty \cos m(\phi - \phi') \sin p\rho \sin p\rho' \,.\, f(\rho', \phi') \frac{K_p^m(\cos\theta)}{K_p^m(\cos\alpha)}\, d\phi'\, dp\, d\rho',$$

for this expression is equal to $\frac{1}{\sqrt{r}} f(\rho, \phi')$ when $\theta = \alpha$.

If V is constant $r = V$, over the cone this expression becomes

$$V = \frac{2}{\pi} \cdot \frac{\sqrt{b}}{\sqrt{r}} \int_0^\infty \int_0^\infty \sin p\rho \sin p\rho' \,.\, \rho^{-\frac{\rho'}{2}} \frac{K_p(\cos\theta)}{K_p(\cos\alpha)}\, dp\, d\rho'$$

$$= \frac{8}{\pi} \sqrt{\frac{b}{r}} \int_0^\infty \frac{p \sin p\rho}{1 + 4p^2} \frac{K_p(\cos\theta)}{K_p(\cos\alpha)}\, dp.$$

16. If there are two conal boundaries $\theta = \alpha$, $\theta = \beta$, over which

$$V = \frac{1}{\sqrt{r}} f_1(\rho, \phi), \quad V = \frac{1}{\sqrt{r}} f_2(\rho, \phi),$$

we obtain by proceeding in a similar manner

$$V = \frac{2}{\pi^2} \cdot \frac{1}{\sqrt{r}} \Sigma\epsilon \int_0^{2\pi} \int_0^\infty \int_0^\infty \cos m(\phi - \phi') \sin p\rho \sin p\rho' f_1(\rho', \phi')$$
$$\frac{K_p^m(\cos\theta)\, K_p^m(-\cos\beta) - K_p^m(-\cos\theta)\, K_p^m(\cos\beta)}{K_p^m(\cos\alpha)\, K_p^m(-\cos\beta) - K_p^m(-\cos\alpha)\, K_p^m(\cos\beta)}\, d\phi'\, dp\, d\rho'$$

$$= \frac{2}{\pi^2} \cdot \frac{1}{\sqrt{r}} \Sigma\epsilon \int_0^{2\pi} \int_0^\infty \int_0^\infty \cos m(\phi - \phi') \sin p\rho \sin p\rho' f_2(\rho', \phi') \,.$$
$$\frac{K_p^m(\cos\theta)\, K_p^m(-\cos\alpha) - K_p^m(-\cos\theta)\, K_p^m(\cos\alpha)}{K_p^m(\cos\alpha)\, K_p^m(-\cos\beta) - K_p^m(-\cos\alpha)\, K_p^m(\cos\beta)}\, d\phi'\, dp'\, d\rho'.$$

Digitized by Google

The cases in which there are also plane boundaries $\phi=0$, $\phi=\gamma$ may be treated as before.

17. If in the preceding cases we supposed b to become infinite we should obtain the solution of the case in which the sole boundary is the infinite half cone, the value of V being zero at infinity; in this case it would be necessary to alter the meaning of ρ, we should put $r=e^{\rho}$ and integrate for ρ between the values $\pm\infty$; for $f(\rho, \phi)$ we should use the formula

$$f(\rho, \phi')=\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cos p(\rho-\rho')f(\rho', \phi')\,dp\,d\rho',$$

instead of the formula used above; we then obtain for V in the case of a single conal boundary, the value

$$\frac{1}{2\pi^2\sqrt{r}}\sum_{m=0}^{\infty}\epsilon\int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cos m(\phi-\phi')\cos p(\rho-\rho')f(\rho', \phi')\frac{K_p^m(\cos\theta)}{K_p^m(\cos\alpha)}d\phi'\,dp\,d\rho',$$

which is given by Heine (*Kugelfunctionen*, Vol. II. p. 243) and is obtained by him in a direct manner.

18. In all the problems discussed above, the value of V has been supposed to be zero over the spherical boundaries, and to have arbitrary given values over the conal boundaries. If the values of V over the spherical boundaries are not zero but given functions of θ and ϕ, we shall have to add to the values of V throughout the internal spaces, which we have obtained, values of V which satisfy the conditions of vanishing over the conal boundaries and having given values over the spherical boundaries; we should then have the complete values of V, which satisfied the conditions of having arbitrarily given values over the whole of the boundaries. The natural way to solve the problem of determining V so that it may vanish over the conal boundaries and have prescribed values over the spherical ones, would be to assume an expression for V in a series of spherical harmonics of degrees and orders so chosen that they vanish for the one or the two values of θ corresponding to the conal boundaries. However the determination of the degrees and orders of the harmonics which satisfy this condition presents such difficulties that I have evaded it by using harmonics of integral degree which express the given temperatures over the whole of the spherical surfaces of which the boundary is a part. If the temperature over the portion of a spherical surface exist by a cone $\theta=\alpha$ is given in the form $F(\theta, \phi)$, I suppose this function so extended as to apply to the whole of the spherical surface; the value of $F(\theta, \phi)$ outside the cone is entirely at choice, and may be taken either as zero or as an analytical function of the same form as that which expresses the given temperature over the portion of the spherical surface cut off by the cone; this arbitrarily assumed form for the value of V over those portions of the spherical surfaces which do not form part of the boundaries of the space considered cannot affect the result, since the value of V inside the space only depends on the boundary values of V. The extension of the function $F(\theta, \phi)$ to the whole spherical surface is merely an analytical artifice.

19. Consider the case of the space bounded by two spheres $r=a$, $r=b$ and a cone $\theta=\alpha$; suppose the temperatures of the portions of the spherical surfaces maintained at

Digitized by Google

given values $F_1(\theta, \phi)$, $F_2(\theta, \phi)$ respectively; we have to determine a value of V which shall have these given values over these boundaries and shall vanish over the conal boundary $\theta = \alpha$. As explained above, I suppose $F_1(\theta, \phi)$, $F_2(\theta, \phi)$ functions given over the *whole* spherical surfaces $r = a$, $r = b$, their values for $\theta > \alpha$ being chosen so as to make the functions most simply expansible in surface harmonics of positive integral degree.

Suppose $F_1(\theta, \phi) = \Sigma Y_n$, $F_2(\theta, \phi) = \Sigma Z_n$, Y_n and Z_n denoting harmonics of integral degree n. Assume

$$V = \Sigma r^n Y_n' + \frac{1}{r^{n+1}} Z_n',$$

then from the conditions

$$a^n Y_n' + \frac{1}{a^{n+1}} Z_n' = Y_n$$

$$b^n Y_n' + \frac{1}{b^{n+1}} Z_n' = Z_n,$$

we obtain Y_n', Z_n', and thence the following value of V,

$$V = \sqrt{\frac{a}{r}}\, \Sigma \frac{\sinh (n+\frac{1}{2})(\sigma - \rho)}{\sinh (n+\frac{1}{2})\sigma} Y_n + \sqrt{\frac{b}{r}}\, \Sigma \frac{\sinh (n+\frac{1}{2})\rho}{\sinh (n+\frac{1}{2})\sigma} Z_n,$$

this value of V which I shall denote by $\frac{1}{\sqrt{r}} \mu(\rho, \phi, \theta)$ has the prescribed values over the spherical boundaries but does not vanish for $\theta = \alpha$, so we must add to it a value of V which vanishes over the spherical boundaries and is equal to $-\frac{1}{\sqrt{r}} . \mu(\rho, \phi, \alpha)$ over the conal boundary.

This is (Art. 12)

$$-\frac{1}{2\pi^2 \sqrt{r}} \sum_{k=1} \sum_{m=0} \epsilon \sin \frac{k\pi\rho}{\sigma} \frac{K^m_{\frac{k\pi}{\sigma}}(\cos\theta)}{K^m_{\frac{k\pi}{\sigma}}(\cos\alpha)} \int_0^\sigma \int_0^{2\pi} \sin \frac{k\pi\rho'}{\sigma} \cos m(\phi - \phi')\, \mu(\rho', \phi', \alpha)\, d\rho'\, d\phi'.$$

Hence the complete value of V which has the prescribed values over the spherical boundaries and vanishes over the conal one is

$$V = \sqrt{\frac{a}{r}}\, \Sigma \frac{\sinh (n+\frac{1}{2})(\sigma - \rho)}{\sinh (n+\frac{1}{2})\sigma} Y_n + \sqrt{\frac{b}{r}}\, \Sigma \frac{\sinh (n+\frac{1}{2})\rho}{\sinh (n+\frac{1}{2})\sigma} Z_n$$

$$-\frac{\sqrt{a}}{2\pi^2 \sqrt{r}} \sum_{k=1} \sum_{m=0} \epsilon \sin \frac{k\pi\rho}{\sigma} . \frac{K^m_{\frac{k\pi}{\sigma}}(\cos\theta)}{K^m_{\frac{k\pi}{\sigma}}(\cos\alpha)}$$

$$\int_0^\sigma \int_0^{2\pi} \sin \frac{k\pi\rho'}{\sigma} \cos m(\phi - \phi')\, \Sigma \frac{\sinh (n+\frac{1}{2})(\sigma - \rho')}{\sinh (n+\frac{1}{2})\sigma} Y_n(\alpha, \phi')\, d\rho'\, d\phi'$$

$$-\frac{\sqrt{b}}{2\pi^2 \sqrt{r}} \sum_{k=1} \sum_{m=0} \epsilon \sin \frac{k\pi\rho}{\sigma} . \frac{K^m_{\frac{k\pi}{\sigma}}(\cos\theta)}{K^m_{\frac{k\pi}{\sigma}}(\cos\alpha)}$$

$$\int_0^\sigma \int_0^{2\pi} \sin \frac{k\pi\rho'}{\sigma} \cos m(\phi - \phi')\, \Sigma \frac{\sinh (n+\frac{1}{2})\rho'}{\sinh (n+\frac{1}{2})\sigma} Z_n(\alpha, \phi')\, d\rho'\, d\phi'.$$

Digitized by Google

We have $\int_0^\sigma \sin\frac{k\pi\rho'}{\sigma}\sinh(n+\frac{1}{2})\rho'\,d\rho' = -\frac{\sqrt{-1}}{2}\int_0^\sigma\left[\cos\left\{\frac{k\pi}{\sigma}-(n+\frac{1}{2})\sqrt{-1}\right\}\rho'\right.$

$$\left.-\cos\left\{\frac{k\pi}{\sigma}+(n+\tfrac{1}{2})\sqrt{-1}\right\}\rho'\right]d\rho'$$

$$=-\frac{\sqrt{-1}\,.\,\sigma}{2}\left[\frac{\sin\{k\pi-(n+\frac{1}{2})\sigma\sqrt{-1}\}}{k\pi-(n+\frac{1}{2})\sigma\sqrt{-1}}-\frac{\sin\{k\pi+(n+\frac{1}{2})\sigma\sqrt{-1}}{k\pi+(n+\frac{1}{2})\sigma\sqrt{-1}}\right]$$

$$=-\frac{\sqrt{-1}\,.\,\sigma}{2}\,\frac{-2k\pi\cos k\pi\sin(n+\frac{1}{2})\sigma\sqrt{-1}}{k^2\pi^2+(n+\frac{1}{2})^2}$$

$=k\pi\sigma\frac{(-1)^{k+1}\sinh(n+\frac{1}{2})\sigma}{k^2\pi^2+(n+\frac{1}{2})^2\sigma^2}$, and similarly $\int_0^\sigma \sin\frac{k\pi\rho'}{\sigma}\sinh(n+\frac{1}{2})(\sigma-\rho')\,d\rho'$

$$=\frac{k\pi\sigma\sinh(n+\frac{1}{2})\sigma}{k^2\pi^2+(m+\frac{1}{2})^2\sigma^2},$$

thus the integration with respect to ρ' in each term has been performed. A further simplification of the expression above is obtained by substituting for Y_n, Z_n their expansions in tesseral harmonics

$$Y_n=\sum_{m_1=0}^{m_1=n}(A_{n,m_1}\cos m_1\phi'+B_{n,m}\sin m_1\phi')\,T_n^{(m)}(\cos\alpha),$$

then since $\int_0^{2\pi}\begin{matrix}\cos\\ \sin\end{matrix}\, m_1\phi'\cos m(\phi-\phi')\,d\phi'$ is zero unless $m_1=m$, and is then equal to $\pi\begin{matrix}\cos\\ \sin\end{matrix}\, m\phi$ we have the following expression for the temperature at any internal point, under the given surface conditions

$$V=\sqrt{\frac{a}{r}}\,\Sigma\,\frac{\sinh(n+\frac{1}{2})(\sigma-\rho)}{\sinh(n+\frac{1}{2})\sigma}\,Y_n+\sqrt{\frac{b}{r}}\,\Sigma\,\frac{\sinh(n+\frac{1}{2})\rho}{\sinh(n+\frac{1}{2})\sigma}\,Z_n$$

$$+\frac{\sqrt{a}\,\sigma}{2\sqrt{r}}\sum_{k=1}\sum_{m=0}(-1)^k\,\Sigma\sin\frac{k\pi\rho}{\sigma}\,.\,\frac{K^m_{\frac{k\pi}{\sigma}}(\cos\theta)}{K^m_{\frac{k\pi}{\sigma}}(\cos\alpha)}\sum_{n=m}^{n=\infty}k\,\frac{T_n^{(m)}\cos\alpha\,(A_{n,m}\cos m\phi+B_{n,m}\sin m\phi)}{k^2\pi^2+(n+\frac{1}{2})^2\sigma^2}$$

$$-\frac{\sqrt{b}\,\sigma}{2\sqrt{r}}\sum_{k=1}\sum_{m=0}\sum\sin\frac{k\pi\rho}{\sigma}\,.\,\frac{K^m_{\frac{k\pi}{\sigma}}(\cos\theta)}{K^m_{\frac{k\pi}{\sigma}}(\cos\alpha)}\sum_{n=m}^{n=\infty}k\,.\,\frac{T_n^{(m)}(\cos\alpha)\,(A'_{n,m}\cos m\phi+B'_{n,m}\sin m\phi)}{k^2\pi^2+(n+\frac{1}{2})^2\sigma^2}$$

where $$A_{n,m}=\frac{\int_{-1}^{1}\int_0^{2\pi}Y_nT_n^m\cos m\phi\,d\mu\,d\phi}{\frac{n+m!}{n-m!}\,\frac{2\pi}{2n+1}}$$ (see Ferrers' *Spherical Harmonics*, p. 86)

$$B_{n,m}=\frac{\int_{-1}^{1}\int_0^{2\pi}Y_nT_n^m\sin m\phi\,d\mu\,d\phi}{\frac{n+m!}{n-m!}\,\frac{2\pi}{2n+1}},$$

$A'_{n,m}=$ an expression similar to $A_{n,m}$ with Z_n for Y_n,

$B'_{n,m}=$ ..... $B_{n,m}$ .....

Digitized by Google

This value of V added to the value in Art. 12 gives the complete values of V throughout the space when the values over the whole boundary are given.

In the particular case in which the temperatures are given constant over the spherical boundaries $V=V_1$ and $V=V_2$, we have

$$V=\frac{bV_2-aV_1}{b-a}+\frac{ab(V_1-V_2)}{r(b-a)}+\frac{\sqrt{a\sigma}}{\sqrt{r}}\sum_{k=1}(-1)^k\sin\frac{k\pi\rho}{\rho}\cdot\frac{K_{\frac{k\pi}{\sigma}}(\cos\theta)}{K_{\frac{k\pi}{\sigma}}(\cos\alpha)}\cdot\frac{k}{\sigma^2+4k^2\pi^2}\cdot(V_1-V_2).$$

20. An important application of the preceding formulae is the determination of the potential of the electricity on the surface of an infinite conductor with a hollow of the shape just considered, due to an electrified point placed inside the hollow space. If we suppose a negative unit of electricity placed at the point, the problem is equivalent to the determination of the value of V throughout the space considered when it is equal over the bounding surface to the reciprocal of the distance from an internal point. This value of V is the Green's function for the space, and when found enables us to determine the potential at any point of the space considered, due to any arbitrarily given surface distribution of matter; for if $G(P)$ denotes the Green's function of any point P, the potential at P of any surface distribution in which σ' is the density at the element dS' of surface is $\int G(P)\sigma' dS$, the integration being taken over the whole boundary.

In this case the value of V over the bounding surface is

$$\frac{1}{a}\cdot\frac{1}{\sqrt{e^{2\rho}+e^{2\rho_0}-2e^{\rho+\rho_0}\{\cos\theta\cos\theta_0+\sin\theta\sin\theta_0\cos(\phi-\phi_0)\}}},$$

if ρ_0, θ_0, ϕ_0 are the coordinates of the internal point. Over the conal boundary the value of V is then to be

$$\frac{1}{a}\cdot\frac{1}{\sqrt{e^{2\rho}+e^{2\rho_0}-2e^{\rho+\rho_0}\{\cos\alpha\cos\theta_0+\sin\alpha\sin\theta_0\cos(\phi-\phi_0)\}}}$$

and we must substitute this value for the function $\frac{1}{\sqrt{r}}f(\rho,\phi)$ in Art. 12.

If $\rho>\rho_0$ this expression is equal to $\frac{1}{a}\Sigma\frac{e^{n\rho_0}}{e^{(n+1)\rho}}P_n\{\cos\alpha\cos\theta_0+\sin\alpha\sin\theta_0\cos(\phi-\phi_0)\}$, and this is equal to

$$\frac{1}{a}\Sigma\frac{e^{n\rho_0}}{e^{(n+1)\rho}}\sum_{m=0}^{m=n}2\Sigma\frac{n-m!}{n+m!}T_n^m(\cos\alpha)\,T_n^m(\cos\theta_0)\cos m(\phi-\phi_0).$$

If $\rho<\rho_0$ the same expression is equal to

$$\frac{1}{a}\Sigma\frac{e^{n\rho}}{e^{(n+1)\rho_0}}\sum_{m=0}^{m=n}2\Sigma\frac{n-m!}{n+m!}T_n^{(m)}(\cos\alpha)\,T_n^{(m)}(\cos\theta_0)\cos m(\phi-\phi_0).$$

We must therefore in accordance with the method of Art. 12 expand a function which is equal to $\frac{e^{n\rho_0}}{e^{(n+\frac{1}{2})\rho}}$ for values of ρ between ρ_0 and σ, and is equal to $\frac{e^{(n-\frac{1}{2})\rho}}{e^{(n+1)\rho_0}}$ for values of ρ

Digitized by Google

between 0 and ρ_0 in a series of sines of multiples of $\frac{\pi\rho}{\sigma}$; the required series will be

$$\Sigma A_{n,k} \sin \frac{k\pi\rho}{\sigma},$$

where $$A_{n,k} = \frac{2}{\pi}\int_0^{\rho_0} e^{-(n+1)\rho_0+(n-\frac{1}{2})\rho} \sin\frac{k\pi\rho}{\sigma}\,d\rho + \frac{2}{\pi}\int_{\rho_0}^{\sigma} e^{n\rho_0-(n+\frac{1}{2})\rho}\sin\frac{k\pi\rho}{\sigma}\,d\rho,$$

carrying out the integrations, we get

$$A_{n,k} = \frac{2\sigma}{\pi} e^{-(n+1)\rho_0}\left\{\frac{k\pi\left(1-e^{(n-\frac{1}{2})\rho_0}\cos\frac{k\pi\rho_0}{\sigma}\right)+(n-\frac{1}{2})\sigma e^{(n-\frac{1}{2})\rho_0}\sin\frac{k\pi\rho_0}{\sigma}}{k^2\pi^2+(n-\frac{1}{2})^2\sigma^2}\right\}$$

$$= \frac{2\sigma}{\pi} e^{n\rho_0}\left\{\frac{k\pi\left(e^{-(n+\frac{1}{2})\sigma}\cos k\pi - e^{-(n+\frac{1}{2})\rho_0}\cos\frac{k\pi\rho_0}{\sigma}\right)-\sigma(n+\frac{1}{2})e^{-(n+\frac{1}{2})\rho_0}\sin\frac{k\pi\rho_0}{\sigma}}{k^2\pi^2+(n+\frac{1}{2})^2\sigma^2}\right\}.$$

The part of V due to this distribution over the cone is then

$$V_1 = \frac{2\sigma}{a\pi\sqrt{r}}\sum_{k=1}^{k=\infty}\sum_{m=0}^{m=\infty}\sum_{n=m}^{n=\infty} 2\Sigma\,.\,e^{-(n+1)\rho_0} A_{n,k}\frac{n-m!}{n+m!}T_n^{(m)}(\cos\alpha)\,T_n^{(m)}(\cos\theta_0)\cos m(\phi-\phi_0)\frac{K^m_{\frac{k\pi}{\sigma}}(\cos\theta)}{K^m_{\frac{k\pi}{\sigma}}(\cos\alpha)}.$$

Over the sphere $\rho = 0$, the value of V is to be

$$\frac{1}{a}\Sigma e^{-(n+1)\rho_0} P_n\{\cos\theta\cos\theta_0+\sin\theta\sin\theta_0\cos(\phi-\phi_0)\},$$

and over the sphere $\rho = \sigma$ it is to be

$$\frac{1}{a}\Sigma e^{n\rho_0-(n+1)\sigma} P_n\{\cos\theta\cos\theta_0+\sin\theta\sin\theta_0\cos(\phi-\phi_0)\},$$

hence using the formula in Art. 19 we have, since in this case

$$A_{m,n}\cos m\phi + B_{m,n}\sin m\phi = \frac{1}{a}e^{-(n+1)\rho_0}\,.\,2\Sigma\,.\,\frac{n-m!}{n+m!}\cos m(\phi-\phi_0),$$

$$A'_{m,n}\cos m\phi + B'_{m,n}\sin m\phi = \frac{1}{a}e^{n\rho_0-(n+1)\sigma}\,.\,2\Sigma\,\frac{n-m!}{n+m!}\cos m(\phi-\phi_0).$$

The value of V_2

$$V_2 = \sqrt{\frac{a}{r}}\Sigma\frac{\sinh(n+\frac{1}{2})(\sigma-\rho)}{\sinh(n+\frac{1}{2})\sigma}\,.\,\frac{1}{a}e^{-(n+1)\rho_0}\sum_{m=0}^{m=n}2\Sigma\,.\,\frac{n-m!}{n+m!}T_n^{(m)}(\cos\theta)\,T_n^{(m)}(\cos\theta_0)\cos m(\phi-\phi_0)$$

$$+\sqrt{\frac{b}{r}}\Sigma\frac{\sinh(n+\frac{1}{2})\rho}{\sinh(n+\frac{1}{2})\sigma}\,.\,\frac{1}{a}e^{n\rho_0-(n+1)\sigma}\sum_{m=0}^{m=n}2\Sigma\,.\,\frac{n-m!}{n+m!}T_n^{(m)}(\cos\theta)\,T_n^{(m)}(\cos\theta_0)\cos m(\phi-\phi_0)$$

$$-\frac{\sqrt{a}\sigma}{2\sqrt{r}}\sum_{k=1}\sum_{m=0}(-1)^k\Sigma\sin\frac{k\pi\rho}{\sigma}\,.\,\frac{K^m_{\frac{k\pi}{\sigma}}(\cos\theta)}{K^m_{\frac{k\pi}{\sigma}}(\cos\alpha)}\sum_{n=m}^{n=\infty}$$

$$\frac{1}{a}\left(e^{-(n+1)\rho_0}\sqrt{a}-\rho^{n\rho_0-(n+1)\sigma}\sqrt{b}\right)2\Sigma\,.\,\frac{n-m!}{n+m!}\frac{kT_n^{(m)}(\cos\alpha)\cos m(\phi-\phi_0)}{k^2\pi^2+(n+\frac{1}{2})^2\sigma^2}.$$

Digitized by Google

The sum of V_1 and V_2 gives the complete value of V, that is the Green's function of the point $\rho_0\theta_0\phi_0$.

The corresponding problem for the case of the space bounded by two cones and two spheres may be solved in the same manner, but it is unnecessary to work out this case in detail as the formulae do not essentially differ from those just obtained.

21. I shall now obtain Green's function for the space bounded by a cone and a single spherical surface, this might be deduced from the last case but is as easily obtained directly by means of the formula in Art. 15. Using the notation in that article, suppose ρ_0, θ_0, ϕ_0 the coordinates of the point for which the Green's function is to be obtained. The reciprocal of the distance between this point and (ρ, θ, ϕ) is

$$\frac{1}{b\sqrt{e^{-2\rho}+e^{-2\rho_0}-2e^{-\rho-\rho_0}\{\cos\theta\cos\theta_0+\sin\theta\sin\theta_0\cos(\phi-\phi_0)\}}}.$$

It is well known that the value of the function over the sphere is that due to a mass e^{ρ_0} at the inverse point whose coordinates are be^{ρ_0}, θ, ϕ, and the value of this at any internal point is

$$\frac{e^{\rho_0}}{b\sqrt{e^{2\rho_0}+e^{-2\rho}-2e^{\rho_0-\rho}\{\cos\theta\cos\theta_0+\sin\theta\sin\theta_0\cos(\phi-\phi_0)\}}}.$$

We must add to this the value obtained from the formula in Art. 15, where we put

$$f(\rho, \phi)=\frac{\sqrt{b}\,e^{-\frac{1}{2}\rho}}{b\sqrt{e^{-2\rho}+e^{-2\rho_0}-2e^{-\rho-\rho_0}\{\cos\alpha\cos\theta_0+\sin\alpha\sin\theta_0\cos(\phi-\phi_0)\}}}$$
$$-\frac{\sqrt{b}\,e^{\rho_0-\frac{1}{2}\rho}}{b\sqrt{e^{2\rho_0}-e^{-2\rho}-2e^{\rho_0-\rho}\{\cos\alpha\cos\theta_0+\sin\alpha\sin\theta_0\cos(\phi-\phi_0)\}}}.$$

The value of Green's function is therefore

$$\frac{1}{b\sqrt{1-e^{-2\rho-2\rho_0}-2e^{-\rho-\rho_0}\{\cos\theta\cos\theta_0+\sin\theta\sin\theta_0\cos(\phi-\phi_0)\}}}$$
$$+\frac{1}{\sqrt{b}}\frac{2}{\pi^2}\frac{1}{\sqrt{r}}\Sigma\Sigma\int_0^{2\pi}\int_0^\infty\int_0^\infty\cos m(\phi-\phi')\sin p\rho\sin p\rho'\frac{K_p^m(\cos\theta)}{K_p^m(\cos\alpha)}$$
$$\left\{\frac{e^{-\frac{1}{2}\rho}}{\sqrt{e^{-2\rho}+e^{-2\rho_0}-2e^{-\rho-\rho_0}\{\cos\alpha\cos\theta_0+\sin\alpha\sin\theta_0\cos(\phi-\phi_0)\}}}\right.$$
$$\left.-\frac{e^{\rho_0-\frac{1}{2}\rho}}{\sqrt{e^{2\rho_0}-e^{-2\rho}-2e^{\rho_0-\rho}\{\cos\alpha\cos\theta_0+\sin\alpha\sin\theta_0\cos(\phi-\phi_0)\}}}\right\}d\phi'\,dp\,d\rho'.$$

This is the potential of the electricity induced by a negative unit placed at the point ρ_0, θ, ϕ. The integration with respect to ϕ' may be carried out by expanding the denominators of the fractions in Tesseral harmonics as in the last case, distinguishing the two cases $\rho \gtrless \rho_0$.

22. I shall now apply the method of inversion to obtain from the preceding cases the solution of the corresponding problems for the spaces bounded by the surfaces which

Digitized by Google

are the inverses with respect to a point on the axis, of the spherical and conal surfaces. The inverse of a spherical surface with respect to any point not on the surface itself is a spherical surface, and that of a cone is the surface generated by the revolution of a circular segment round its chord.

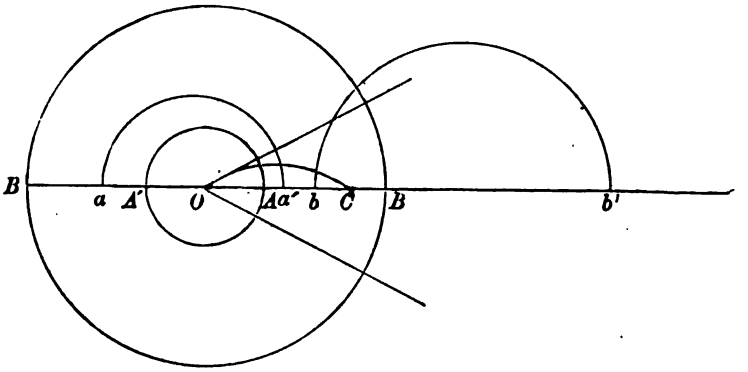


Fig. 1.

Let O be the centres of two spheres and the vertex of a cone as in Art. 12; take a point C on the axis in the space between the spheres and invert the system with respect to C, taking $CO = k$ for the constant of inversion, so that O is its own inverse point. The two spheres of which AA', BB' are diameters invert into two spheres on aa', bb' as diameters and the cone inverts into the surface formed by the revolution of a segment of a circle of which OC is chord, and which contains an angle equal to the semi-vertical angle of the cone.

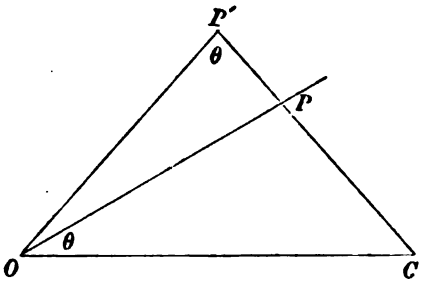


Fig. 2.

Let P' be the inverse of any point P, then we have, if ρ, θ, ϕ are the coordinates of P, $\angle OP'C = \angle POC = \theta$, also $\frac{OP'}{P'C} = \frac{OP}{OC} = \frac{a}{k}e^{\rho}$; thus the locus of P' for which ρ is constant is a series of spheres for which O and C are common inverse points. Corresponding to $\rho = 0$, $\rho = \sigma$ we have the two spheres for which $\frac{OP'}{P'C}$ is equal to $\frac{a}{k}$ and $\frac{a}{k}e^{\sigma}$, and the radii of these spheres are easily found to be $\frac{k^2 a}{k^2 - a^2}$, $\frac{k^2 a e^{\sigma}}{a^2 e^{2\sigma} - k^2}$. In the inverse figure

Digitized by Google

we regard ρ, θ, ϕ as the coordinates of any point P'; we have

$$CP'^2 = a^2 (e^{2\rho} + e^{2\tau} - 2e^{\rho+\tau} \cos\theta) = a^2 . e^{\rho+\tau} \{2 \cosh(\rho - \tau) - 2\cos\theta\}$$

where $$k = ae^{\tau}.$$

According to the method of inversion if V is the potential at P due to any distribution of matter, the potential V' at P' due to the inverse distribution is given by

$$V' = V . \frac{k}{CP'},$$

$$\therefore \; V' = V . e^{\frac{\rho-\tau}{2}} \sqrt{2\cosh(\rho+\tau) - 2\cos\theta}.$$

23. To the problem of determining the potential throughout the space bounded by two spheres and a cone corresponds that of finding the potential at any point in the space external to two spheres aa', bb' connected by the figure OC formed by the rotation of a segment of a circle. In the case in which the two spheres aa', bb' are equal and the segment is flattened down we have a body approximately the shape of a dumb-bell slightly thickened in the middle.

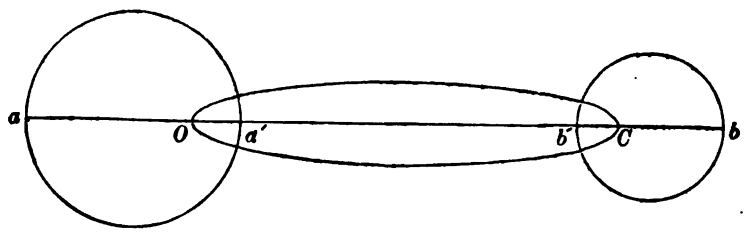


Fig. 3.

Suppose that it is required to find a potential V' throughout the external space so that at the surface of the body just described

$$V' = F(\rho, \theta, \phi).$$

We have $$V' = V . e^{\frac{\rho-\tau}{2}} \sqrt{2\cosh(\rho+\tau) - 2\cos\theta},$$

where V is determined by the formula in Art. 12, substituting for $f(\rho, \theta, \phi)$ the value

$$\frac{F(\rho, \theta, \phi) . e^{\frac{1}{2}\tau}}{\sqrt{2\cosh(\rho+\tau) - 2\cos\theta}}.$$

24. First suppose it required to find the potential at any point due to the distribution of electricity on a conductor of this form charged to potential C.

For the spheres $\rho = 0$, $\rho = \sigma$ the values of V are

$$\frac{Ce^{\frac{1}{2}\tau}}{\sqrt{2\cosh\tau - 2\cos\theta}}, \quad \frac{Ce^{\frac{\tau-\sigma}{2}}}{\sqrt{2\cosh(\sigma+\tau) - 2\cos\theta}};$$

Digitized by Google

and for the cone $\theta = \alpha$ V is

$$\frac{\sqrt{a}}{\sqrt{r}} \frac{Ce^{\frac{1}{2}\tau}}{\sqrt{2\cosh(\rho+\tau) - 2\cos\alpha}}.$$

Hence the part of V due to the value of V over the cone is

$$\frac{Ce^{\frac{1}{2}\tau}\sqrt{a}}{4\pi\sqrt{r}} \sum_{k=1} \sin\frac{k\pi\rho}{\sigma} \frac{K_{\frac{k\pi}{\sigma}}(\cos\theta)}{K_{\frac{k\pi}{\sigma}}(\cos\alpha)} \cdot \int_0^\sigma \frac{\sin\frac{k\pi\rho'}{\sigma}}{\sqrt{2\cosh(\rho'+\tau) - 2\cos\alpha}} d\rho'.$$

To find the parts of V due to the values of V over the spheres we must in accordance with Art. 19 expand these values in harmonics of integral degree, we have at once

$$\frac{Ce^{\frac{1}{2}\tau}}{\sqrt{2\cosh\tau - 2\cos\theta}} = C\sum_{n=0}^{n=\infty} e^{-n\tau} P_n(\cos\theta),$$

$$\frac{Ce^{\frac{1}{2}(\tau-\sigma)}}{\sqrt{2\cosh(\tau+\sigma) - 2\cos\theta}} = Ce^{-\sigma}\Sigma e^{-n(\tau+\sigma)} P_n(\cos\theta).$$

Hence the corresponding part of V is

$$\sqrt{\frac{a}{r}}\Sigma \frac{\sinh(n+\frac{1}{2})(\sigma-\rho)}{\sinh(n+\frac{1}{2})\sigma} . Ce^{-n\tau} P_n(\cos\theta) + \sqrt{\frac{b}{r}}\Sigma \frac{\sinh(n+\frac{1}{2})\rho}{\sinh(n+\frac{1}{2})\sigma} Ce^{-n\tau-(n+1)\sigma} P_n(\cos\theta)$$

$$+ \frac{\sqrt{a}\,\sigma}{4\sqrt{r}} C \sum_{k=1} (-1)^k \sin\frac{k\pi\rho}{\sigma} . \frac{K_{\frac{k\pi}{\sigma}}(\cos\theta)}{K_{\frac{k\pi}{\sigma}}(\cos\alpha)} \sum_{n=0}^{n=\infty} \frac{k.e^{-n\tau} P_n(\cos\alpha)}{k^2\pi^2 + \sigma^2(n+\frac{1}{2})^2}$$

$$- \frac{\sqrt{b}\,\sigma}{4\sqrt{r}} C \sum_{k=1} \sin\frac{k\pi\rho}{\sigma} . \frac{K_{\frac{k\pi}{\sigma}}(\cos\theta)}{K_{\frac{k\pi}{\sigma}}(\cos\alpha)} \sum_{n=0}^{n=\infty} \frac{ke^{-n\tau-(n+1)\sigma} P_n(\cos\alpha)}{k^2\pi^2 + \sigma^2(n+\frac{1}{2})^2}.$$

The integral $\int_0^\sigma \frac{\sin\frac{k\pi\rho'}{\sigma}}{\sqrt{2\cosh(\rho'+\tau) - 2\cos\alpha}} d\rho'$ can be expanded in a series, thus:

$$\int_0^\sigma \frac{\sin\frac{k\pi\rho'}{\sigma}}{\sqrt{2\cosh(\rho'+\tau) - 2\cos\alpha}} d\rho' = \int_0^\sigma \sin\frac{k\pi\rho'}{\sigma} \Sigma e^{-(n+\frac{1}{2})(\rho'+\tau)} P_n(\cos\alpha)\, d\rho'$$

$$= \Sigma \frac{\frac{k\pi}{\sigma} P_n(\cos\alpha)}{(n+\frac{1}{2})^2 + \frac{k^2\pi^2}{\sigma^2}} [e^{-(n+\frac{1}{2})\tau} - \cos k\pi . e^{-(n+\frac{1}{2})(\tau+\sigma)}].$$

Digitized by Google

Hence the value of V' is

$$C\sqrt{2\cosh(\rho+\tau)-2\cos\theta}\left[\sqrt{\frac{a}{r}}\Sigma\frac{\sinh(n+\frac{1}{2})(\sigma-\rho)}{\sinh(n+\frac{1}{2})\sigma}e^{-n\tau}P_n(\cos\theta)\right.$$
$$\left.+\sqrt{\frac{b}{r}}\Sigma\frac{\sinh(n+\frac{1}{2})\rho}{\sinh(n+\frac{1}{2})\sigma}e^{-n\tau-(n+1)\sigma}P_n(\cos\theta)\right]$$
$$+\frac{C}{4}\sqrt{2\cosh(\rho+\tau)-2\cos\theta}\sqrt{\frac{a}{r}}\Sigma\sin\frac{k\pi\rho}{\sigma}\frac{K_{\frac{k\pi}{\sigma}}(\cos\theta)}{K_{\frac{k\pi}{\sigma}}(\cos\alpha)}\sum_{n=0}\frac{k\sigma P_n(\cos\alpha)}{k^2\pi^2+(n+\frac{1}{2})^2\sigma^2}$$
$$\{e^{-n\tau}(1-\cos k\pi)-e^{-n\tau-(n+\frac{1}{2})\sigma}(1+\cos k\pi)\}.$$

This is the potential at any point in external space of the electricity on the conductor when it is at potential C.

25. Next suppose it required to find the electricity induced on a conductor of the same shape induced by a negative unit of electricity at any point in the external space.

We shall obtain the solution of this problem by inverting the case solved in Art. 21, supposing that a mass $-\frac{CP}{K}$ is placed at the point P, the inverse of the given point $(\rho_0\theta_0\phi_0)$. In order to find V we must therefore multiply the value of V in Art. 21 by $e^{\frac{1}{2}(\rho_0-\tau)}\sqrt{2\cosh(\rho_0+\tau)-2\cos\theta_0}$, hence the value of V' is

$$e^{\frac{1}{2}(\rho+\rho_0)-\tau}\sqrt{2\cosh(\rho_0+\tau)-2\cos\theta_0}\sqrt{2\cosh(\rho+\tau)-2\cos\theta}\,(V_1+V_2),$$

where V_1 and V_2 are the expressions in Art. 19.

26. For the case of the infinite half cone with no bounding spheres, if $r=e^\rho$ denote the distance of any point from the origin, it is known (Heine, Vol. II. p. 243) that the value of V at any point in the cone corresponding to $V=\frac{1}{\sqrt{r}}f(\rho,\phi)$ over the surface is

$$V=\frac{1}{2\pi^2\sqrt{r}}\sum_{m=0}^{m=\infty}\int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\rho',\phi')\frac{K_\mu^m(\cos\theta)}{K_\mu^m(\cos\alpha)}\cos\mu(\rho-\rho')\cos m(\phi-\phi').\,d\phi d\rho' d\mu.$$

This may easily be deduced from Art. 15, by making the radius b of the bounding sphere infinite and altering the meaning of ρ.

By inversion we find the potential V' of the space external to a figure bounded by the surface of revolution generated by the revolution of a segment of a circle round its chord, the value of V' which is such that the potential at the bounding surface is $F(\rho,\phi)$ is

$$\frac{\sqrt{2\cosh\rho-2\cos\theta}}{2\pi^2}\sum_{m=0}^{m=\infty}\int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{F(\rho',\phi')}{\sqrt{2\cosh\rho'-2\cos\alpha}}\cdot\frac{K_\mu^m(\cos\theta)}{K_\mu^m(\cos\alpha)}\cos\mu(\rho-\rho')\cos m(\phi-\phi')\,d\phi d\rho' d\mu.$$

27. As an example of the use of this formula I shall find the current function (ψ) in an infinite fluid when a solid of this shape is moving with a velocity V in the direction of its axis.

Digitized by Google

Let y be distance of any point P from the axis and x the distance from C measured along the axis, then μ satisfies the equation

$$\frac{d^2\psi}{dx^2}+\frac{d^2\psi}{dy^2}-\frac{1}{y}\frac{\psi^2 d}{dy}=0.$$

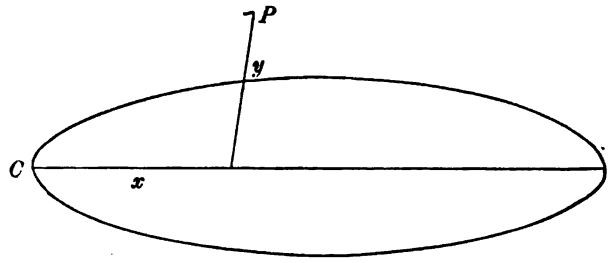


Fig. 4.

In a paper on the potentials of the surfaces generated by the revolution of Limaçons recently read before the Philosophical Society, Mr A. B. Basset has remarked that this equation is of such a form that $\frac{\psi}{y}\sin\phi$ is a potential; in fact the equation may be written

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{1}{y}\frac{d}{dy}+\frac{1}{y^2}\frac{d^2}{d\phi^2}\right)\left(\frac{\psi\sin\phi}{y}\right)=0,$$

and since at the surface $\psi=\frac{1}{2}Vy^2$, $\frac{\psi\sin\phi}{y}$ is a potential of the external space and is equal at the surface to $\frac{1}{2}Vy\sin\phi$.

Now referring to the figure in Art. 22, we see that

$$ky=\sin\theta\,.\,\frac{a}{k}\,.\,e^{\rho}\,.\,\frac{k^2}{a^2e^{\rho+\tau}\{2\cosh(\rho+\tau)-2\cos\theta\}}\quad\text{or}\quad y=\frac{\sin\theta}{2\cosh\rho-2\cos\theta},$$

putting $k=1$, $a=0$ to suit the new notation. Hence the value of $\frac{\psi\sin\phi}{y}$ over the surface is $\frac{1}{2}V\frac{\sin\theta\sin\phi}{2\cosh\rho-2\cos\theta}$. Thus referring to the formula in the last article we have for the value of the current function

$$\frac{V\sin\theta\sin\alpha}{4\pi^2\sqrt{2\cosh\rho-2\cos\theta}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{(2\cosh\rho'-2\cos\theta)^{\frac{3}{2}}}\frac{K_\mu(\cos\theta)}{K_\mu(\cos\alpha)}\cos\mu(\rho-\rho')\,d\rho'd\mu.$$

28. It may seem worthy of remark that corresponding problems in two dimensions may be solved by means of two dimensional harmonics of imaginary degree.

If $\left(Ar^n+\frac{B}{r^n}\right)\begin{matrix}\sin\\ \cos\end{matrix}\,n\theta$ vanishes for two values $r=a$, $r=b$ of r, we must have

$$Aa^n+\frac{B}{a^n}=Ab^n+\frac{B}{b^n}=0,$$

Digitized by Google

therefore
$$a^{2n} = b^{2n}, \text{ or } 2n \log \frac{b}{a} = 2\sqrt{-1}\, k\pi,$$
$$n = \frac{k\pi\sqrt{-1}}{\log \frac{b}{a}}.$$

If $b = ae^{\sigma}$, $r = ae^{\rho}$, then $\sin \frac{k\pi\rho}{\sigma}\left(A_k e^{\frac{k\pi\theta}{\sigma}} + B_k e^{-\frac{k\pi\theta}{\sigma}}\right)$ vanishes for $\rho = 0$ and $\rho = \sigma$.

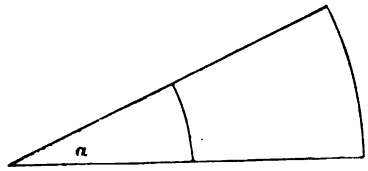


Fig. 5.

For example to determine the steady motion of heat in the space bounded by $r = a$, $r = b$, $\theta = 0$, $\theta = \alpha$ when the temperature is $f_1(\rho)$ for $\theta = 0$, $f_2(\rho)$ for $\theta = \alpha$ and vanishes for $r = a$, $r = b$, we have to determine A_k and B_k from the equations
$$(A_k + B_k)\frac{\sigma}{2} = \int_0^{\sigma} f_1(\rho') \sin \frac{k\pi\rho'}{\sigma}\, d\rho,$$
$$\left(A_k e^{\frac{k\pi\alpha}{\sigma}} + B_k e^{-\frac{k\pi\alpha}{\sigma}}\right)\frac{\sigma}{2} = \int_0^{\sigma} f_2(\rho') \sin \frac{k\pi\rho'}{\sigma}\, d\rho,$$

hence the temperature is
$$\frac{2}{\sigma}\sum_{k=1}^{k=\infty} \frac{\sin \frac{k\pi\rho}{\sigma}}{\sinh \frac{k\pi\alpha}{\sigma}} \int_0^{\sigma} \left\{\sinh \frac{k\pi\theta}{\sigma} f_2(\rho') + \sinh \frac{k\pi(\alpha - \theta)}{\sigma} \cdot f_1(\rho')\right\} \sin \frac{k\pi\rho'}{\sigma}\, d\rho'.$$

Digitized by Google

XI. *Table of the Exponential Function e^x to twelve places of Decimals.* By F. W. Newman, Emeritus Professor of University College, London.

[Communicated by Prof. Adams. Read October 31, 1887.]

This Table gives the value of e^x for values of x from $x = 0{\cdot}001$ to $x = 2{\cdot}000$, taken at intervals of 0·001.

The mode of forming it is exactly similar to that described at length by the author in the introduction to his Table of the Descending Exponential, which was published in Part III. of Vol. XIII. of the Society's *Transactions*. The object in both cases was to secure a systematic verification of the results.

First from e, we may find e^2 by the formula

$$e^2 = e \,.\, e \;\; = e\left\{1 + \frac{1}{1} + \frac{1}{1\,.\,2} + \frac{1}{1\,.\,2\,.\,3} + \&c.\right\},$$

also

$$1 = e \,.\, e^{-1} = e\left\{1 - \frac{1}{1} + \frac{1}{1\,.\,2} - \frac{1}{1\,.\,2\,.\,3} + \&c.\right\}.$$

Hence, if M denote the sum of the odd terms and N that of the even terms of the above series for e^2, we have

$$e^2 = M + N,$$

also

$$1 = M - N,$$

which sufficiently tests the accuracy of both M and N and therefore that of e^2.

Similarly e^3 may be found with verification from e^2.

For

$$e^3 = e^2 \,.\, e \;\; = e^2\left(1 + \frac{1}{1} + \frac{1}{1\,.\,2} + \frac{1}{1\,.\,2\,.\,3} + \&c.\right),$$

also

$$e \;\; = e^3 \,.\, e^{-1} = e^2\left(1 - \frac{1}{1} + \frac{1}{1\,.\,2} - \frac{1}{1\,.\,2\,.\,3} + \&c.\right).$$

And if M' denote the sum of the odd terms, and N' the sum of the even terms of the series for e^3, we have

$$e^3 = M' + N',$$

and

$$e \;\; = M' - N';$$

and since e is already known, this tests the accuracy of both M' and N' and therefore that of $M' + N'$ or e^3.

After this the intervals are halved, and e^x is obtained when x is taken successively $= 2{\cdot}5$, $1{\cdot}5$, and $0{\cdot}5$ by means of

Digitized by Google

$$e^{\pm\frac{1}{2}} = \left(1 \pm \frac{1}{2} + \frac{1}{2.4} \pm \frac{1}{2.4.6} + \&c.\right),$$

since
$$e^3 . e^{-\frac{1}{2}} = e^{2\cdot5}, \quad e^{2\cdot5} . e^{-\frac{1}{2}} = e^2, \quad e^{2\cdot5} . e^{\frac{1}{2}} = e^3, \quad e^3 . e^{\frac{1}{2}} = e^{3\cdot5},$$
$$e^2 . e^{-\frac{1}{2}} = e^{1\cdot5}, \quad e^{1\cdot5} . e^{-\frac{1}{2}} = e^1, \quad e^{1\cdot5} . e^{\frac{1}{2}} = e^2,$$
$$e^1 . e^{-\frac{1}{2}} = e^{\cdot5},$$

by which all are doubly verified.

Further, by means of

$$e^{\pm\frac{1}{10}} = \left(1 \pm \frac{1}{10} + \frac{1}{10.20} \pm \frac{1}{10.20.30} + \&c.\right),$$

still working downwards from e^3, we may obtain in succession, with verification at every fifth result,

e^x when x is 2·9, 2·8, 2·7, 2·6, 2·5, ... 2, 1·9, 1·8, ... 1, 0·9, 0·8, ... 0·1 and 0.

The table thus formed is called by the author his skeleton table, in which the calculations are carried to 16 places of decimals, but from $x=3$ to $x=2$ it is left as a mere skeleton.

Then between $x=2$ and $x=0$ the entries were filled in, making x vary at each step by ·01. First, all the intervals were halved by means of

$$e^{\pm\cdot05} = \left(1 \pm \frac{1}{20} + \frac{1}{20.40} \pm \frac{1}{20.40.60} + \&c.\right).$$

Thus
$$e^{1\cdot55} = e^2 \left(1 - \frac{1}{20} + \frac{1}{20.40} - \frac{1}{20.40.60} + \&c.\right),$$

and again
$$e^{1\cdot95} = e^{1\cdot9} \left(1 + \frac{1}{20} + \frac{1}{20.40} + \frac{1}{20.40.60} + \&c.\right),$$

and
$$e^{1\cdot85} = e^{1\cdot9} \left(1 - \frac{1}{20} + \frac{1}{20.40} - \frac{1}{20.40.60} + \&c.\right),$$

so that if M'' denote the sum of the odd terms of either of these latter series, and N'' the sum of the even terms, then

$$M'' + N'' = e^{1\cdot95},$$

and
$$M'' - N'' = e^{1\cdot85};$$

and since $e^{1\cdot95}$ has been already calculated independently, this tests the accuracy of both M'' and N'', and therefore that of $M'' - N''$ or of $e^{1\cdot85}$.

In this way may be obtained with perpetual verification, the value of e^x from $x=2$ to $x=1\cdot95$, 1·9, 1·85, &c. ... down to ·1, ·05, 0.

After this, by means of

$$e^{\pm\cdot01} = \left(1 \pm \frac{1}{100} + \frac{1}{100.200} \pm \frac{1}{100.200.300} + \&c.\right),$$

the process is continued from $x=2$, so that x may descend at every step by ·01, and verification is obtained after every fifth entry.

Digitized by Google

Finally the table was enlarged, without any change in the process, so as to make x descend by only ·001 at each step.

Thus, the intervals are first halved by means of

$$e^{\pm \cdot 005} = 1 \pm \frac{1}{200} + \frac{1}{200.400} \pm \frac{1}{200.400.600} + \&c.,$$

which gives
$$e^{1\cdot 995} = e^{2}\left\{1 - \frac{1}{200} + \frac{1}{200.400} - \frac{1}{200.400.600} + \&c.\right\},$$

from which may be derived in succession the values of e^x for $x = 1\cdot985$, $1\cdot975$, &c. ... down to ·005, with verification at every step.

After thus halving the intervals, the table is completed by employing in a similar manner the formula

$$e^{\pm \cdot 001} = 1 \pm \frac{1}{1000} + \frac{1}{1000.2000} \pm \frac{1}{1000.2000.3000} + \&c.,$$

and verification is obtained after every fifth entry in the table.

The table has been thoroughly revised by Professor Adams, who has thus detected several errors which had almost exclusively arisen in copying. With the exception of these, the revision showed that the errors of the table as given by the author rarely exceeded a unit in the last place of decimals.

In the table as printed the figure in the 12th decimal place has been altered when requisite in order to make the table correct to the *nearest* unit in the last decimal place. In one or two cases the figures which would follow immediately after the 12th decimal are so close to 500 that it is not possible to be certain which of two consecutive figures in the 12th place is the nearest to the truth, without carrying the testing calculations beyond 15 places of decimals, which it would not be worth while to do.

SKELETON TABLE OF e^x TO 16 DECIMALS, FROM $x=0$ TO $x=3$.

x	e^x	x	e^x	x	e^x
·1	1·1051 7091 8075 6476	1·1	3·0041 6602 3946 4331	2·1	8·1661 6991 2567 6501
·2	1·2214 0275 8160 1698	1·2	3·3201 1692 2736 5474	2·2	9·0250 1349 9434 1209
·3	1·3498 5880 7576 0031	1·3	3·6692 9666 7619 2442	2·3	9·9741 8245 4814 7208
·4	1·4918 2469 7641 2703	1·4	4·0551 9996 6844 6745	2·4	11·0231 7638 0641 6017
·5	1·6487 2127 0700 1281	1·5	4·4816 8907 0338 0648	2·5	12·1824 9396 0703 4735
·6	1·8221 1880 0390 5090	1·6	4·9530 3242 4395 1148	2·6	13·4637 3803 5001 6904
·7	2·0137 5270 7470 4765	1·7	5·4739 4739 1727 1998	2·7	14·8797 3172 4872 8341
·8	2·2255 4092 8492 4676	1·8	6·0496 4746 4412 9461	2·8	16·4446 4677 1097 0499
·9	2·4596 0311 1156 9496	1·9	6·6858 9444 2279 2694	2·9	18·1741 4536 9443 0610
1·0	2·7182 8182 8459 0452	2·0	7·3890 5609 8930 6502	3·0	20·0855 3692 3187 6677

Digitized by Google

x	e^x	x	e^x	x	e^x	x	e^x
·000	1·	·050	1·0512 7109 6376	·100	1·1051 7091 8076	·150	1·1618 3424 2728
·001	1·0010 0050 0167	·051	1·0523 2289 3283	·101	1·1062 7664 1763	·151	1·1629 9665 8082
·002	1·0020 0200 1334	·052	1·0533 7574 2513	·102	1·1073 8347 1728	·152	1·1641 6023 6432
·003	1·0030 0450 4503	·053	1·0544 2964 5119	·103	1·1084 9140 9276	·153	1·1653 2497 8943
·004	1·0040 0801 0677	·054	1·0554 8460 2155	·104	1·1096 0045 4916	·154	1·1664 9088 6778
·005	1·0050 1252 0859	·055	1·0565 4061 4675	·105	1·1107 1061 0356	·155	1·1676 5796 1105
·006	1·0060 1803 6054	·056	1·0575 9768 3737	·106	1·1118 2187 6507	·156	1·1688 2620 3090
·007	1·0070 2455 7267	·057	1·0586 5581 0395	·107	1·1129 3425 4479	·157	1·1699 9561 3901
·008	1·0080 3208 5504	·058	1·0597 1499 5710	·108	1·1140 4774 5386	·158	1·1711 6619 4708
·009	1·0090 4062 1774	·059	1·0607 7524 0740	·109	1·1151 6235 0341	·159	1·1723 3794 6681
·010	1·0100 5016 7084	·060	1·0618 3654 6545	·110	1·1162 7807 0459	·160	1·1735 1087 0992
·011	1·0110 6072 2445	·061	1·0628 9891 4187	·111	1·1173 9490 6854	·161	1·1746 8496 8814
·012	1·0120 7228 8866	·062	1·0639 6234 4728	·112	1·1185 1286 0645	·162	1·1758 6024 1321
·013	1·0130 8486 7360	·063	1·0650 2683 9231	·113	1·1196 3193 2949	·163	1·1770 3668 9688
·014	1·0140 9845 8938	·064	1·0660 9239 8762	·114	1·1207 5212 4884	·164	1·1782 1431 5093
·015	1·0151 1306 4615	·065	1·0671 5902 4384	·115	1·1218 7343 7572	·165	1·1793 9311 8711
·016	1·0161 2868 5406	·066	1·0682 2671 7166	·116	1·1229 9587 2133	·166	1·1805 7310 1723
·017	1·0171 4532 2325	·067	1·0692 9547 8175	·117	1·1241 1942 9691	·167	1·1817 5426 5308
·018	1·0181 6297 6390	·068	1·0703 6530 8479	·118	1·1252 4411 1367	·168	1·1829 3661 0648
·019	1·0191 8164 8617	·069	1·0714 3620 9148	·119	1·1263 6991 8288	·169	1·1841 2013 8924
·020	1·0202 0134 0027	·070	1·0725 0818 1254	·120	1·1274 9685 1579	·170	1·1853 0485 1320
·021	1·0212 2205 1638	·071	1·0735 8122 5868	·121	1·1286 2491 2367	·171	1·1864 9074 9022
·022	1·0222 4378 4470	·072	1·0746 5534 4064	·122	1·1297 5410 1780	·172	1·1876 7783 3214
·023	1·0232 6653 9547	·073	1·0757 3053 6915	·123	1·1308 8442 0947	·173	1·1888 6610 5084
·024	1·0242 9031 7891	·074	1·0768 0680 5496	·124	1·1320 1587 0999	·174	1·1900 5556 5820
·025	1·0253 1512 0524	·075	1·0778 8415 0885	·125	1·1331 4845 3067	·175	1·1912 4621 6612
·026	1·0263 4094 8473	·076	1·0789 6257 4157	·126	1·1342 8216 8283	·176	1·1924 3805 8651
·027	1·0273 6780 2763	·077	1·0800 4207 6393	·127	1·1354 1701 7781	·177	1·1936 3109 3127
·028	1·0283 9568 4421	·078	1·0811 2265 8670	·128	1·1365 5300 2697	·178	1·1948 2532 1235
·029	1·0294 2459 4475	·079	1·0822 0432 2070	·129	1·1376 9012 4166	·179	1·1960 2074 4168
·030	1·0304 5453 3954	·080	1·0832 8706 7675	·130	1·1388 2838 3325	·180	1·1972 1736 3122
·031	1·0314 8550 3887	·081	1·0843 7089 6567	·131	1·1399 6778 1312	·181	1·1984 1517 9293
·032	1·0325 1750 5305	·082	1·0854 5580 9830	·132	1·1411 0831 9267	·182	1·1996 1419 3880
·033	1·0335 5053 9241	·083	1·0865 4180 8548	·133	1·1422 4999 8331	·183	1·2008 1440 8081
·034	1·0345 8460 6728	·084	1·0876 2889 3809	·134	1·1433 9281 9645	·184	1·2020 1582 3096
·035	1·0356 1970 8800	·085	1·0887 1706 6698	·135	1·1445 3678 4351	·185	1·2032 1844 0128
·036	1·0366 5584 6491	·086	1·0898 0632 8305	·136	1·1456 8189 3595	·186	1·2044 2226 0378
·037	1·0376 9302 0838	·087	1·0908 9667 9718	·137	1·1468 2814 8520	·187	1·2056 2728 5050
·038	1·0387 3123 2878	·088	1·0919 8812 2028	·138	1·1479 7555 0274	·188	1·2068 3351 5350
·039	1·0397 7048 3650	·089	1·0930 8065 6326	·139	1·1491 2410 0004	·189	1·2080 4095 2483
·040	1·0408 1077 4192	·090	1·0941 7428 3705	·140	1·1502 7379 8857	·190	1·2092 4959 7657
·041	1·0418 5210 5545	·091	1·0952 6900 5258	·141	1·1514 2464 7985	·191	1·2104 5945 2081
·042	1·0428 9447 8751	·092	1·0963 6482 2081	·142	1·1525 7664 8537	·192	1·2116 7051 6965
·043	1·0439 3789 4851	·093	1·0974 6173 5268	·143	1·1537 2980 1666	·193	1·2128 8279 3519
·044	1·0449 8235 4888	·094	1·0985 5974 5917	·144	1·1548 8410 8525	·194	1·2140 9628 2956
·045	1·0460 2785 9909	·095	1·0996 5885 5126	·145	1·1560 3957 0268	·195	1·2153 1098 6490
·046	1·0470 7441 0957	·096	1·1007 5906 3994	·146	1·1571 9618 8051	·196	1·2165 2690 5334
·047	1·0481 2200 9080	·097	1·1018 6037 3621	·147	1·1583 5396 3030	·197	1·2177 4404 0706
·048	1·0491 7065 5324	·098	1·1029 6278 5109	·148	1·1595 1289 6363	·198	1·2189 6239 3822
·049	1·0502 2035 0740	·099	1·1040 6629 9559	·149	1·1606 7298 9209	·199	1·2201 8196 5900

Digitized by Google

x	e^x	x	e^x	x	e^x	x	e^x
·200	1·2214 0275 8160	·250	1·2840 2541 6688	·300	1·3498 5880 7576	·350	1·4190 6754 8593
·201	1·2226 2477 1823	·251	1·2853 1008 4331	·301	1·3512 0934 1538	·351	1·4204 8732 5912
·202	1·2238 4800 8111	·252	1·2865 9603 7285	·302	1·3525 6122 6709	·352	1·4219 0852 3719
·203	1·2250 7246 8247	·253	1·2878 8327 6835	·303	1·3539 1446 4442	·353	1·4233 3114 3434
·204	1·2262 9815 3456	·254	1·2891 7180 4268	·304	1·3552 6905 6090	·354	1·4247 5518 6480
·205	1·2275 2506 4963	·255	1·2904 6162 0873	·305	1·3566 2500 3006	·355	1·4261 8065 4281
·206	1·2287 5320 3995	·256	1·2917 5272 7940	·306	1·3579 8230 6548	·356	1·4276 0754 8264
·207	1·2299 8257 1781	·257	1·2930 4512 6759	·307	1·3593 4096 8072	·357	1·4290 3586 9854
·208	1·2312 1316 9549	·258	1·2943 3881 8624	·308	1·3607 0098 8937	·358	1·4304 6562 0480
·209	1·2324 4499 8530	·259	1·2956 3380 4828	·309	1·3620 6237 0503	·359	1·4318 9680 1572
·210	1·2336 7805 9957	·260	1·2969 3008 6666	·310	1·3634 2511 4132	·360	1·4333 2941 4560
·211	1·2349 1235 5061	·261	1·2982 2766 5434	·311	1·3647 8922 1186	·361	1·4347 6346 0879
·212	1·2361 4788 5079	·262	1·2995 2654 2429	·312	1·3661 5469 3029	·362	1·4361 9894 1960
·213	1·2373 8465 1244	·263	1·3008 2671 8952	·313	1·3675 2153 1028	·363	1·4376 3585 9241
·214	1·2386 2265 4793	·264	1·3021 2819 6301	·314	1·3688 8973 6547	·364	1·4390 7421 4158
·215	1·2398 6189 6966	·265	1·3034 3097 5778	·315	1·3702 5931 0957	·365	1·4405 1400 8149
·216	1·2411 0237 9001	·266	1·3047 3505 8687	·316	1·3716 3025 5626	·366	1·4419 5524 2655
·217	1·2423 4410 2138	·267	1·3060 4044 6331	·317	1·3730 0257 1925	·367	1·4433 9791 9115
·218	1·2435 8706 7619	·268	1·3073 4714 0015	·318	1·3743 7626 1228	·368	1·4448 4203 8974
·219	1·2448 3127 6688	·269	1·3086 5514 1046	·319	1·3757 5132 4906	·369	1·4462 8760 3675
·220	1·2460 7673 0587	·270	1·3099 6445 0733	·320	1·3771 2776 4336	·370	1·4477 3461 4663
·221	1·2473 2343 0564	·271	1·3112 7507 0385	·321	1·3785 0558 0894	·371	1·4491 8307 3387
·222	1·2485 7137 7864	·272	1·3125 8700 1311	·322	1·3798 8477 5957	·372	1·4506 3298 1293
·223	1·2498 2057 3736	·273	1·3139 0024 4825	·323	1·3812 6535 0906	·373	1·4520 8433 9833
·224	1·2510 7101 9428	·274	1·3152 1480 2239	·324	1·3826 4730 7119	·374	1·4535 3715 0457
·225	1·2523 2271 6192	·275	1·3165 3067 4868	·325	1·3840 3064 5981	·375	1·4549 9141 4618
·226	1·2535 7566 5278	·276	1·3178 4786 4027	·326	1·3854 1536 8873	·376	1·4564 4713 3771
·227	1·2548 2986 7940	·277	1·3191 6637 1035	·327	1·3868 0147 7180	·377	1·4579 0430 9371
·228	1·2560 8532 5432	·278	1·3204 8619 7209	·328	1·3881 8897 2289	·378	1·4593 6294 2876
·229	1·2573 4203 9010	·279	1·3218 0734 3870	·329	1·3895 7785 5588	·379	1·4608 2303 5743
·230	1·2586 0000 9929	·280	1·3231 2981 2337	·330	1·3909 6812 8464	·380	1·4622 8458 9434
·231	1·2598 5923 9449	·281	1·3244 5360 3935	·331	1·3923 5979 2308	·381	1·4637 4760 5410
·232	1·2611 1972 8828	·282	1·3257 7871 9987	·332	1·3937 5284 8513	·382	1·4652 1208 5133
·233	1·2623 8147 9327	·283	1·3271 0516 1817	·333	1·3951 4729 8470	·383	1·4666 7803 0068
·234	1·2636 4449 2208	·284	1·3284 3293 0753	·334	1·3965 4314 3575	·384	1·4681 4544 1682
·235	1·2649 0876 8733	·285	1·3297 6202 8121	·335	1·3979 4038 5222	·385	1·4696 1432 1441
·236	1·2661 7431 0167	·286	1·3310 9245 5252	·336	1·3993 3902 4811	·386	1·4710 8467 0815
·237	1·2674 4111 7775	·287	1·3324 2421 3476	·337	1·4007 3906 3739	·387	1·4725 5649 1273
·238	1·2687 0919 2825	·288	1·3337 5730 4123	·338	1·4021 4050 3405	·388	1·4740 2978 4288
·239	1·2699 7853 6584	·289	1·3350 9172 8529	·339	1·4035 4334 5213	·389	1·4755 0455 1333
·240	1·2712 4915 0321	·290	1·3364 2748 8025	·340	1·4049 4759 0564	·390	1·4769 8079 3883
·241	1·2725 2103 5308	·291	1·3377 6458 3950	·341	1·4063 5324 0862	·391	1·4784 5851 3413
·242	1·2737 9419 2816	·292	1·3391 0301 7639	·342	1·4077 6029 7514	·392	1·4799 3771 1402
·243	1·2750 6862 4118	·293	1·3404 4279 0432	·343	1·4091 6876 1926	·393	1·4814 1838 9329
·244	1·2763 4433 0489	·294	1·3417 8390 3667	·344	1·4105 7863 5508	·394	1·4829 0054 8675
·245	1·2776 2131 3205	·295	1·3431 2635 8686	·345	1·4119 8991 9668	·395	1·4843 8419 0921
·246	1·2788 9957 3542	·296	1·3444 7015 6832	·346	1·4134 0261 5818	·396	1·4858 6931 7551
·247	1·2801 7911 2778	·297	1·3458 1529 9448	·347	1·4148 1672 5370	·397	1·4873 5593 0051
·248	1·2814 5993 2194	·298	1·3471 6178 7880	·348	1·4162 3224 9740	·398	1·4888 4402 9907
·249	1·2827 4203 3070	·299	1·3485 0962 3473	·349	1·4176 4919 0342	·399	1·4903 3361 8607

Digitized by Google

x	e^x	x	e^x	x	e^x	x	e^x
·400	1·4918 2469 7641	·450	1·5683 1218 5490	·500	1·6487 2127 0700	·550	1·7332 5301 7867
·401	1·4933 1726 8500	·451	1·5698 8128 2093	·501	1·6503 7081 6606	·551	1·7349 8713 7801
·402	1·4948 1133 2676	·452	1·5714 5194 8578	·502	1·6520 2201 2883	·552	1·7367 2299 2721
·403	1·4963 0689 1664	·453	1·5730 2418 6514	·503	1·6536 7486 1183	·553	1·7384 6058 4365
·404	1·4978 0394 6958	·454	1·5745 9799 7475	·504	1·6553 2936 3157	·554	1·7401 9991 4470
·405	1·4993 0250 0057	·455	1·5761 7338 3034	·505	1·6569 8552 0461	·555	1·7419 4098 4774
·406	1·5008 0255 2458	·456	1·5777 5034 4766	·506	1·6586 4333 4750	·556	1·7436 8379 7020
·407	1·5023 0410 5662	·457	1·5793 2888 4249	·507	1·6603 0280 7683	·557	1·7454 2835 2949
·408	1·5038 0716 1170	·458	1·5809 0900 3061	·508	1·6619 6394 0919	·558	1·7471 7465 4307
·409	1·5053 1172 0486	·459	1·5824 9070 2783	·509	1·6636 2673 6119	·559	1·7489 2270 2840
·410	1·5068 1778 5113	·460	1·5840 7398 4994	·510	1·6652 9119 4946	·560	1·7506 7250 0296
·411	1·5083 2535 6558	·461	1·5856 5885 1281	·511	1·6669 5731 9064	·561	1·7524 2404 8424
·412	1·5098 3443 6329	·462	1·5872 4530 3226	·512	1·6686 2511 0140	·562	1·7541 7734 8977
·413	1·5113 4502 5934	·463	1·5888 3334 2416	·513	1·6702 9456 9841	·563	1·7559 3240 3707
·414	1·5128 5712 6884	·464	1·5904 2297 0440	·514	1·6719 6569 9836	·564	1·7576 8921 4370
·415	1·5143 7074 0692	·465	1·5920 1418 8887	·515	1·6736 3850 1798	·565	1·7594 4778 2722
·416	1·5158 8586 8871	·466	1·5936 0699 9348	·516	1·6753 1297 7398	·566	1·7612 0811 0522
·417	1·5174 0251 2935	·467	1·5952 0140 3417	·517	1·6769 8912 8311	·567	1·7629 7019 9530
·418	1·5189 2067 4402	·468	1·5967 9740 2687	·518	1·6786 6695 6213	·568	1·7647 3405 1508
·419	1·5204 4035 4790	·469	1·5983 9499 8755	·519	1·6803 4646 2783	·569	1·7664 9966 8221
·420	1·5219 6155 5619	·470	1·5999 9419 3217	·520	1·6820 2764 9699	·570	1·7682 6705 1434
·421	1·5234 8427 8409	·471	1·6015 9498 7674	·521	1·6837 1051 8643	·571	1·7700 3620 2913
·422	1·5250 0852 4683	·472	1·6031 9738 3727	·522	1·6853 9507 1297	·572	1·7718 0712 4430
·423	1·5265 3429 5966	·473	1·6048 0138 2976	·523	1·6870 8130 9347	·573	1·7735 7981 7753
·424	1·5280 6159 3784	·474	1·6064 0698 7027	·524	1·6887 6923 4478	·574	1·7753 5428 4656
·425	1·5295 9041 9663	·475	1·6080 1419 7486	·525	1·6904 5884 8379	·575	1·7771 3052 6914
·426	1·5311 2077 5133	·476	1·6096 2301 5958	·526	1·6921 5015 2739	·576	1·7789 0854 6302
·427	1·5326 5266 1724	·477	1·6112 3344 4054	·527	1·6938 4314 9249	·577	1·7806 8834 4600
·428	1·5341 8608 0968	·478	1·6128 4548 3384	·528	1·6955 3783 9602	·578	1·7824 6992 3585
·429	1·5357 2103 4397	·479	1·6144 5913 5559	·529	1·6972 3422 5493	·579	1·7842 5328 5041
·430	1·5372 5752 3548	·480	1·6160 7440 2193	·530	1·6989 3230 8619	·580	1·7860 3843 0750
·431	1·5387 9554 9957	·481	1·6176 9128 4902	·531	1·7006 3209 0677	·581	1·7878 2536 2498
·432	1·5403 3511 5161	·482	1·6193 0978 5302	·532	1·7023 3357 3367	·582	1·7896 1408 2071
·433	1·5418 7622 0701	·483	1·6209 2990 5012	·533	1·7040 3675 8391	·583	1·7914 0459 1258
·434	1·5434 1886 8116	·484	1·6225 5164 5652	·534	1·7057 4164 7452	·584	1·7931 9689 1851
·435	1·5449 6305 8951	·485	1·6241 7500 8844	·535	1·7074 4824 2254	·585	1·7949 9098 5640
·436	1·5465 0879 4749	·486	1·6257 9999 6211	·536	2·7091 5654 4505	·586	1·7967 8687 4420
·437	1·5480 5607 7056	·487	1·6274 2660 9379	·537	1·7108 6655 5913	·587	1·7985 8455 9988
·438	1·5496 0490 7420	·488	1·6290 5484 9973	·538	1·7125 7827 8187	·588	1·8003 8404 4140
·439	1·5511 5528 7388	·489	1·6306 8471 9622	·539	1·7142 9171 3040	·589	1·8021 8532 8676
·440	1·5527 0721 8511	·490	1·6323 1621 9955	·540	1·7160 0686 2185	·590	1·8039 8841 5398
·441	1·5542 6070 2342	·491	1·6339 4935 2606	·541	1·7177 2372 7337	·591	1·8057 9330 6108
·442	1·5558 1574 0434	·492	1·6355 8411 9205	·542	1·7194 4231 0212	·592	1·8076 0000 2612
·443	1·5573 7233 4342	·493	1·6372 2052 1389	·543	1·7211 6261 2530	·593	1·8094 0850 6716
·444	1·5589 3048 5622	·494	1·6388 5856 0794	·544	1·7228 8463 6011	·594	1·8112 1882 0229
·445	1·5604 9019 5833	·495	1·6404 9823 9057	·545	1·7246 0838 2376	·595	1·8130 3094 4960
·446	1·5620 5146 6534	·496	1·6421 3955 7819	·546	1·7263 3385 3351	·596	1·8148 4488 2723
·447	1·5636 1429 9286	·497	1·6437 8251 8720	·547	1·7280 6105 0659	·597	1·8166 6063 5331
·448	1·5651 7869 5654	·498	1·6454 2712 3404	·548	1·7297 8997 6028	·598	1·8184 7820 4599
·449	1·5667 4465 7199	·499	1·6470 7337 3515	·549	1·7315 2063 1187	·599	1·8202 9759 2346

Digitized by Google

x	e^x	x	e^x	x	e^x	x	e^x
·600	1·8221 1880 0391	·650	1·9155 4082 9014	·700	2·0137 5270 7470	·750	2·1170 0001 6613
·601	1·8239 4183 0554	·651	1·9174 5732 7933	·701	2·0157 6746 7390	·751	2·1191 1807 5482
·602	1·8257 6668 4660	·652	1·9193 7574 4309	·702	2·0177 8424 3077	·752	2·1212 3825 3470
·603	1·8275 9336 4532	·653	1·9212 9608 0061	·703	2·0198 0303 6549	·753	2·1233 6055 2696
·604	1·8294 2187 1998	·654	1·9232 1833 7109	·704	2·0218 2384 9824	·754	2·1254 8497 5283
·605	1·8312 5220 8886	·655	1·9251 4251 7376	·705	2·0238 4668 4922	·755	2·1276 1152 3355
·606	1·8330 8437 7026	·656	1·9270 6862 2786	·706	2·0258 7154 3868	·756	2·1297 4019 9039
·607	1·8349 1837 8251	·657	1·9289 9665 5264	·707	2·0278 9842 8685	·757	2·1318 7100 4463
·608	1·8367 5421 4394	·658	1·9309 2661 6739	·708	2·0299 2734 1401	·758	2·1340 0394 1759
·609	1·8385 9188 7292	·659	1·9328 5850 9141	·709	2·0319 5828 4045	·759	2·1361 3901 3058
·610	1·8404 3139 8782	·660	1·9347 9233 4402	·710	2·0339 9125 8647	·760	2·1382 7622 0497
·611	1·8422 7275 0703	·661	1·9367 2809 4455	·711	2·0360 2626 7240	·761	2·1404 1556 6212
·612	1·8441 1594 4897	·662	1·9386 6579 1237	·712	2·0380 6331 1860	·762	2·1425 5705 2343
·613	1·8459 6098 3207	·663	1·9406 0542 6684	·713	2·0401 0239 4543	·763	2·1447 0068 1031
·614	1·8478 0786 7479	·664	1·9425 4700 2737	·714	2·0421 4351 7329	·764	2·1468 4645 4420
·615	1·8496 5659 9558	·665	1·9444 9052 1337	·715	2·0441 8668 2259	·765	2·1489 9437 4655
·616	1·8515 0718 1295	·666	1·9464 3598 4428	·716	2·0462 3189 1375	·766	2·1511 4444 3885
·617	1·8533 5961 4538	·667	1·9483 8339 3954	·717	2·0482 7914 6723	·767	2·1532 9666 4260
·618	1·8552 1390 1141	·668	1·9503 3275 1865	·718	2·0503 2845 0351	·768	2·1554 5103 7932
·619	1·8570 7004 2959	·669	1·9522 8406 0108	·719	2·0523 7980 4308	·769	2·1576 0756 7054
·620	1·8589 2804 1846	·670	1·9542 3732 0636	·720	2·0544 3321 0644	·770	2·1597 6625 3785
·621	1·8607 8789 9662	·671	1·9561 9253 5401	·721	2·0564 8867 1414	·771	2·1619 2710 0282
·622	1·8626 4961 8266	·672	1·9581 4970 6359	·722	2·0585 4618 8672	·772	2·1640 9010 8706
·623	1·8645 1319 9520	·673	1·9601 0883 5466	·723	2·0606 0576 4477	·773	2·1662 5528 1221
·624	1·8663 7864 5286	·674	1·9620 6992 4683	·724	2·0626 6740 0888	·774	2·1684 2261 9991
·625	1·8682 4595 7432	·675	1·9640 3297 5970	·725	2·0647 3109 9966	·775	2·1705 9212 7183
·626	1·8701 1513 7824	·676	1·9659 9799 1290	·726	2·0667 9686 3776	·776	2·1727 6380 4969
·627	1·8719 8618 8331	·677	1·9679 6497 2608	·727	2·0688 6469 4383	·777	2·1749 3765 5518
·628	1·8738 5911 0825	·678	1·9699 3392 1891	·728	2·0709 3459 3855	·778	2·1771 1368 1005
·629	1·8757 3390 7178	·679	1·9719 0484 1108	·729	2·0730 0656 4261	·779	2·1792 9188 3605
·630	1·8776 1057 9264	·680	1·9738 7773 2230	·730	2·0750 8060 7674	·780	2·1814 7226 5498
·631	1·8794 8912 8962	·681	1·9758 5259 7231	·731	2·0771 5672 6168	·781	2·1836 5482 8864
·632	1·8813 6955 8149	·682	1·9778 2943 8084	·732	2·0792 3492 1819	·782	2·1858 3957 5884
·633	1·8832 5186 8705	·683	1·9798 0825 6766	·733	2·0813 1519 6705	·783	2·1880 2650 8744
·634	1·8851 3606 2514	·684	1·9817 8905 5257	·734	2·0833 9755 2906	·784	2·1902 1562 9631
·635	1·8870 2214 1459	·685	1·9837 7183 5537	·735	2·0854 8199 2505	·785	2·1924 0694 0733
·636	1·8889 1010 7426	·686	1·9857 5659 9589	·736	2·0875 6851 7586	·786	2·1946 0044 4243
·637	1·8907 9996 2303	·687	1·9877 4334 9398	·737	2·0896 5713 0236	·787	2·1967 9614 2353
·638	1·8926 9170 7981	·688	1·9897 3208 6951	·738	2·0917 4783 2543	·788	2·1989 9403 7260
·639	1·8945 8534 6350	·689	1·9917 2281 4235	·739	2·0938 4062 6598	·789	2·2011 9413 1161
·640	1·8964 8087 9305	·690	1·9937 1553 3243	·740	2·0959 3551 4494	·790	2·2033 9642 6256
·641	1·8983 7830 8741	·691	1·9957 1024 5966	·741	2·0980 3249 8326	·791	2·2056 0092 4748
·642	1·9002 7763 6555	·692	1·9977 0695 4400	·742	2·1001 3158 0190	·792	2·2078 0762 8841
·643	1·9021 7886 4647	·693	1·9997 0566 0541	·743	2·1022 3276 2186	·793	2·2100 1654 0742
·644	1·9040 8199 4919	·694	2·0017 0636 6388	·744	2·1043 3604 6415	·794	2·2122 2766 2659
·645	1·9059 8702 9272	·695	2·0037 0907 3941	·745	2·1064 4143 4981	·795	2·2144 4099 6804
·646	1·9078 9396 9612	·696	2·0057 1378 5204	·746	2·1085 4892 9988	·796	2·2166 5654 5391
·647	1·9098 0281 7847	·697	2·0077 2050 2180	·747	2·1106 5853 3544	·797	2·2188 7431 0634
·648	1·9117 1357 5885	·698	2·0097 2922 6877	·748	2·1127 7024 7758	·798	2·2210 9429 4751
·649	1·9136 2624 5636	·699	2·0117 3996 1304	·749	2·1148 8407 4743	·799	2·2233 1649 9964

Digitized by Google

x	e^x	x	e^x	x	e^x	x	e^x
·800	2·2255 4092 8492	·850	2·3396 4685 1926	·900	2·4596 0311 1157	·950	2·5857 0965 9316
·801	2·2277 6758 2562	·851	2·3419 8766 8991	·901	2·4620 6394 4480	·951	2·5882 9666 2261
·802	2·2299 9646 4400	·852	2·3443 3082 8045	·902	2·4645 2723 9867	·952	2·5908 8625 3503
·803	2·2322 2757 6235	·853	2·3466 7633 1429	·903	2·4669 9299 9781	·953	2·5934 7843 5632
·804	2·2344 6092 0297	·854	2·3490 2418 1490	·904	2·4694 6122 6688	·954	2·5960 7321 1239
·805	2·2366 9649 8820	·855	2·3513 7438 0575	·905	2·4719 3192 3057	·955	2·5986 7058 2920
·806	2·2389 3431 4040	·856	2·3537 2693 1035	·906	2·4744 0509 1359	·956	2·6012 7055 3271
·807	2·2411 7436 8194	·857	2·3560 8183 5222	·907	2·4768 8073 4065	·957	2·6038 7312 4893
·808	2·2434 1666 3523	·858	2·3584 3909 5490	·908	2·4793 5885 3652	·958	2·6064 7830 0389
·809	2·2456 6120 2269	·859	2·3607 9871 4199	·909	2·4818 3945 2599	·959	2·6090 8608 2363
·810	2·2479 0798 6676	·860	2·3631 6069 3706	·910	2·4843 2253 3385	·960	2·6116 9647 3423
·811	2·2501 5701 8992	·861	2·3655 2503 6374	·911	2·4868 0809 8494	·961	2·6143 0947 6180
·812	2·2524 0830 1465	·862	2·3678 9174 4567	·912	2·4892 9615 0411	·962	2·6169 2509 3247
·813	2·2546 6183 6346	·863	2·3702 6082 0652	·913	2·4917 8669 1624	·963	2·6195 4332 7239
·814	2·2569 1762 5889	·864	2·3726 3226 6998	·914	2·4942 7972 4625	·964	2·6221 6418 0775
·815	2·2591 7567 2350	·865	2·3750 0608 5977	·915	2·4967 7525 1905	·965	2·6247 8765 6475
·816	2·2614 3597 7987	·866	2·3773 8227 9962	·916	2·4992 7327 5961	·966	2·6274 1375 6962
·817	2·2636 9854 5059	·867	2·3797 6085 1329	·917	2·5017 7379 9290	·967	2·6300 4248 4864
·818	2·2659 6337 5831	·868	2·3821 4180 2458	·918	2·5042 7682 4393	·968	2·6326 7384 2809
·819	2·2682 3047 2566	·869	2·3845 2513 5728	·919	2·5067 8235 3773	·969	2·6353 0783 3427
·820	2·2704 9983 7532	·870	2·3869 1085 3524	·920	2·5092 9038 9936	·970	2·6379 4445 9354
·821	2·2727 7147 2998	·871	2·3892 9895 8231	·921	2·5118 0093 5390	·971	2·6405 8372 3226
·822	2·2750 4538 1236	·872	2·3916 8945 2237	·922	2·5143 1399 2644	·972	2·6432 2562 7681
·823	2·2773 2156 4519	·873	2·3940 8233 7933	·923	2·5168 2956 4213	·973	2·6458 7017 5362
·824	2·2796 0002 5124	·874	2·3964 7761 7711	·924	2·5193 4765 2612	·974	2·6485 1736 8913
·825	2·2818 8076 5329	·875	2·3988 7529 3967	·925	2·5218 6826 0358	·975	2·6511 6721 0983
·826	2·2841 6378 7415	·876	2·4012 7536 9099	·926	2·5243 9138 9973	·976	2·6538 1970 4219
·827	2·2864 4909 3666	·877	2·4036 7784 5506	·927	2·5269 1704 3980	·977	2·6564 7485 1276
·828	2·2887 3668 6365	·878	2·4060 8272 5591	·928	2·5294 4522 4903	·978	2·6591 3265 4807
·829	2·2910 2656 7801	·879	2·4084 9001 1759	·929	2·5319 7593 5273	·979	2·6617 9311 7472
·830	2·2933 1874 0264	·880	2·4108 9970 6417	·930	2·5345 0917 7618	·980	2·6644 5624 1929
·831	2·2956 1320 6046	·881	2·4133 1181 1975	·931	2·5370 4495 4473	·981	2·6671 2203 0844
·832	2·2979 0996 7441	·882	2·4157 2633 0846	·932	2·5395 8326 8372	·982	2·6697 9048 6880
·833	2·3002 0902 6747	·883	2·4181 4326 5442	·933	2·5421 2412 1856	·983	2·6724 6161 2707
·834	2·3025 1038 6262	·884	2·4205 6261 8183	·934	2·5446 6751 7464	·984	2·6751 3541 0996
·835	2·3048 1404 8287	·885	2·4229 8439 1486	·935	2·5472 1345 7739	·985	2·6778 1188 4421
·836	2·3071 2001 5127	·886	2·4254 0858 7773	·936	2·5497 6194 5228	·986	2·6804 9103 5658
·837	2·3094 2828 9086	·887	2·4278 3520 9470	·937	2·5523 1298 2479	·987	2·6831 7286 7386
·838	2·3117 3887 2475	·888	2·4302 6425 9001	·938	2·5548 6657 2044	·988	2·6858 5738 2287
·839	2·3140 5176 7602	·889	2·4326 9573 8798	·939	2·5574 2271 6475	·989	2·6885 4458 3046
·840	2·3163 6697 6781	·890	2·4351 2965 1290	·940	2·5599 8141 8329	·990	2·6912 3447 2349
·841	2·3186 8450 2328	·891	2·4375 6599 8912	·941	2·5625 4268 0165	·991	2·6939 2705 2887
·842	2·3210 0434 6559	·892	2·4400 0478 4100	·942	2·5651 0650 4544	·992	2·6966 2232 7353
·843	2·3233 2651 1794	·893	2·4424 4600 9293	·943	2·5676 7289 4029	·993	2·6993 2029 8441
·844	2·3256 5100 0357	·894	2·4448 8967 6933	·944	2·5702 4185 1188	·994	2·7020 2096 8850
·845	2·3279 7781 4570	·895	2·4473 3578 9462	·945	2·5728 1337 8588	·995	2·7047 2434 1279
·846	2·3303 0695 6762	·896	2·4497 8434 9328	·946	2·5753 8747 8803	·996	2·7074 3041 8434
·847	2·3326 3842 9261	·897	2·4522 3535 8978	·947	2·5779 6415 4404	·997	2·7101 3920 3019
·848	2·3349 7223 4398	·898	2·4546 8882 0863	·948	2·5805 4340 7971	·998	2·7128 5069 7743
·849	2·3373 0837 4508	·899	2·4571 4473 7438	·949	2·5831 2524 2081	·999	2·7155 6490 5319

Digitized by Google

x	e^x	x	e^x	x	e^x	x	e^x
1·000	2·7182 8182 8459	1·050	2·8576 5111 8063	1·100	3·0041 6602 3946	1·150	3·1581 9290 9690
1·001	2·7210 0146 9882	1·051	2·8605 1019 8483	1·101	3·0071 7169 2554	1·151	3·1613 5268 2222
1·002	2·7237 2383 2306	1·052	2·8633 7213 9414	1·102	3·0101 8036 8334	1·152	3·1645 1561 6108
1·003	2·7264 4891 8454	1·053	2·8662 3694 3716	1·103	3·0131 9205 4294	1·153	3·1676 8171 4509
1·004	2·7291 7673 1052	1·054	2·8691 0461 4256	1·104	3·0162 0675 3447	1·154	3·1708 5098 0593
1·005	2·7319 0727 2826	1·055	2·8719 7515 3901	1·105	3·0192 2446 8807	1·155	3·1740 2341 7528
1·006	2·7346 4054 6508	1·056	2·8748 4856 5522	1·106	3·0222 4520 3391	1·156	3·1771 9902 8486
1·007	2·7373 7655 4830	1·057	2·8777 2485 1991	1·107	3·0252 6896 0221	1·157	3·1803 7781 6644
1·008	2·7401 1530 0530	1·058	2·8806 0401 6185	1·108	3·0282 9574 2320	1·158	3·1835 5978 5179
1·009	2·7428 5678 6345	1·059	2·8834 8606 0983	1·109	3·0313 2555 2715	1·159	3·1867 4493 7275
1·010	2·7456 0101 5017	1·060	2·8863 7098 9268	1·110	3·0343 5839 4436	1·160	3·1899 3327 6116
1·011	2·7483 4798 9290	1·061	2·8892 5880 3924	1·111	3·0373 9427 0515	1·161	3·1931 2480 4891
1·012	2·7510 9771 1912	1·062	2·8921 4950 7839	1·112	3·0404 3318 3989	1·162	3·1963 1952 6790
1·013	2·7538 5018 5631	1·063	2·8950 4310 3904	1·113	3·0434 7513 7897	1·163	3·1995 1744 5010
1·014	2·7566 0541 3201	1·064	2·8979 3959 5012	1·114	3·0465 2013 5279	1·164	3·2027 1856 2747
1·015	2·7593 6339 7376	1·065	2·9008 3898 4060	1·115	3·0495 6817 9183	1·165	3·2059 2288 3203
1·016	2·7621 2414 0915	1·066	2·9037 4127 3947	1·116	3·0526 1927 2654	1·166	3·2091 3040 9582
1·017	2·7648 8764 6579	1·067	2·9066 4646 7575	1·117	3·0556 7341 8746	1·167	3·2123 4114 5092
1·018	2·7676 5391 7130	1·068	2·9095 5456 7851	1·118	3·0587 3062 0510	1·168	3·2155 5509 2943
1·019	2·7704 2295 5336	1·069	2·9124 6557 7681	1·119	3·0617 9088 1006	1·169	3·2187 7225 6349
1·020	2·7731 9476 3964	1·070	2·9153 7949 9977	1·120	3·0648 5420 3293	1·170	3·2219 9263 8529
1·021	2·7759 6934 5788	1·071	2·9182 9633 7653	1·121	3·0679 2059 0434	1·171	3·2252 1624 2700
1·022	2·7787 4670 3581	1·072	2·9212 1609 3625	1·122	3·0709 9004 5497	1·172	3·2284 4307 2089
1·023	2·7815 2684 0121	1·073	2·9241 3877 0814	1·123	3·0740 6257 1549	1·173	3·2316 7312 9921
1·024	2·7843 0975 8189	1·074	2·9270 6437 2141	1·124	3·0771 3817 1664	1·174	3·2349 0641 9426
1·025	2·7870 9546 0566	1·075	2·9299 9290 0534	1·125	3·0802 1684 8918	1·175	3·2381 4294 3838
1·026	2·7898 8395 0039	1·076	2·9329 2435 8919	1·126	3·0832 9860 6389	1·176	3·2413 8270 6393
1·027	2·7926 7522 9396	1·077	2·9358 5875 0229	1·127	3·0863 8344 7159	1·177	3·2446 2571 0331
1·028	2·7954 6930 1428	1·078	2·9387 9607 7398	1·128	3·0894 7137 4312	1·178	3·2478 7195 8895
1·029	2·7982 6616 8931	1·079	2·9417 3634 3364	1·129	3·0925 6239 0937	1·179	3·2511 2145 5332
1·030	2·8010 6583 4699	1·080	2·9446 7955 1066	1·130	3·0956 5650 0125	1·180	3·2543 7420 2890
1·031	2·8038 6830 1534	1·081	2·9476 2570 3447	1·131	3·0987 5370 4969	1·181	3·2576 3020 4822
1·032	2·8066 7357 2237	1·082	2·9505 7480 3455	1·132	3·1018 5400 8568	1·182	3·2608 8946 4385
1·033	2·8094 8164 9614	1·083	2·9535 2685 4038	1·133	3·1049 5741 4020	1·183	3·2641 5198 4838
1·034	2·8122 9253 6472	1·084	2·9564 8185 8148	1·134	3·1080 6392 4431	1·184	3·2674 1776 9443
1·035	2·8151 0623 5624	1·085	2·9594 3981 8739	1·135	3·1111 7354 2905	1·185	3·2706 8682 1466
1·036	2·8179 2274 9882	1·086	2·9624 0073 8772	1·136	3·1142 8627 2553	1·186	3·2739 5914 4176
1·037	2·8207 4208 2063	1·087	2·9653 6462 1205	1·137	3·1174 0211 6488	1·187	3·2772 3474 0846
1·038	2·8235 6423 4986	1·088	2·9683 3146 9002	1·138	3·1205 2107 7826	1·188	3·2805 1361 4750
1·039	2·8263 8921 1474	1·089	2·9713 0128 5132	1·139	3·1236 4315 9684	1·189	3·2837 9576 9169
1·040	2·8292 1701 4352	1·090	2·9742 7407 2563	1·140	3·1267 6836 5186	1·190	3·2870 8120 7383
1·041	2·8320 4764 6446	1·091	2·9772 4983 4268	1·141	3·1298 9669 7457	1·191	3·2903 6993 2679
1·042	2·8348 8111 0588	1·092	2·9802 2857 3224	1·142	3·1330 2815 9624	1·192	3·2936 6194 8345
1·043	2·8377 1740 9612	1·093	2·9832 1029 2408	1·143	3·1361 6275 4820	1·193	3·2969 5725 7674
1·044	2·8405 5654 6354	1·094	2·9861 9499 4803	1·144	3·1393 0048 6179	1·194	3·3002 5586 3960
1·045	2·8433 9852 3652	1·095	2·9891 8268 3393	1·145	3·1424 4135 6839	1·195	3·3035 5777 0502
1·046	2·8462 4334 4349	1·096	2·9921 7336 1166	1·146	3·1455 8536 9941	1·196	3·3068 6298 0602
1·047	2·8490 9101 1289	1·097	2·9951 6703 1113	1·147	3·1487 3252 8628	1·197	3·3101 7149 7565
1·048	2·8519 4152 7321	1·098	2·9981 6369 6227	1·148	3·1518 8283 6047	1·198	3·3134 8332 4700
1·049	2·8547 9489 5295	1·099	3·0011 6335 9505	1·149	3·1550 3629 5350	1·199	3·3167 9846 5319

Digitized by Google

x	e^x	x	e^x	x	e^x	x	e^x
1·200	3·3201 1692 2737	1·250	3·4903 4295 7462	1·300	3·6692 9666 7619	1·350	3·8574 2553 0697
1·201	3·3234 3870 0271	1·251	3·4938 3504 6173	1·301	3·6729 6779 9547	1·351	3·8612 8488 5584
1·202	3·3267 6380 1245	1·252	3·4973 3062 8719	1·302	3·6766 4260 4443	1·352	3·8651 4810 1755
1·203	3·3300 9222 8983	1·253	3·5008 2970 8596	1·303	3·6803 2108 5981	1·353	3·8690 1518 3075
1·204	3·3334 2398 6813	1·254	3·5043 3228 9303	1·304	3·6840 0324 7841	1·354	3·8728 8613 3411
1·205	3·3367 5907 8067	1·255	3·5078 3837 4343	1·305	3·6876 8909 3705	1·355	3·8767 6095 6633
1·206	3·3400 9750 6081	1·256	3·5113 4796 7221	1·306	3·6913 7862 7258	1·356	3·8806 3965 6617
1·207	3·3434 3927 4193	1·257	3·5148 6107 1447	1·307	3·6950 7185 2190	1·357	3·8845 2223 7240
1·208	3·3467 8438 5744	1·258	3·5183 7769 0535	1·308	3·6987 6877 2194	1·358	3·8884 0870 2386
1·209	3·3501 3284 4080	1·259	3·5218 9782 8001	1·309	3·7024 6939 0967	1·359	3·8922 9905 5941
1·210	3·3534 8465 2549	1·260	3·5254 2148 7365	1·310	3·7061 7371 2210	1·360	3·8961 9330 1795
1·211	3·3568 3981 4503	1·261	3·5289 4867 2151	1·311	3·7098 8173 9627	1·361	3·9000 9144 3843
1·212	3·3601 9833 3297	1·262	3·5324 7938 5886	1·312	3·7135 9347 6926	1·362	3·9039 9348 5983
1·213	3·3635 6021 2290	1·263	3·5360 1363 2100	1·313	3·7173 0892 7819	1·363	3·9078 9943 2116
1·214	3·3669 2545 4843	1·264	3·5395 5141 4329	1·314	3·7210 2809 6021	1·364	3·9118 0928 6150
1·215	3·3702 9406 4322	1·265	3·5430 9273 6109	1·315	3·7247 5098 5251	1·365	3·9157 2305 1993
1·216	3·3736 6604 4095	1·266	3·5466 3760 0982	1·316	3·7284 7759 9233	1·366	3·9196 4073 3559
1·217	3·3770 4139 7534	1·267	3·5501 8601 2493	1·317	3·7322 0794 1693	1·367	3·9235 6233 4766
1·218	3·3804 2012 8016	1·268	3·5537 3797 4191	1·318	3·7359 4201 6360	1·368	3·9274 8785 9536
1·219	3·3838 0223 8917	1·269	3·5572 9348 9626	1·319	3·7396 7982 6971	1·369	3·9314 1731 1795
1·220	3·3871 8773 3621	1·270	3·5608 5256 2356	1·320	3·7434 2137 7261	1·370	3·9353 5069 5470
1·221	3·3905 7661 5513	1·271	3·5644 1519 5938	1·321	3·7471 6667 0973	1·371	3·9392 8801 4497
1·222	3·3939 6888 7983	1·272	3·5679 8139 3936	1·322	3·7509 1571 1852	1·372	3·9432 2927 2813
1·223	3·3973 6455 4421	1·273	3·5715 5115 9915	1·323	3·7546 6850 3647	1·373	3·9471 7447 4357
1·224	3·4007 6361 8224	1·274	3·5751 2449 7446	1·324	3·7584 2505 0111	1·374	3·9511 2362 3077
1·225	3·4041 6608 2791	1·275	3·5787 0141 0102	1·325	3·7621 8535 5000	1·375	3·9550 7672 2921
1·226	3·4075 7195 1524	1·276	3·5822 8190 1459	1·326	3·7659 4942 2075	1·376	3·9590 3377 7841
1·227	3·4109 8122 7830	1·277	3·5858 6597 5099	1·327	3·7697 1725 5099	1·377	3·9629 9479 1796
1·228	3·4143 9391 5117	1·278	3·5894 5363 4604	1·328	3·7734 8885 7842	1·378	3·9669 5976 8746
1·229	3·4178 1001 6798	1·279	3·5930 4488 3564	1·329	3·7772 6423 4073	1·379	3·9709 2871 2656
1·230	3·4212 2953 6290	1·280	3·5966 3972 5569	1·330	3·7810 4338 7569	1·380	3·9749 0162 7495
1·231	3·4246 5247 7011	1·281	3·6002 3816 4214	1·331	3·7848 2632 2108	1·381	3·9788 7851 7236
1·232	3·4280 7884 2385	1·282	3·6038 4020 3098	1·332	3·7886 1304 1475	1·382	3·9828 5938 5856
1·233	3·4315 0863 5839	1·283	3·6074 4584 5822	1·333	3·7924 0354 9454	1·383	3·9868 4423 7335
1·234	3·4349 4186 0801	1·284	3·6110 5509 5992	1·334	3·7961 9784 9838	1·384	3·9908 3307 5659
1·235	3·4383 7852 0705	1·285	3·6146 6795 7217	1·335	3·7999 9594 6419	1·385	3·9948 2590 4817
1·236	3·4418 1861 8988	1·286	3·6182 8443 3111	1·336	3·8037 9784 2997	1·386	3·9988 2272 8800
1·237	3·4452 6215 9090	1·287	3·6219 0452 7290	1·337	3·8076 0354 3373	1·387	4·0028 2355 1607
1·238	3·4487 0914 4455	1·288	3·6255 2824 3373	1·338	3·8114 1305 1353	1·388	4·0068 2837 7238
1·239	3·4521 5957 8529	1·289	3·6291 5558 4985	1·339	3·8152 2637 0746	1·389	4·0108 3720 9697
1·240	3·4556 1346 4763	1·290	3·6327 8655 5753	1·340	3·8190 4350 5366	1·390	4·0148 5005 2994
1·241	3·4590 7080 6610	1·291	3·6364 2115 9307	1·341	3·8228 6445 9030	1·391	4·0188 6691 1142
1·242	3·4625 3160 7529	1·292	3·6400 5939 9284	1·342	3·8266 8923 5559	1·392	4·0228 8778 8156
1·243	3·4659 9587 0979	1·293	3·6437 0127 9319	1·343	3·8305 1783 8777	1·393	4·0269 1268 8059
1·244	3·4694 6360 0426	1·294	3·6473 4680 3057	1·344	3·8343 5027 2513	1·394	4·0309 4161 4875
1·245	3·4729 3479 9337	1·295	3·6509 9597 4141	1·345	3·8381 8654 0600	1·395	4·0349 7457 2632
1·246	3·4764 0947 1183	1·296	3·6546 4879 6222	1·346	3·8420 2664 6874	1·396	4·0390 1156 5365
1·247	3·4798 8761 9438	1·297	3·6583 0527 2952	1·347	3·8458 7059 5174	1·397	4·0430 5259 7109
1·248	3·4833 6924 7581	1·298	3·6619 6540 7987	1·348	3·8497 1838 9346	1·398	4·0470 9767 1907
1·249	3·4868 5435 9094	1·299	3·6656 2920 4989	1·349	3·8535 7003 3236	1·399	4·0511 4679 3802

Digitized by Google

x	e^x	x	e^x	x	e^x	x	e^x
1·400	4·0551 9996 6845	1·450	4·2631 1451 5169	1·500	4·4816 8907 0338	1·550	4·7114 7018 2591
1·401	4·0592 5719 5088	1·451	4·2673 7976 1952	1·501	4·4861 7300 1000	1·551	4·7161 8400 9294
1·402	4·0633 1848 2588	1·452	4·2716 4927 6115	1·502	4·4906 6141 7835	1·552	4·7209 0255 2181
1·403	4·0673 8383 3407	1·453	4·2759 2306 1928	1·503	4·4951 5432 5333	1·553	4·7256 2581 5972
1·404	4·0714 5325 1611	1·454	4·2802 0112 3664	1·504	4·4996 5172 7985	1·554	4·7303 5380 5389
1·405	4·0755 2674 1268	1·455	4·2844 8346 5602	1·505	4·5041 5363 0288	1·555	4·7350 8652 5159
1·406	4·0796 0430 6452	1·456	4·2887 7009 2024	1·506	4·5086 6003 6746	1·556	4·7398 2398 0017
1·407	4·0836 8595 1240	1·457	4·2930 6100 7216	1·507	4·5131 7095 1865	1·557	4·7445 6617 4699
1·408	4·0877 7167 9715	1·458	4·2973 5621 5469	1·508	4·5176 8638 0155	1·558	4·7493 1311 3948
1·409	4·0918 6149 5962	1·459	4·3016 5572 1079	1·509	4·5222 0632 6131	1·559	4·7540 6480 2510
1·410	4·0959 5540 4071	1·460	4·3059 5952 8345	1·510	4·5267 3079 4314	1·560	4·7588 2124 5138
1·411	4·1000 5340 8136	1·461	4·3102 6764 1571	1·511	4·5312 5978 9229	1·561	4·7635 8244 6587
1·412	4·1041 5551 2254	1·462	4·3145 8006 5065	1·512	4·5357 9331 5403	1·562	4·7683 4841 1619
1·413	4·1082 6172 0528	1·463	4·3188 9680 3139	1·513	4·5403 3137 7371	1·563	4·7731 1914 5000
1·414	4·1123 7203 7065	1·464	4·3232 1786 0111	1·514	4·5448 7397 9671	1·564	4·7778 9465 1500
1·415	4·1164 8646 5973	1·465	4·3275 4324 0301	1·515	4·5494 2112 6846	1·565	4·7826 7493 5895
1·416	4·1206 0501 1369	1·466	4·3318 7294 8034	1·516	4·5539 7282 3441	1·566	4·7874 6000 2966
1·417	4·1247 2767 7370	1·467	4·3362 0698 7641	1·517	4·5585 2907 4011	1·567	4·7922 4985 7497
1·418	4·1288 5446 8098	1·468	4·3405 4536 3455	1·518	4·5630 8988 3109	1·568	4·7970 4450 4278
1·419	4·1329 8538 7682	1·469	4·3448 8807 9815	1·519	4·5676 5525 5298	1·569	4·8018 4394 8105
1·420	4·1371 2044 0251	1·470	4·3492 3514 1063	1·520	4·5722 2519 5142	1·570	4·8066 4819 3775
1·421	4·1412 5962 9942	1·471	4·3535 8655 1546	1·521	4·5767 9970 7212	1·571	4·8114 5724 6094
1·422	4·1454 0296 0892	1·472	4·3579 4231 5617	1·522	4·5813 7879 6082	1·572	4·8162 7110 9871
1·423	4·1495 5043 7245	1·473	4·3623 0243 7630	1·523	4·5859 6246 6331	1·573	4·8210 8978 9920
1·424	4·1537 0206 3149	1·474	4·3666 6692 1946	1·524	4·5905 5072 2543	1·574	4·8259 1329 1058
1·425	4·1578 5784 2756	1·475	4·3710 3577 2930	1·525	4·5951 4356 9307	1·575	4·8307 4161 8110
1·426	4·1620 1778 0221	1·476	4·3754 0899 4949	1·526	4·5997 4101 1214	1·576	4·8355 7477 5905
1·427	4·1661 8187 9704	1·477	4·3797 8659 2378	1·527	4·6043 4305 2862	1·577	4·8404 1276 9274
1·428	4·1703 5014 5369	1·478	4·3841 6856 9594	1·528	4·6089 4969 8854	1·578	4·8452 5560 3057
1·429	4·1745 2258 1385	1·479	4·3885 5493 0979	1·529	4·6135 6095 3796	1·579	4·8501 0328 2095
1·430	4·1786 9919 1923	1·480	4·3929 4568 0919	1·530	4·6181 7682 2300	1·580	4·8549 5581 1237
1·431	4·1828 7998 1161	1·481	4·3973 4082 3805	1·531	4·6227 9730 8980	1·581	4·8598 1319 5336
1·432	4·1870 6495 3280	1·482	4·4017 4036 4032	1·532	4·6274 2241 8459	1·582	4·8646 7543 9248
1·433	4·1912 5411 2464	1·483	4·4061 4430 6000	1·533	4·6320 5215 5360	1·583	4·8695 4254 7836
1·434	4·1954 4746 2902	1·484	4·4105 5265 4113	1·534	4·6366 8652 4313	1·584	4·8744 1452 5967
1·435	4·1996 4500 8788	1·485	4·4149 6541 2779	1·535	4·6413 2552 9954	1·585	4·8792 9137 8513
1·436	4·2038 4675 4319	1·486	4·4193 8258 6410	1·536	4·6459 6917 6920	1·586	4·8841 7311 0350
1·437	4·2080 5270 3698	1·487	4·4238 0417 9425	1·537	4·6506 1746 9856	1·587	4·8890 5972 6361
1·438	4·2122 6286 1129	1·488	4·4282 3019 6244	1·538	4·6552 7041 3410	1·588	4·8939 5123 1433
1·439	4·2164 7723 0824	1·489	4·4326 6064 1293	1·539	4·6599 2801 2235	1·589	4·8988 4763 0456
1·440	4·2206 9581 6997	1·490	4·4370 9551 9004	1·540	4·6645 9027 0988	1·590	4·9037 4892 8327
1·441	4·2249 1862 3865	1·491	4·4415 3483 3810	1·541	4·6692 5719 4332	1·591	4·9086 5512 9947
1·442	4·2291 4565 5653	1·492	4·4459 7859 0152	1·542	4·6739 2878 6933	1·592	4·9135 6624 0223
1·443	4·2333 7691 6586	1·493	4·4504 2679 2472	1·543	4·6786 0505 3464	1·593	4·9184 8226 4065
1·444	4·2376 1241 0897	1·494	4·4548 7944 5220	1·544	4·6832 8599 8600	1·594	4·9234 0320 6390
1·445	4·2418 5214 2820	1·495	4·4593 3655 2848	1·545	4·6879 7162 7022	1·595	4·9283 2907 2119
1·446	4·2460 9611 6596	1·496	4·4637 9811 9812	1·546	4·6926 6194 3416	1·596	4·9332 5986 6177
1·447	4·2503 4433 6469	1·497	4·4682 6415 0575	1·547	4·6973 5695 2473	1·597	4·9381 9559 3496
1·448	4·2545 9680 6686	1·498	4·4727 3464 9603	1·548	4·7020 5665 8887	1·598	4·9431 3625 9010
1·449	4·2588 5353 1501	1·499	4·4772 0962 1366	1·549	4·7067 6106 7358	1·599	4·9480 8186 7662

Digitized by Google

x	e^x	x	e^x	x	e^x	x	e^x
1·600	4·9530 3242 4395	1·650	5·2069 7982 7180	1·700	5·4739 4739 1727	1·750	5·7546 0267 6006
1·601	4·9579 8793 4161	1·651	5·2121 8941 1365	1·701	5·4794 2407 7005	1·751	5·7603 6015 6942
1·602	4·9629 4840 1916	1·652	5·2174 0420 7740	1·702	5·4849 0624 1708	1·752	5·7661 2339 8240
1·603	4·9679 1383 2620	1·653	5·2226 2422 1520	1·703	5·4903 9389 1317	1·753	5·7718 9240 5661
1·604	4·9728 8423 1237	1·654	5·2278 4945 7924	1·704	5·4958 8703 1320	1·754	5·7776 6718 4975
1·605	4·9778 5960 2740	1·655	5·2330 7992 2178	1·705	5·5013 8566 7211	1·755	5·7834 4774 1957
1·606	4·9828 3995 2102	1·656	5·2383 1561 9513	1·706	5·5068 8980 4489	1·756	5·7892 3408 2387
1·607	4·9878 2528 4305	1·657	5·2435 5655 5163	1·707	5·5123 9944 8656	1·757	5·7950 2621 2051
1·608	4·9928 1560 4333	1·658	5·2488 0273 4371	1·708	5·5179 1460 5223	1·758	5·8008 2413 6743
1·609	4·9978 1091 7178	1·659	5·2540 5416 2382	1·709	5·5234 3527 9706	1·759	5·8066 2786 2258
1·610	5·0028 1122 7834	1·660	5·2593 1084 4447	1·710	5·5289 6147 7624	1·760	5·8124 3739 4403
1·611	5·0078 1654 1301	1·661	5·2645 7278 5824	1·711	5·5344 9320 4504	1·761	5·8182 5273 8985
1·612	5·0128 2686 2585	1·662	5·2698 3999 1773	1·712	5·5400 3046 5878	1·762	5·8240 7390 1820
1·613	5·0178 4219 6697	1·663	5·2751 1246 7564	1·713	5·5455 7326 7283	1·763	5·8299 0088 8730
1·614	5·0228 6254 8651	1·664	5·2803 9021 8467	1·714	5·5511 2161 4261	1·764	5·8357 3370 5541
1·615	5·0278 8792 3469	1·665	5·2856 7324 9761	1·715	5·5566 7551 2361	1·765	5·8415 7235 8086
1·616	5·0329 1832 6174	1·666	5·2909 6156 6728	1·716	5·5622 3496 7138	1·766	5·8474 1685 2204
1·617	5·0379 5376 1799	1·667	5·2962 5517 4658	1·717	5·5677 9998 4150	1·767	5·8532 6719 3740
1·618	5·0429 9423 5377	1·668	5·3015 5407 8843	1·718	5·5733 7056 8962	1·768	5·8591 2338 8543
1·619	5·0480 3975 1950	1·669	5·3068 5828 4583	1·719	5·5789 4672 7145	1·769	5·8649 8544 2470
1·620	5·0530 9031 6564	1·670	5·3121 6779 7181	1·720	5·5845 2846 4276	1·770	5·8708 5336 1383
1·621	5·0581 4593 4268	1·671	5·3174 8262 1948	1·721	5·5901 1578 5936	1·771	5·8767 2715 1149
1·622	5·0632 0661 0118	1·672	5·3228 0276 4198	1·722	5·5957 0869 7711	1·772	5·8826 0681 7644
1·623	5·0682 7234 9176	1·673	5·3281 2822 9250	1·723	5·6013 0720 5196	1·773	5·8884 9236 6746
1·624	5·0733 4315 6506	1·674	5·3334 5902 2432	1·724	5·6069 1131 3989	1·774	5·8943 8380 4340
1·625	5·0784 1903 7180	1·675	5·3387 9514 9073	1·725	5·6125 2102 9693	1·775	5·9002 8113 6319
1·626	5·0834 9999 6273	1·676	5·3441 3661 4510	1·726	5·6181 3635 7919	1·776	5·9061 8436 8580
1·627	5·0885 8603 8867	1·677	5·3494 8342 4084	1·727	5·6237 5730 4282	1·777	5·9120 9350 7025
1·628	5·0936 7717 0047	1·678	5·3548 3558 3141	1·728	5·6293 8387 4402	1·778	5·9180 0855 7564
1·629	5·0987 7339 4905	1·679	5·3601 9309 7035	1·729	5·6350 1607 3907	1·779	5·9239 2952 6113
1·630	5·1038 7471 8537	1·680	5·3655 5597 1122	1·730	5·6406 5390 8428	1·780	5·9298 5641 8591
1·631	5·1089 8114 6043	1·681	5·3709 2421 0766	1·731	5·6462 9738 3604	1·781	5·9357 8924 0927
1·632	5·1140 9268 2532	1·682	5·3762 9782 1334	1·732	5·6519 4650 5078	1·782	5·9417 2799 9052
1·633	5·1192 0933 3113	1·683	5·3816 7680 8200	1·733	5·6576 0127 8498	1·783	5·9476 7269 8905
1·634	5·1243 3110 2904	1·684	5·3870 6117 6744	1·734	5·6632 6170 9520	1·784	5·9536 2334 6432
1·635	5·1294 5799 7027	1·685	5·3924 5093 2350	1·735	5·6689 2780 3805	1·785	5·9595 7994 7583
1·636	5·1345 9002 0608	1·686	5·3978 4608 0406	1·736	5·6745 9956 7018	1·786	5·9655 4250 8314
1·637	5·1397 2717 8780	1·687	5·4032 4662 6310	1·737	5·6802 7700 4831	1·787	5·9715 1103 4588
1·638	5·1448 6947 6679	1·688	5·4086 5257 5460	1·738	5·6859 6012 2921	1·788	5·9774 8553 2373
1·639	5·1500 1691 9448	1·689	5·4140 6393 3263	1·739	5·6916 4892 6972	1·789	5·9834 6600 7645
1·640	5·1551 6951 2235	1·690	5·4194 8070 5131	1·740	5·6973 4342 2672	1·790	5·9894 5246 6383
1·641	5·1603 2726 0191	1·691	5·4249 0289 6480	1·741	5·7030 4361 5716	1·791	5·9954 4491 4574
1·642	5·1654 9016 8475	1·692	5·4303 3051 2732	1·742	5·7087 4951 1804	1·792	6·0014 4335 8211
1·643	5·1706 5824 2250	1·693	5·4357 6355 9316	1·743	5·7144 6111 6643	1·793	6·0074 4780 3291
1·644	5·1758 3148 6683	1·694	5·4412 0204 1663	1·744	5·7201 7843 5943	1·794	6·0134 5825 5820
1·645	5·1810 0990 6949	1·695	5·4466 4596 5213	1·745	5·7259 0147 5421	1·795	6·0194 7472 1807
1·646	5·1861 9350 8224	1·696	5·4520 9533 5409	1·746	5·7316 3024 0802	1·796	6·0254 9720 7270
1·647	5·1913 8229 5694	1·697	5·4575 5015 7701	1·747	5·7373 6473 7813	1·797	6·0315 2571 8230
1·648	5·1965 7627 4546	1·698	5·4630 1043 7544	1·748	5·7431 0497 2130	1·798	6·0375 6026 0717
1·649	5·2017 7544 9975	1·699	5·4684 7618 0397	1·749	5·7488 5094 9672	1·799	6·0436 0084 0764

Digitized by Google

x	e^x	x	e^x	x	e^x	x	e^x
1·800	6·0496 4746 4413	1·850	6·3598 1952 2602	1·900	6·6858 9444 2279	1·950	7·0286 8758 0589
1·801	6·0557 0013 7710	1·851	6·3661 8252 3094	1·901	6·6925 8368 0783	1·951	7·0357 1978 3685
1·802	6·0617 5886 6707	1·852	6·3725 5188 9770	1·902	6·6992 7961 1872	1·952	7·0427 5902 2502
1·803	6·0678 2365 7464	1·853	6·3789 2762 8998	1·903	6·7059 8224 2240	1·953	7·0498 0530 4078
1·804	6·0738 9451 6044	1·854	6·3853 0974 7154	1·904	6·7126 9157 8592	1·954	7·0568 5863 5460
1·805	6·0799 7144 8520	1·855	6·3916 9825 0621	1·905	6·7194 0762 7635	1·955	7·0639 1902 3701
1·806	6·0860 5446 0968	1·856	6·3980 9314 5786	1·906	6·7261 3039 6087	1·956	7·0709 8647 5862
1·807	6·0921 4355 9471	1·857	6·4044 9443 9045	1·907	6·7328 5989 0669	1·957	7·0780 6099 9010
1·808	6·0982 3875 0118	1·858	6·4109 0213 6799	1·908	6·7395 9611 8112	1·958	7·0851 4260 0219
1·809	6·1043 4003 9004	1·859	6·4173 1624 5456	1·909	6·7463 3908 5152	1·959	7·0922 3128 6572
1·810	6·1104 4743 2231	1·860	6·4237 3677 1429	1·910	6·7530 8879 8531	1·960	7·0993 2706 5157
1·811	6·1165 6093 5905	1·861	6·4301 6372 1140	1·911	6·7598 4526 5000	1·961	7·1064 2994 3069
1·812	6·1226 8055 6141	1·862	6·4365 9710 1015	1·912	6·7666 0849 1315	1·962	7·1135 3992 7412
1·813	6·1288 0629 9058	1·863	6·4430 3691 7487	1·913	6·7733 7848 4238	1·963	7·1206 5702 5295
1·814	6·1349 3817 0782	1·864	6·4494 8317 6997	1·914	6·7801 5525 0541	1·964	7·1277 8124 3836
1·815	6·1410 7617 7445	1·865	6·4559 3588 5991	1·915	6·7869 3879 6999	1·965	7·1349 1259 0158
1·816	6·1472 2032 5184	1·866	6·4623 9505 0921	1·916	6·7937 2913 0397	1·966	7·1420 5107 1394
1·817	6·1533 7062 0144	1·867	6·4688 6067 8247	1·917	6·8005 2625 7525	1·967	7·1491 9669 4682
1·818	6·1595 2706 8476	1·868	6·4753 3277 4434	1·918	6·8073 3018 5179	1·968	7·1563 4946 7167
1·819	6·1656 8967 6335	1·869	6·4818 1134 5955	1·919	6·8141 4092 0164	1·969	7·1635 0939 6002
1·820	6·1718 5844 9884	1·870	6·4882 9639 9287	1·920	6·8209 5846 9291	1·970	7·1706 7648 8347
1·821	6·1780 3339 5292	1·871	6·4947 8794 0916	1·921	6·8277 8283 9376	1·971	7·1778 5075 1369
1·822	6·1842 1451 8733	1·872	6·5012 8597 7333	1·922	6·8346 1403 7245	1·972	7·1850 3219 2242
1·823	6·1904 0182 6390	1·873	6·5077 9051 5038	1·923	6·8414 5206 9729	1·973	7·1922 2081 8148
1·824	6·1965 9532 4450	1·874	6·5143 0156 0533	1·924	6·8482 9694 3665	1·974	7·1994 1663 6276
1·825	6·2027 9501 9105	1·875	6·5208 1912 0330	1·925	6·8551 4866 5899	1·975	7·2066 1965 3820
1·826	6·2090 0091 6556	1·876	6·5273 4320 0947	1·926	6·8620 0724 3282	1·976	7·2138 2987 7986
1·827	6·2152 1302 3008	1·877	6·5338 7380 8908	1·927	6·8688 7268 2673	1·977	7·2210 4731 5981
1·828	6·2214 3134 4673	1·878	6·5404 1095 0743	1·928	6·8757 4499 0937	1·978	7·2282 7197 5024
1·829	6·2276 5588 7771	1·879	6·5469 5463 2989	1·929	6·8826 2417 4947	1·979	7·2355 0386 2341
1·830	6·2338 8665 8525	1·880	6·5535 0486 2191	1·930	6·8895 1024 1581	1·980	7·2427 4298 5161
1·831	6·2401 2366 3166	1·881	6·5600 6164 4898	1·931	6·8964 0319 7727	1·981	7·2499 8935 0725
1·832	6·2463 6690 7931	1·882	6·5666 2498 7668	1·932	6·9033 0305 0276	1·982	7·2572 4296 6279
1·833	6·2526 1639 9064	1·883	6·5731 9489 7063	1·933	6·9102 0980 6128	1·983	7·2645 0383 9077
1·834	6·2588 7214 2813	1·884	6·5797 7137 9653	1·934	6·9171 2347 2191	1·984	7·2717 7197 6379
1·835	6·2651 3414 5436	1·885	6·5863 5444 2015	1·935	6·9240 4405 5378	1·985	7·2790 4738 5453
1·836	6·2714 0241 3193	1·886	6·5929 4409 0732	1·936	6·9309 7156 2610	1·986	7·2863 3007 3576
1·837	6·2776 7695 2353	1·887	6·5995 4033 2394	1·937	6·9379 0600 0814	1·987	7·2936 2004 8029
1·838	6·2839 5776 9190	1·888	6·6061 4317 3597	1·938	6·9448 4737 6924	1·988	7·3009 1731 6103
1·839	6·2902 4486 9986	1·889	6·6127 5262 0943	1·939	6·9517 9569 7883	1·989	7·3082 2188 5095
1·840	6·2965 3826 1027	1·890	6·6193 6868 1043	1·940	6·9587 5097 0637	1·990	7·3155 3376 2310
1·841	6·3028 3794 8606	1·891	6·6259 9136 0512	1·941	6·9657 1320 2143	1·991	7·3228 5295 5058
1·842	6·3091 4393 9025	1·892	6·6326 2066 5973	1·942	6·9726 8239 9363	1·992	7·3301 7947 0661
1·843	6·3154 5623 8588	1·893	6·6392 5660 4055	1·943	6·9796 5856 9266	1·993	7·3375 1331 6443
1·844	6·3217 7485 3607	1·894	6·6458 9918 1394	1·944	6·9866 4171 8829	1·994	7·3448 5449 9739
1·845	6·3280 9979 0402	1·895	6·6525 4840 4633	1·945	6·9936 3185 5033	1·995	7·3522 0302 7891
1·846	6·3344 3105 5297	1·896	6·6592 0428 0421	1·946	7·0006 2898 4870	1·996	7·3595 5890 8246
1·847	6·3407 6865 4624	1·897	6·6658 6681 5413	1·947	7·0076 3311 5336	1·997	7·3669 2214 8160
1·848	6·3471 1259 4720	1·898	6·6725 3601 6274	1·948	7·0146 4425 3436	1·998	7·3742 9275 4998
1·849	6·3534 6288 1929	1·899	6·6792 1188 9670	1·949	7·0216 6240 6181	1·999	7·3816 7073 6129

2·000 | 7·3890 5609 8930 6502

Digitized by Google

XII. *The Equations of an Isotropic Elastic Solid in Polar and Cylindrical Co-ordinates, their Solution and Application.* By C. CHREE, M.A., Fellow of King's College, Cambridge.

[*Read October* 31, 1887.]

CONTENTS.

SECT.
I. §§ 1—6. Fundamental Equations in Polar Co-ordinates.
II. ,, 7—19. Equilibrium of Solid Sphere or Spherical Shell.
III. ,, 20—22. Tendency to Rupture.
IV. ,, 23—31. Gravitating nearly Spherical Mass.
V. ,, 32—39. Solid Rotating Sphere.
VI. ,, 40—47. Rotating Spherical Shell.
VII. ,, 48—50. Sphere or Spherical Shell with given surface displacements.
VIII. ,, 51—62. Vibrations of Sphere or Spherical Shell.
SECT.
IX. §§ 63—69. Fundamental Equations in Cylindrical Co-ordinates.
X. ,, 70—77. Equilibrium of Cylinder, solution in ascending powers of r and z.
XI. ,, 78—83. Equilibrium of Infinite Cylinder or Cylindrical Shell.
XII. ,, 84—90. Vibrations of Infinite Solid Cylinder or Cylindrical Shell.
XIII. ,, 91—99. Equilibrium of a Finite Cylinder under purely surface forces.

SECTION I.

FUNDAMENTAL EQUATIONS IN POLAR CO-ORDINATES.

§ 1. IN considering the form taken in polar co-ordinates by the equations for the equilibrium or state of vibration of an isotropic elastic solid, it is convenient to deduce the result from the formula for the energy expressed in Cartesian co-ordinates. Using Thomson and Tait's notation, let α, β, γ denote displacements parallel to three fixed rectangular axes, and let

$$\left.\begin{aligned} \frac{d\alpha}{dx} &= e, & \frac{d\beta}{dz} + \frac{d\gamma}{dy} &= a, \\ \frac{d\beta}{dy} &= f, & \frac{d\gamma}{dx} + \frac{d\alpha}{dz} &= b, \\ \frac{d\gamma}{dz} &= g, & \frac{d\alpha}{dy} + \frac{d\beta}{dx} &= c, \\ & & \frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz} &= \delta, \end{aligned}\right\} \quad \text{.....(1).}$$

while

Digitized by Google

If m and n are the elastic constants of a homogeneous isotropic material, the potential energy W_0 per unit volume answering to the displacement α, β, γ is given by

$$2W_0 = (m+n)\,\delta^2 + n\,\{a^2 + b^2 + c^2 - 4\,(ef + eg + fg)\} \quad \text{.....(2)*}.$$

We must transform this expression to polar co-ordinates. Moreover it is most convenient to take the axes not in fixed directions but specially for each point considered.

Thus if r, θ, ϕ denote the co-ordinates of any point P in a sphere referred to the centre O of the sphere as origin, to a fixed plane from which to measure ϕ, and to a fixed axis from which to measure θ, it is best to consider the displacements of P as

u along OP,

v along a perpendicular to OP in that plane which contains OP and the fixed diameter from which θ is measured [this plane we shall call the meridian plane at P],

w in a direction perpendicular to the former two directions, and so perpendicular to the meridian plane.

u, v, and w are of course functions of r, θ, and ϕ, and we have to express e, f, g, a, b, c, δ, W_0 in terms of r, θ, ϕ, u, v, w.

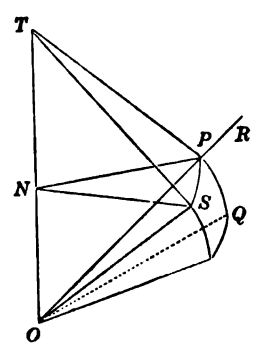
§ 2. To do so we require certain elementary geometrical relations between the directions of the fundamental axes at P and at adjacent points. The following figure shows these at once.

Let P be the point r, θ, ϕ,

Q ..... r, $\theta + \partial\theta$, ϕ,

R ..... $r + \partial r$, θ, ϕ,

S ..... r, θ, $\phi + \partial\phi$.



Let PN, SN be perpendiculars on the axis from which θ is measured, and PT, ST tangents to the meridian sections at P and S; i.e. PT is perpendicular to OP in the meridian plane at P, and similarly for ST. Then

$$\widehat{PNS} = \partial\phi,$$

$$\widehat{POS} = \widehat{PNS}\,\frac{SN}{SO} = \sin\theta\,\partial\phi,$$

$$\widehat{PTS} = \widehat{PNS}\,\frac{SN}{ST} = \cos\theta\,\partial\phi.$$

At R the fundamental axes are parallel to those at P.

At Q $\begin{cases} \text{the axis of } u \text{ makes angle } \partial\theta \text{ with axis of } u \text{ at } P, \\ \text{.....} v \text{ ..... } \partial\theta \text{ ..... } v \text{ .....}, \\ \text{.....} w \text{ is parallel to axis of } w \text{ at } P. \end{cases}$

At S $\begin{cases} \text{the axis of } u \text{ makes angle } \sin\theta\,\partial\phi \text{ with axis of } u \text{ at } P, \\ \text{.....} v \text{ ..... } \cos\theta\,\partial\phi \text{ ..... } v \text{ .....}, \\ \text{.....} w \text{ ..... } \partial\phi \text{ ..... } w \text{ .....} \end{cases}$

* Cf. Thomson and Tait's *Nat. Phil.*, Part II., Equation (7), p. 232, putting $k = m - \frac{1}{3}n$.

Digitized by Google

If now we understand exactly what is meant by $\frac{du}{d\theta}$ and similar differential coefficients, we can deduce at once what we want.

If u_1 denote the displacement at Q along the direction OQ, then $\frac{du}{d\theta} = \text{Lt.} \frac{u_1 - u}{\partial\theta}$, when Q moves up to P.

Suppose however we took the axes of x, y, z, which are fixed in space, to coincide in direction with the axes of u, v, w at P; then $u = \alpha$, but $\frac{d\alpha}{dy}$ is not $= \frac{du}{r d\theta}$. But

$$\frac{d\alpha}{dy} = \text{Lt.} \frac{\text{component of displacement at } Q \textit{ parallel to } OP - u}{PQ},$$

when Q moves up to P;

$$\text{i.e.} \quad \frac{d\alpha}{dy} = \text{Lt.} \frac{\left(u + \frac{du}{d\theta}\partial\theta\right)\cos\partial\theta - \left(v + \frac{dv}{d\theta}\partial\theta\right)\sin\partial\theta - u}{r\partial\theta}, \text{ when } \partial\theta \text{ indefinitely small,}$$

$$= \frac{du}{d\theta} - \frac{v}{r}.$$

The other differentials are to be found exactly in the same way. This method is really that employed in Rigid Dynamics in the case of moving axes, and is practically the same as that given by Mr Webb in the *Messenger of Mathematics*, February, 1882.

We thus easily find $e = \frac{d\alpha}{dx} = \frac{du}{dr}$ ..... (3),

$$f = \frac{d\beta}{dy} = \frac{u}{r} + \frac{1}{r}\frac{dv}{d\theta} \quad \text{.....(4)},$$

$$g = \frac{d\gamma}{dz} = \frac{u}{r} + \frac{v}{r}\cot\theta + \frac{1}{r\sin\theta}\frac{dw}{d\phi} \quad \text{..... (5)},$$

$$a = \frac{d\gamma}{dy} + \frac{d\beta}{dz} = \frac{1}{r\sin\theta}\frac{dv}{d\phi} + \frac{1}{r}\frac{dw}{d\theta} - \frac{w}{r}\cot\theta \quad \text{..... (6)},$$

$$b = \frac{d\alpha}{dz} + \frac{d\gamma}{dx} = \frac{dw}{dr} - \frac{w}{r} + \frac{1}{r\sin\theta}\frac{du}{d\phi} \quad \text{..... (7)},$$

$$c = \frac{d\beta}{dx} + \frac{d\alpha}{dy} = \frac{1}{r}\frac{du}{d\theta} + \frac{dv}{dr} - \frac{v}{r} \quad \text{..... (8)}.$$

It is to be noticed that most writers on Elastic Solids measure θ from the equator so that they take v opposite in direction to the above, also θ of our notation is equivalent to $\frac{\pi}{2} - \phi$ of Lamé's. Taking this into account the above values agree with those of Mr Webb.

We thus find $\delta \equiv e + f + g = \frac{du}{dr} + \frac{2u}{r} + \frac{1}{r}\frac{dv}{d\theta} + \frac{v\cot\theta}{r} + \frac{1}{r\sin\theta}\frac{dw}{d\phi}$ .....(9),

or, as is often more convenient,

$$\delta = \frac{1}{r^2}\left\{\frac{d.ur^2}{dr} + \frac{1}{\sin\theta}\frac{d.vr\sin\theta}{d\theta} + \frac{1}{\sin^2\theta}\frac{d.wr\sin\theta}{d\phi}\right\} \quad \text{..... (10)};$$

Digitized by Google

$$2W_0 = (m+n)\left\{\frac{1}{r^2}\frac{d.ur^2}{dr} + \frac{1}{r\sin\theta}\frac{d.v\sin\theta}{d\theta} + \frac{1}{r\sin\theta}\frac{dw}{d\phi}\right\}^2$$

$$+ n\left\{-4\frac{du}{dr}\left(\frac{2u}{r} + \frac{1}{r}\frac{dv}{d\theta} + \frac{v\cot\theta}{r} + \frac{1}{r\sin\theta}\frac{dw}{d\phi}\right) - 4\left(\frac{u}{r} + \frac{1}{r}\frac{dv}{d\theta}\right)\left(\frac{u}{r} + \frac{v\cot\theta}{r} + \frac{1}{r\sin\theta}\frac{dw}{d\phi}\right)\right.$$

$$\left. + \left(\frac{1}{r}\frac{dw}{d\theta} - \frac{w\cot\theta}{r} + \frac{1}{r\sin\theta}\frac{dv}{d\phi}\right)^2 + \left(\frac{dw}{dr} - \frac{w}{r} + \frac{1}{r\sin\theta}\frac{du}{d\phi}\right)^2 + \left(\frac{dv}{dr} - \frac{v}{r} + \frac{1}{r}\frac{du}{d\theta}\right)^2\right\} \ldots\ldots (11).$$

§ 3. It seems on the whole the best way of finding the equations of motion or of equilibrium to employ the principle of the conservation of energy along with D'Alembert's principle. Thus the increase of the potential energy of the system must be equal to the work done by the forces, other than elastic, which act throughout the mass or over the surface and by the reversed effective forces. Of course this is merely another way of saying that the sum of the increases in the potential and in the kinetic energies is equal to the work done by the external forces. Let us denote the non-elastic forces which act throughout the interior of the mass, i.e. "bodily forces," by R, Θ, Φ parallel to u, v, w respectively, per unit mass; and denote the surface forces similarly resolved by F, G, H per unit area.

We have then to find the increase in the potential energy of the entire mass due to an arbitrary displacement ∂u, ∂v, ∂w, and also the work done by the systems R, Θ, Φ, F, G, H and the reversed effective forces, and equate the two. Thus

$$\iiint \rho r^2 \sin\theta\, dr\, d\theta\, d\phi \left[\left(R - \frac{d^2u}{dt^2}\right)\partial u + \left(\Theta - \frac{d^2v}{dt^2}\right)\partial v + \left(\Phi - \frac{d^2w}{dt^2}\right)\partial w\right] + \iint ds\,(F\partial u + G\partial v + H\partial w)$$

= change in potential energy of solid due to displacements ∂u, ∂v, ∂w, for all possible values of the ratios $\partial u : \partial v : \partial w$ ..... (12).

The triple integral applies to the whole of the interior of the solid, the double to the whole surface.

Owing to the length of the expression it seems best to write out separately the changes in the potential energy due to ∂u, ∂v and ∂w.

If then W denote the potential energy of the entire solid,

$$2W = \iiint r^2 \sin\theta\, dr\, d\theta\, d\phi \left[(m+n)\left(\frac{1}{r^2}\frac{d.ur^2}{dr} + \frac{1}{r\sin\theta}\frac{d.v\sin\theta}{d\theta} + \frac{1}{r\sin\theta}\frac{dw}{d\phi}\right)^2\right.$$

$$+ n\left\{-4\frac{du}{dr}\left(\frac{2u}{r} + \frac{1}{r}\frac{dv}{d\theta} + \frac{v\cot\theta}{r} + \frac{1}{r\sin\theta}\frac{dw}{d\phi}\right) - 4\left(\frac{u}{r} + \frac{1}{r}\frac{dv}{d\theta}\right)\left(\frac{u}{r} + \frac{v\cot\theta}{r} + \frac{1}{r\sin\theta}\frac{dw}{d\phi}\right)\right.$$

$$\left.\left. + \left(\frac{1}{r}\frac{dw}{d\theta} - \frac{w\cot\theta}{r} + \frac{1}{r\sin\theta}\frac{dv}{d\phi}\right)^2 + \left(\frac{dw}{dr} - \frac{w}{r} + \frac{1}{r\sin\theta}\frac{du}{d\phi}\right)^2 + \left(\frac{dv}{dr} - \frac{v}{r} + \frac{1}{r}\frac{du}{d\theta}\right)^2\right\}\right] \ldots (13).$$

Denote the variation in W due to displacement ∂u by ∂W_1,
..... to ∂v by ∂W_2,
..... and to ∂w by ∂W_3.

Then we shall find ∂W_1, ∂W_2, ∂W_3 each to consist of two series of terms; one for the interior, the other for the surface. To find these, only elementary knowledge of the Calculus of Variations, or rather integration by parts, is required.

Digitized by Google

To show clearly the method employed we shall find the variation of the term in δ due to ∂u fully. It is

$$\iiint r^2 \sin\theta\, dr\, d\theta\, d\phi \left[2(m+n)\left(\frac{1}{r^2}\frac{d.ur^2}{dr}+\frac{1}{r\sin\theta}\frac{d.v\sin\theta}{d\theta}+\frac{1}{r\sin\theta}\frac{dw}{d\phi}\right)\left(\frac{1}{r^2}\frac{d.r^2\partial u}{dr}\right)\right],$$

i.e.

$$\iiint \sin\theta\, dr\, d\theta\, d\phi \left\{2(m+n)\,\delta\frac{d.r^2\partial u}{dr}\right\}.$$

Integrating by parts, this is equal to

$$2\iint (m+n)\,\delta.r^2.\partial u\, d\theta\, d\phi - 2\iiint (m+n)\frac{d\delta}{dr}\partial u\, r^2 \sin\theta\, dr\, d\theta\, d\phi.$$

To obtain the entire change ∂W in the potential energy, we have only to add together the changes ∂W_1, ∂W_2 and ∂W_3 which are to be obtained exactly in the way illustrated above. This involves no absolute difficulty but algebra of some length. Introducing the values so obtained for ∂W in (12), supposing the normal at any point of the surface to make with the directions of u, v, w at that point angles $\cos^{-1}\lambda$, $\cos^{-1}\mu$, $\cos^{-1}\nu$ respectively,

so that

$$r^2 \sin\theta\, d\phi\, d\theta = \lambda\, dS \qquad (14),$$

$$r\sin\theta\, d\phi\, dr = \mu\, dS \qquad (15),$$

and

$$r\, d\theta\, dr = \nu\, dS \qquad (16),$$

we find

$$\begin{aligned}
&\iint dS\Bigg[\partial u\left\{F-\lambda\left((m+n)\delta-2n\left(\delta-\frac{du}{dr}\right)\right)-\mu n\left(\frac{dv}{dr}-\frac{v}{r}+\frac{1}{r}\frac{du}{d\theta}\right)-\nu n\left(\frac{dw}{dr}-\frac{w}{r}+\frac{1}{r\sin\theta}\frac{du}{d\phi}\right)\right\}\\
&\quad+\partial v\left\{G-\lambda n\left(\frac{dv}{dr}-\frac{v}{r}+\frac{1}{r}\frac{du}{d\theta}\right)-\mu\left((m+n)\delta-2n\left(\delta-\frac{u}{r}-\frac{1}{r}\frac{dv}{d\theta}\right)\right)\right.\\
&\qquad\qquad\left.-\nu n\left(\frac{1}{r}\frac{dw}{d\theta}-\frac{w\cot\theta}{r}+\frac{1}{r\sin\theta}\frac{dv}{d\phi}\right)\right\}\\
&\quad+\partial w\left\{H-\lambda n\left(\frac{dw}{dr}-\frac{w}{r}+\frac{1}{r\sin\theta}\frac{du}{d\phi}\right)-\mu n\left(\frac{1}{r}\frac{dw}{d\theta}-\frac{w\cot\theta}{r}+\frac{1}{r\sin\theta}\frac{dv}{d\phi}\right)\right.\\
&\qquad\qquad\left.-\nu\left((m+n)\delta-2n\left(\delta-\frac{u}{r}-\frac{v\cot\theta}{r}-\frac{1}{r\sin\theta}\frac{dw}{d\phi}\right)\right)\right\}\Bigg]\\
&-\iiint r^2\sin\theta\, dr\, d\theta\, d\phi\Bigg[\partial u\left\{\rho\left(\frac{d^2u}{dt^2}-R\right)-(m+n)\frac{d\delta}{dr}+\frac{n}{r\sin\theta}\frac{d}{d\theta}.\sin\theta\left(\frac{dv}{dr}+\frac{v}{r}-\frac{1}{r}\frac{du}{d\theta}\right)\right.\\
&\qquad\qquad\left.-\frac{n}{r\sin\theta}\frac{d}{d\phi}\left(\frac{1}{r\sin\theta}\frac{du}{d\phi}-\frac{dw}{dr}-\frac{w}{r}\right)\right\}\\
&\quad+\partial v\left\{\rho\left(\frac{d^2v}{dt^2}-\Theta\right)-\frac{m+n}{r}\frac{d\delta}{d\theta}+\frac{n}{r^2\sin\theta}\frac{d}{d\phi}\left(\frac{dw}{d\theta}+w\cot\theta-\frac{1}{\sin\theta}\frac{dv}{d\phi}\right)-\frac{n}{r}\frac{d}{dr}\left(r\frac{dv}{dr}+v-\frac{du}{d\theta}\right)\right\}\\
&\quad+\partial w\left\{\rho\left(\frac{d^2w}{dt^2}-\Phi\right)-\frac{m+n}{r\sin\theta}\frac{d\delta}{d\phi}+\frac{n}{r}\frac{d}{dr}\left(\frac{1}{\sin\theta}\frac{du}{d\phi}-r\frac{dw}{dr}-w\right)\right.\\
&\qquad\qquad\left.-\frac{n}{r^2}\frac{d}{d\theta}\left(\frac{dw}{d\theta}+w\cot\theta-\frac{1}{\sin\theta}\frac{du}{d\phi}\right)\right\}\Bigg]=0 \quad \ldots (17).
\end{aligned}$$

Digitized by Google

§ 4. The displacements ∂u, ∂v, ∂w are arbitrary and independent, thus we must have the coefficients of ∂u, of ∂v and of ∂w in (17) in the interior, and at the surface, all zero. We thus get three equations at each surface (if more than one), and three internal equations. The latter are

$$\rho\left(\frac{d^2u}{dt^2}-R\right)=(m+n)\frac{d\delta}{dr}-\frac{n}{r\sin\theta}\frac{d}{d\theta}.\sin\theta\left(\frac{dv}{dr}+\frac{v}{r}-\frac{1}{r}\frac{du}{d\theta}\right)+\frac{n}{r\sin\theta}\frac{d}{d\phi}\left(\frac{1}{r\sin\theta}\frac{du}{d\phi}-\frac{dw}{dr}-\frac{w}{r}\right)\text{..(18)},$$

$$\rho\left(\frac{d^2v}{dt^2}-\Theta\right)=\frac{m+n}{r}\frac{d\delta}{d\theta}-\frac{n}{r^2\sin\theta}\frac{d}{d\phi}\left(\frac{dw}{d\theta}+w\cot\theta-\frac{1}{\sin\theta}\frac{dv}{d\phi}\right)+\frac{n}{r}\frac{d}{dr}\left(r\frac{dv}{dr}+v-\frac{du}{d\theta}\right)\text{.....(19)},$$

$$\rho\left(\frac{d^2w}{dt^2}-\Phi\right)=\frac{m+n}{r\sin\theta}\frac{d\delta}{d\phi}-\frac{n}{r}\frac{d}{dr}\left(\frac{1}{\sin\theta}\frac{du}{d\phi}-r\frac{dw}{dr}-w\right)+\frac{n}{r^2}\frac{d}{d\theta}\left(\frac{dw}{d\theta}+w\cot\theta-\frac{1}{\sin\theta}\frac{du}{d\phi}\right)\text{.....(20)}.$$

These equations may be put in a very neat form. Write

$$\frac{1}{r^2\sin\theta}\left(\frac{d\,.\,wr\sin\theta}{d\theta}-\frac{d\,.\,vr}{d\phi}\right)=\mathfrak{A}\text{.....(21)},$$

$$\frac{1}{\sin\theta}\left(\frac{du}{d\phi}-\frac{d\,.\,wr\sin\theta}{dr}\right)=\mathfrak{B}\text{.....(22)},$$

$$\sin\theta\left(\frac{d\,.\,vr}{dr}-\frac{du}{d\theta}\right)=\mathfrak{C}\text{.....(23)}.$$

Multiply both sides of the first internal equation by $r^2\sin\theta$,

..... second ..... $r\sin\theta$,

.....and of the third..... r,

and they become respectively

$$(m+n)\,r^2\sin\theta\frac{d\delta}{dr}-n\frac{d\mathfrak{C}}{d\theta}+n\frac{d\mathfrak{B}}{d\phi}=\rho r^2\sin\theta\left(\frac{d^2u}{dt^2}-R\right)\text{.....(24)},$$

$$(m+n)\sin\theta\frac{d\delta}{d\theta}-n\frac{d\mathfrak{A}}{d\phi}+n\frac{d\mathfrak{C}}{dr}=\rho r\sin\theta\left(\frac{d^2v}{dt^2}-\Theta\right)\text{.....(25)},$$

$$(m+n)\operatorname{cosec}\theta\frac{d\delta}{d\phi}-n\frac{d\mathfrak{B}}{dr}+n\frac{d\mathfrak{A}}{d\theta}=\rho r\left(\frac{d^2w}{dt^2}-\Phi\right)\text{.....(26)};$$

while the surface conditions are

$$F=\lambda\left\{(m+n)\,\delta-2n\left(\delta-\frac{du}{dr}\right)\right\}+\mu n\left(\frac{dv}{dr}-\frac{v}{r}+\frac{1}{r}\frac{du}{d\theta}\right)+\nu n\left(\frac{dw}{dr}-\frac{w}{r}+\frac{1}{r\sin\theta}\frac{du}{d\phi}\right)\text{.....(27)},$$

$$G=\lambda n\left(\frac{dv}{dr}-\frac{v}{r}+\frac{1}{r}\frac{du}{d\theta}\right)+\mu\left\{(m+n)\,\delta-2n\left(\delta-\frac{1}{r}\frac{dv}{d\theta}-\frac{u}{r}\right)\right\}+\nu n\left(\frac{1}{r}\frac{dw}{d\theta}-\frac{w}{r}\cot\theta+\frac{1}{r\sin\theta}\frac{dv}{d\phi}\right)\text{...(28)},$$

$$H=\lambda n\left(\frac{dw}{dr}-\frac{w}{r}+\frac{1}{r\sin\theta}\frac{du}{d\phi}\right)+\mu n\left(\frac{1}{r}\frac{dw}{d\theta}-\frac{w}{r}\cot\theta+\frac{1}{r\sin\theta}\frac{dv}{d\phi}\right)$$
$$+\nu\left\{(m+n)\,\delta-2n\left(\delta-\frac{u}{r}-\frac{v}{r}\cot\theta-\frac{1}{r\sin\theta}\frac{dw}{d\phi}\right)\right\}\text{.....(29)}.$$

Digitized by Google

Usually the surface is that of a sphere and so $\mu = 0 = \nu$ while $\lambda = 1$. In this case we get

$$F = (m+n)\,\delta - 2n\left(\delta - \frac{du}{dr}\right) \quad \text{(30)},$$

$$G = n\left(r\frac{d.\frac{v}{r}}{dr} + \frac{1}{r}\frac{du}{d\theta}\right) \quad \text{(31)},$$

$$H = n\left(r\frac{d.\frac{w}{r}}{dr} + \frac{1}{r\sin\theta}\frac{du}{d\phi}\right) \quad \text{(32)}.$$

The equations both internal and at the surface are really identical with those given in Lamé, page 200, etc. He however gives merely the results and we have nowhere seen them actually worked out for the general case. The *form* above is slightly different from Lamé's, because he uses as his angles of reference ψ and ϕ, answering to our ϕ and $\frac{\pi}{2} - \theta$ respectively, and his v is measured in the opposite direction to ours.

§ 5. The reason that has led to the use of a different notation here is that it seems objectionable to introduce in Elastic Solids angles different from those generally employed in solid geometry, at least in English books.

Also since the development of these equations depends largely on the properties of spherical harmonics, it is awkward to employ a notation different from that which occurs in the standard English works on that subject. This awkwardness is due to the two facts that it would make reference to the ordinary spherical harmonic properties more difficult, and that it would make it harder to recognise these properties when they turned up.

§ 6. The surface conditions may be employed in finding the elastic forces or "stresses" at any point in the interior of the solid. For we can draw an imaginary surface through the point considered so as to include a limited quantity of the elastic material whose conditions of equilibrium can be separately studied. It is in equilibrium under the action of the system R, Θ, Φ, of the reversed effective forces if motion exist, and of the surface tractions. These surface tractions are as before given by the values found above for F, G, H, but they are now really the stresses existing at this boundary. We can suppose the boundary moved about so that its normal lies in any required direction and so can find the whole series of stresses at the point considered. Suppose we denote these stresses by letters indicating their direction. Thus let

R_r = Normal traction parallel to u on the face $r^2 \sin\theta\, d\phi\, d\theta$,
R_θ = Tangential ..... u ..... $r\sin\theta\, d\phi\, dr$,
R_ϕ = Tangential ..... u ..... $r\, d\theta\, dr$,
Θ_θ = Normal ..... v ..... $r\sin\theta\, d\phi\, dr$,
Θ_r = Tangential ..... v ..... $r^2 \sin\theta\, d\phi\, d\theta$,
Θ_ϕ = Tangential ..... v ..... $r\, d\theta\, dr$,
Φ_ϕ = Normal ..... w ..... $r\, d\theta\, dr$,
Φ_r = Tangential ..... w ..... $r^2 \sin\theta\, d\phi\, d\theta$,
Φ_θ = Tangential ..... w ..... $r\sin\theta\, d\phi\, dr$.

Digitized by Google

Putting $\lambda = 1$ in the surface equations (27), (28), (29) we get the tractions on the face $r^2 \sin\theta \, d\phi \, d\theta$.

They are F, G, H parallel to u, v, w in the one notation,

R_r, Θ_r, Φ_r ..... in the other .....;

$$\therefore \; R_r = (m+n)\,\delta - 2n\left(\delta - \frac{du}{dr}\right) \quad \text{(33)},$$

$$\Theta_r = n\left(r\frac{d.\frac{v}{r}}{dr} + \frac{1}{r}\frac{du}{d\theta}\right) \quad \text{(34)},$$

$$\Phi_r = n\left(r\frac{d.\frac{w}{r}}{dr} + \frac{1}{r\sin\theta}\frac{du}{d\phi}\right) \quad \text{(35)}.$$

Putting $\mu = 1$, we get the tractions on the face $r\sin\theta \, d\phi \, dr$,

$$R_\theta = n\left(r\frac{d.\frac{v}{r}}{dr} + \frac{1}{r}\frac{du}{d\theta}\right) \quad \text{(36)},$$

$$\Theta_\theta = (m+n)\,\delta - 2n\left(\delta - \frac{u}{r} - \frac{1}{r}\frac{dv}{d\theta}\right) \quad \text{(37)},$$

$$\Phi_\theta = n\left(\frac{1}{r}\frac{dw}{d\theta} - \frac{w}{r}\cot\theta + \frac{1}{r\sin\theta}\frac{dv}{d\phi}\right) \quad \text{(38)}.$$

So putting $\nu = 1$, we get the tractions on the face $r\,d\theta\,dr$,

$$R_\phi = n\left(r\frac{d.\frac{w}{r}}{dr} + \frac{1}{r\sin\theta}\frac{du}{d\phi}\right) \quad \text{(39)},$$

$$\Theta_\phi = n\left(\frac{1}{r}\frac{dw}{d\theta} - \frac{w}{r}\cot\theta + \frac{1}{r\sin\theta}\frac{dv}{d\phi}\right) \quad \text{(40)},$$

$$\Phi_\phi = (m+n)\,\delta - 2n\left(\delta - \frac{u}{r} - \frac{v}{r}\cot\theta - \frac{1}{r\sin\theta}\frac{dw}{d\phi}\right) \quad \text{(41)}.$$

From these values we get at once the well-known relations

$$\Theta_r = R_\theta; \qquad R_\phi = \Phi_r; \qquad \Theta_\phi = \Phi_\theta \quad \text{(42)}.$$

Of course we might deduce these at once from the form of W, putting

$$R_r = \frac{dW}{de}, \qquad \Phi_\theta = \frac{dW}{da}, \text{ etc.}$$

We have thus deduced all the circumstances whether of equilibrium or of motion from the expression in Cartesians for the potential energy. This expression requires only the knowledge of the expression for the relative displacements as given by the equation to the "strain ellipsoid," the invariants of which, viz.

$$e + f + g \quad \text{and} \quad ef + eg + fg - \tfrac{1}{4}(a^2 + b^2 + c^2),$$

are all that is required.

Digitized by Google

SECTION II.

EQUILIBRIUM OF SOLID SPHERE OR SPHERICAL SHELL.

§ 7. We shall now apply these polar equations to the following problem.

A solid sphere, or a shell bounded by concentric spherical surfaces when undisturbed, is acted on in its substance by a system of forces for which a potential exists, the potential being expressed by a series of solid spherical harmonics, and over its surface or surfaces forces are applied which are expressed as a series of surface spherical harmonics. It is required to find the displacement at every point of its substance.

This problem has been solved by Sir W. Thomson in Cartesian co-ordinates, see Thomson and Tait, Vol. I. Part II., beginning at § 737. Lamé also has given a solution in polars in Liouville's *Journal* for 1854; but it is extremely complex, and is entirely different from what follows. Professor Pearson in the *Quarterly Journal* for 1879 has solved the case of normal surface forces on a solid sphere expressed as Legendre's coefficients. Professor Darwin has converted Sir W. Thomson's solution into polars for the case $\frac{n}{m}=0$, see *Phil. Transactions*, 1879. The following method is entirely different from any of the above, it solves the most general form of the problem, and seems less difficult than Sir W. Thomson's method, especially in the case when the surface forces are purely normal. The same or closely analogous methods are applied hereafter to the case of equilibrium or vibration of a cylinder, and to the vibrations of a sphere.

§ 8. If V denote the potential of the non-elastic or bodily forces, then $R=\frac{dV}{dr}$, $\Theta=\frac{1}{r}\frac{dV}{d\theta}$, $\Phi=\frac{1}{r\sin\theta}\frac{dV}{d\phi}$.

We have, see (9) and (10) Section I.,

$$\delta=\frac{du}{dr}+\frac{2u}{r}+\frac{1}{r}\frac{dv}{d\theta}+\frac{v\cot\theta}{r}+\frac{1}{r\sin\theta}\frac{dw}{d\phi} \quad\text{.....(1)}$$

$$\equiv\frac{1}{r^2}\left(\frac{d\,.\,ur^2}{dr}+\frac{1}{\sin\theta}\frac{d\,.\,vr\sin\theta}{d\theta}+\frac{1}{\sin^2\theta}\frac{d\,.\,wr\sin\theta}{d\phi}\right)\quad\text{.....(1 a).}$$

The surface conditions are (30), (31), (32) Section I., viz.

$$(m-n)\delta+2n\frac{du}{dr}=\text{normal traction per unit area} \quad\text{.....(2),}$$

$$n\left(\frac{rd\,.\,\frac{v}{r}}{dr}+\frac{1}{r}\frac{du}{d\theta}\right)=\text{tangential surface traction along tangent to meridian} \quad\text{.....(3),}$$

$$n\left(\frac{rd\,.\,\frac{w}{r}}{dr}+\frac{1}{r\sin\theta}\frac{du}{d\phi}\right)=\text{tangential surface traction along perpendicular to meridian...(4),}$$

putting $r=a$ for a solid sphere,

and $=a$ and a' for a shell.

Digitized by Google

The internal equations are (24), (25), (26), Section I. viz.

$$(m+n)\,r^2 \sin\theta \frac{d\delta}{dr} - n\frac{d\mathfrak{C}}{d\theta} + n\frac{d\mathfrak{B}}{d\phi} = -\rho r^2 \sin\theta \frac{dV}{dr} \quad \text{.....(5)},$$

$$(m+n)\sin\theta \frac{d\delta}{d\theta} - n\frac{d\mathfrak{A}}{d\phi} + n\frac{d\mathfrak{C}}{dr} = -\rho \sin\theta \frac{dV}{d\theta} \quad \text{.....(6)},$$

$$(m+n)\operatorname{cosec}\theta \frac{d\delta}{d\phi} - n\frac{d\mathfrak{B}}{dr} + n\frac{d\mathfrak{A}}{d\theta} = -\rho \operatorname{cosec}\theta \frac{dV}{d\phi} \quad \text{.....(7)};$$

where

$$\mathfrak{A} = \frac{1}{r^2 \sin\theta}\left(\frac{d\,.\,wr\sin\theta}{d\theta} - \frac{d\,.\,vr}{d\phi}\right) \quad \text{.....(8)},$$

$$\mathfrak{B} = \frac{1}{\sin\theta}\left(\frac{du}{d\phi} - \frac{d\,.\,wr\sin\theta}{dr}\right) \quad \text{.....(9)},$$

$$\mathfrak{C} = \sin\theta\left(\frac{d\,.\,vr}{dr} - \frac{du}{d\theta}\right) \quad \text{.....(10)}.$$

We shall first consider a solid sphere, and afterwards find the additional terms to be introduced in the solution for the case of a shell.

Differentiate (5) with respect to r, (6) with respect to θ, and (7) with respect to ϕ; adding we get

$$(m+n)\,\nabla^2\delta = -\rho\nabla^2 V \quad \text{.....(11)},$$

where as usual

$$\nabla^2\delta \equiv \frac{d^2\delta}{dr^2} + \frac{2}{r}\frac{d\delta}{dr} + \frac{1}{\sin\theta}\frac{d}{d\theta}\cdot\sin\theta\frac{d\delta}{d\theta} + \frac{1}{\sin^2\theta}\frac{d^2\delta}{d\phi^2} \quad \text{.....(12)}.$$

According to the data of the problem $V = \Sigma(r^i V_i)$, where V_i is a surface spherical harmonic of degree i, and so $r^i V_i$ a solid harmonic of the same degree.

At present we shall retain only one term $r^i V_i$ as a type of the solution.

The solution of (11) is obviously

$$\delta = -\frac{\rho}{m+n} r^i V_i + r^i Y_i \quad \text{.....(13)},$$

where Y_i is an arbitrary surface harmonic of degree i, retained as a type.

Substituting this value of δ in (5), (6) and (7) we get

$$\frac{d\mathfrak{C}}{d\theta} - \frac{d\mathfrak{B}}{d\phi} = \frac{m+n}{n} r^2 \sin\theta \frac{d}{dr}\cdot r^i Y_i \quad \text{.....(14)},$$

$$\frac{d\mathfrak{A}}{d\phi} - \frac{d\mathfrak{C}}{dr} = \frac{m+n}{n}\sin\theta\frac{d}{d\theta}\cdot r^i Y_i \quad \text{.....(15)},$$

$$\frac{d\mathfrak{B}}{dr} - \frac{d\mathfrak{A}}{d\theta} = \frac{m+n}{n}\frac{1}{\sin\theta}\frac{d}{d\phi}\cdot r^i Y_i \quad \text{.....(16)}.$$

Substituting in (14) the values of $\mathfrak{C}$ and $\mathfrak{B}$ given in (10) and (9), we get

$$\frac{d}{d\theta}\cdot\sin\theta\left(\frac{d\,.\,vr}{dr} - \frac{du}{d\theta}\right) - \frac{1}{\sin\theta}\left(\frac{d^2u}{d\phi^2} - \frac{d}{dr}\cdot\frac{d\,.\,wr\sin\theta}{d\phi}\right) = \frac{m+n}{n} r^2 \sin\theta \frac{d}{dr}\cdot r^i Y_i.$$

Digitized by Google

In this substitute from (1 a),

$$\frac{d\,.\,wr\sin\theta}{d\phi} = -\sin^2\theta\left(\frac{d\,.\,ur^2}{dr} + \frac{1}{\sin\theta}\frac{d\,.\,vr\sin\theta}{d\theta}\right) + r^2\delta\sin^2\theta,$$

and we get

$$-\frac{d}{d\theta}.\sin\theta\frac{du}{d\theta} - \frac{1}{\sin\theta}\frac{d^2u}{d\phi^2} + \frac{d}{d\theta}.\sin\theta\frac{d\,.\,vr}{dr} - \sin\theta\frac{d}{dr}\left(\frac{d\,.\,ur^2}{dr} + \frac{1}{\sin\theta}\frac{d\,.\,vr\sin\theta}{d\theta}\right)$$

$$+\sin\theta\frac{d}{dr}.r^2\delta = \frac{m+n}{n}r^2\sin\theta\frac{d}{dr}.r^iY_i.$$

Substituting for δ from (13) and reducing, the terms in v cut out. Bringing over the term in δ to the second side, performing the differentiations with respect to r, and dividing both sides by $\sin\theta$, we get

$$\frac{d^2}{dr^2}.ur^2 + \frac{1}{\sin\theta}\frac{d}{d\theta}.\sin\theta\frac{du}{d\theta} + \frac{1}{\sin^2\theta}\frac{d^2u}{d\phi^2} = Y_ir^{i+1}\left(i+2-\frac{m+n}{n}i\right) - V_ir^{i+1}\frac{\rho}{m+n}(i+2)\ldots(17).$$

The left-hand side $= 0$ is the general equation in u; the right-hand side gives the typical terms, treating the ith surface harmonics as typical. We shall find that part of the solution of (17) depending on r^{i+1} by assuming

$$u = u_ir^{i+1} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18);$$

this gives to determine u_i the equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}.\sin\theta\frac{du_i}{d\theta} + \frac{1}{\sin^2\theta}\frac{d^2u_i}{d\phi^2} + (i+3)(i+2)u_i = -\left(\frac{mi}{n}-2\right)Y_i - \frac{(i+2)\rho}{m+n}V_i\ldots(19).$$

Since the equation satisfied by a Laplace's function X_i of the ith degree is

$$\frac{1}{\sin\theta}\frac{d}{d\theta}.\sin\theta\frac{dX_i}{d\theta} + \frac{1}{\sin^2\theta}\frac{d^2X_i}{d\phi^2} + i(i+1)X_i = 0,$$

the particular solution of (19) is

$$-\frac{\left(\frac{mi}{n}-2\right)}{2(2i+3)}Y_i - \frac{(i+2)\rho V_i}{2(2i+3)(m+n)},$$

because $$(i+3)(i+2) - i(i+1) = 2(2i+3).$$

The general solution is Z_{i+2}, an arbitrary surface harmonic of degree $i+2$;

$$\therefore\ u_i \equiv -\frac{(i+2)\rho V_i}{2(2i+3)(m+n)} - \frac{\left(\frac{mi}{n}-2\right)}{2(2i+3)}Y_i + Z_{i+2} \quad\ldots\ldots\ldots\ldots\ldots\ldots(20)$$

is the coefficient of r^{i+1} in the complete value of u.

This shows that the part of the solution of (17) which involves surface harmonics of degree i, is

$$u = -r^{i+1}\left\{\frac{\rho(i+2)V_i}{2(2i+3)(m+n)} + \frac{\frac{mi}{n}-2}{2(2i+3)}Y_i\right\} + r^{i-1}Z_i\ldots\ldots\ldots\ldots\ (21),$$

where Z_i is an arbitrary surface harmonic of degree i, which however must be zero if $i = 0$.

Digitized by Google

§ 9. Again in (15) substitute their values for $\mathfrak{A}$ and $\mathfrak{C}$ from (8) and (10), and for

$$\frac{d.wr\sin\theta}{d\phi} \equiv \left(\delta - \frac{1}{r^2}\frac{d.ur^2}{dr} - \frac{1}{r^2\sin\theta}\frac{d.vr\sin\theta}{d\theta}\right) r^2\sin^2\theta,$$

i.e.
$$\equiv \frac{-\rho}{m+n} r^{i+2}\sin^2\theta V_i + r^{i+2}\sin^2\theta Y_i - \sin^2\theta\frac{d.ur^2}{dr} - \sin\theta\frac{d.vr\sin\theta}{d\theta},$$

retaining typical terms.

We thus get

$$\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left[\frac{-\rho}{m+n} r^{i+2}\sin^2\theta V_i + r^{i+2}\sin^2\theta Y_i - \sin^2\theta\frac{d.ur^2}{dr} - \sin\theta\frac{d.vr\sin\theta}{d\theta}\right]$$
$$-\frac{1}{r^2\sin\theta}\frac{d^2.vr}{d\phi^2} - \sin\theta\left(\frac{d^2.vr}{dr^2} - \frac{d^2u}{dr\,d\theta}\right) = \frac{m+n}{n}\sin\theta r^i\frac{dY_i}{d\theta},$$

whence

$$\frac{d^2.vr\sin\theta}{dr^2} + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}.\sin\theta\frac{d.vr\sin\theta}{d\theta} + \frac{1}{r^2\sin^2\theta}\frac{d^2.vr\sin\theta}{d\phi^2} = -\frac{1}{r^2\sin\theta}\frac{d}{d\theta}.\sin^2\theta\frac{d.ur^2}{dr}$$
$$+\sin\theta\frac{d^2u}{dr\,d\theta} - \frac{\rho}{m+n}\frac{r^i}{\sin\theta}\frac{d}{d\theta}(\sin^2\theta V_i) + \frac{r^i}{\sin\theta}\frac{d}{d\theta}(\sin^2\theta Y_i) - \frac{m+n}{n} r^i\sin\theta\frac{dY_i}{d\theta}.$$

Substitute for u the value given in (18) and (20) so as to obtain that part of v which has r^{i+1} for its coefficient. This is the same end as we obtained above in our choice of the particular part of δ which we put in. Reducing now the second side of the last equation by performing the differentiations and putting the terms together we get

$$\frac{1}{r^i}\left[\frac{d^2.vr\sin\theta}{dr^2} + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}.\sin\theta\frac{d.vr\sin\theta}{d\theta} + \frac{1}{r^2\sin^2\theta}\frac{d^2.vr\sin\theta}{d\phi^2}\right] = \frac{-\rho(i+1)}{(m+n)(2i+3)}\left(\sin\theta\frac{dV_i}{d\theta} - iV_i\cos\theta\right)$$
$$-\frac{\frac{m}{n}(i+3)+2}{2i+3}\left(\sin\theta\frac{dY_i}{d\theta} - i\cos\theta Y_i\right) - 2\left(\sin\theta\frac{dZ_{i+2}}{d\theta} + (i+3)\cos\theta Z_{i+2}\right) \text{.....(22).}$$

To obtain the coefficient of r^{i+1} in v, put $v\sin\theta = r^{i+1}v_i$, and (22) gives

$$(i+1)(i+2)v_i + \frac{1}{\sin\theta}\frac{d}{d\theta}.\sin\theta\frac{dv_i}{d\theta} + \frac{1}{\sin^2\theta}\frac{d^2v_i}{d\phi^2} = \frac{-\rho}{m+n}\frac{i+1}{2i+3}\left(\sin\theta\frac{dV_i}{d\theta} - i\cos\theta V_i\right)$$
$$-\frac{\frac{m}{n}(i+3)+2}{2i+3}\left(\sin\theta\frac{dY_i}{d\theta} - i\cos\theta Y_i\right) - 2\left(\sin\theta\frac{dZ_{i+2}}{d\theta} + (i+3)\cos\theta Z_{i+2}\right) \text{... (23).}$$

To find the particular solution of (23), substitute for v_i in the first side $\sin\theta\frac{dY_i}{d\theta}$, and we find as the result of the operations on the first side

$$(i+1)(i+2)\sin\theta\frac{dY_i}{d\theta} + \frac{1}{\sin\theta}\frac{d}{d\theta}.\sin\theta\frac{d}{d\theta}.\sin\theta\frac{dY_i}{d\theta} + \frac{1}{\sin\theta}\frac{d}{d\theta}.\frac{d^2Y_i}{d\phi^2}.$$

Substitute in this $\frac{d^2Y_i}{d\phi^2} \equiv -\left\{i(i+1)\sin^2\theta Y_i + \sin\theta\frac{d}{d\theta}.\sin\theta\frac{dY_i}{d\theta}\right\}$, and it immediately reduces to

$$2(i+1)\left(\sin\theta\frac{dY_i}{d\theta} - i\cos\theta Y_i\right).$$

Digitized by Google

Thus the part of the particular solution of (23) that depends on Y_i is

$$-\frac{\frac{m}{n}(i+3)+2}{2(i+1)(2i+3)}\sin\theta\frac{dY_i}{d\theta},$$

and the part depending on V_i is similarly $-\dfrac{\rho\sin\theta}{2(m+n)(2i+3)}\dfrac{dV_i}{d\theta}$.

In exactly the same way we find that the part of the particular solution depending on Z_{i+2} is

$$\frac{1}{i+2}\sin\theta\frac{dZ_{i+2}}{d\theta}.$$

The general solution of (23) is $v_i = X_{i+1}$, an arbitrary surface harmonic of degree $i+1$, which our surface conditions will show to be zero if the surface forces be purely normal.

The complete solution of (23) is thus

$$v_i = -\frac{\rho\sin\theta}{2(m+n)(2i+3)}\frac{dV_i}{d\theta} - \frac{\frac{m}{n}(i+3)+2}{2(i+1)(2i+3)}\sin\theta\frac{dY_i}{d\theta} + \frac{\sin\theta}{i+2}\frac{dZ_{i+2}}{d\theta} + X_{i+1} \quad\ldots\ldots(24).$$

But $\dfrac{v_i}{\sin\theta}$ is the coefficient of r^{i+1} in the complete solution for v, therefore that part of the solution of v which depends on those surface harmonics of the ith degree occurring in (21), adding the term in X_i to remind us of the existence of a series of that nature given by (24), is

$$v = -\frac{r^{i+1}}{2(2i+3)}\frac{d}{d\theta}\left\{\frac{\rho}{m+n}V_i + \frac{\frac{m}{n}(i+3)+2}{i+1}Y_i\right\} + \frac{r^{i-1}}{i}\frac{dZ_i}{d\theta} + \frac{r^i}{\sin\theta}X_i \quad\ldots\ldots\ldots(25).$$

The complete solution for v would require us to write a Σ before each of the typical terms in (25), because these in themselves are not surface harmonics. We use this form of solution for its convenience as it shows at once what terms in the solution depend on the same surface harmonics V_i, Y_i, Z_i. This is especially the case when the surface harmonics are purely normal, and so $X_i = 0$ as we shall see shortly.

§ 10. We might determine w in like manner from (16), but it is obviously simpler to deduce it from (1 a).

Substituting in (1 a) the values already obtained for u and v in (21) and (25), we get

$$\frac{d.wr\sin\theta}{d\phi} = r^{i+2}\sin^2\theta\left(\frac{-\rho}{m+n}V_i + Y_i\right) + \frac{(i+3)\sin^2\theta}{2(2i+3)}r^{i+2}\left\{\frac{\rho}{m+n}(i+2)V_i + \left(\frac{mi}{n}-2\right)Y_i\right\}$$

$$-(i+1)\sin^2\theta r^i Z_i + \sin\theta\frac{d}{d\theta}\left[\frac{\sin\theta r^{i+2}}{2(2i+3)}\left\{\frac{\rho}{m+n}\frac{dV_i}{d\theta} + \frac{\frac{m}{n}(i+3)+2}{i+1}\frac{dY_i}{d\theta}\right\} + \frac{\sin\theta}{i}r^i\frac{dZ_i}{d\theta}\right] - \sin\theta r^{i+1}\frac{dX_i}{d\theta}$$

as the equation giving those terms in w which depend on V_i, Y_i, Z_i, X_i.

Digitized by Google

On reduction this at once becomes

$$\frac{d.wr\sin\theta}{d\phi}=\frac{\rho\sin^2\theta r^{i+2}}{2(m+n)(2i+3)}\left\{i(i+1)V_i+\frac{1}{\sin\theta}\frac{d}{d\theta}.\sin\theta\frac{dV_i}{d\theta}\right\}$$

$$+r^{i+2}\sin^2\theta\frac{\frac{m}{n}(i+3)+2}{2(i+1)(2i+3)}\left\{i(i+1)Y_i+\frac{1}{\sin\theta}\frac{d}{d\theta}.\sin\theta\frac{dY_i}{d\theta}\right\}$$

$$-\frac{r^i\sin^2\theta}{i}\left\{i(i+1)Z_i+\frac{1}{\sin\theta}\frac{d}{d\theta}.\sin\theta\frac{dZ_i}{d\theta}\right\}-r^{i+1}\sin\theta\frac{dX_i}{d\theta}.$$

Employing the well-known differential equation for surface harmonics of degree i, and dividing out by r, we get

$$\frac{d.w\sin\theta}{d\phi}=\frac{-\rho r^{i+1}}{2(m+n)(2i+3)}\frac{d^2V_i}{d\phi^2}-\frac{\frac{m}{n}(i+3)+2}{2(i+1)(2i+3)}\frac{d^2Y_i}{d\phi^2}+\frac{r^{i-1}}{i}\frac{d^2Z_i}{d\phi^2}-r^i\sin\theta\frac{dX_i}{d\theta}\ldots(26),$$

whence $$w=\frac{-r^{i+1}}{2(2i+3)}\frac{1}{\sin\theta}\frac{d}{d\phi}\left\{\frac{\rho}{m+n}V_i+\frac{\frac{m}{n}(i+3)+2}{i+1}Y_i\right\}+\frac{r^{i-1}}{i\sin\theta}\frac{dZ_i}{d\phi}+w_ir^i\ \ldots\ldots(27),$$

where w_i is given by $$\frac{dw_i}{d\phi}=-\frac{dX_i}{d\theta}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(27'').$$

The form of w_i is given at once by that of X_i. Thus for the Tesseral* harmonic expression, if $CT_i^\sigma\cos\sigma\phi+C'T_i^\sigma\sin\sigma\phi$ denote the part of X_i depending on $\cos\sigma\phi$ and $\sin\sigma\phi$, we have

$$-w_i=\Sigma\frac{1}{\sigma}\frac{d.T_i^\sigma}{d\theta}(C\sin\sigma\phi-C'\cos\sigma\phi),$$

where Σ denotes summation with respect to σ.

(26) would also admit of a solution $w=f(r,\theta)$† because in the expression for δ given by (1) w occurs only as varying with ϕ.

§ 11. To determine Y_i, Z_i, X_i... we have the surface conditions (2), (3), (4).

Let us first consider the case when the surface forces are purely normal, as this is much the most manageable case and seems most useful. It also will best bring out the parallelism of the typical solutions (21), (25) and (27).

If a be the radius of the sphere, we get from (2),

$$\frac{-\rho}{m+n}a^iV_i\frac{m(2i+3)+n(i^2+i-1)}{2i+3}-a^iY_i\frac{m(i^2-i-3)+n}{2i+3}+2n(i-1)a^{i-2}Z_i=S_i\ldots(28),$$

where S_i represents the normal tractions depending on surface harmonics of degree i.

From (3) in like manner, leaving out the common factor n,

$$\frac{d}{d\theta}\left\{\frac{-\rho}{m+n}a^i\frac{i+1}{2i+3}V_i-\frac{mi(i+2)-n}{n(i+1)(2i+3)}Y_ia^i+\frac{2(i-1)}{i}Z_ia^{i-2}\right\}+\frac{(i-1)X_ia^{i-1}}{\sin\theta}=0\ldots(29).$$

* Ferrers' *Spherical Harmonics*, p. 80.

† See Ibbetson's *Mathematical Theory of Elasticity*, Example (18), p. 380, noticing misprint $\frac{r}{A}$ for $\left(\frac{r}{A}\right)^i$.

Digitized by Google

So obviously from (4),

$$\frac{1}{\sin\theta}\frac{d}{d\phi}\left\{\frac{-\rho}{m+n}a^i\frac{i+1}{2i+3}V_i-\frac{mi(i+2)-n}{n(i+1)(2i+3)}Y_ia^i+\frac{2(i-1)}{i}Z_ia^{i-2}\right\}+(i-1)w_ia^{i-1}=0\ldots(30),$$

where w_i has the value given by (27″).

Since (29) and (30) are to hold for all values of θ and ϕ, i.e. all over the surface, they are satisfied by $X_i=0$, and therefore $w_i=0$, and by

$$\frac{-\rho}{m+n}\frac{i+1}{2i+3}a^iV_i-\frac{\frac{mi}{n}(i+2)-1}{(i+1)(2i+3)}a^iY_i+\frac{2(i-1)}{i}a^{i-2}Z_i=0\ \ldots\ldots\ldots\ldots\ldots\ldots(31).$$

For this satisfies the surface conditions (29) and (30); also (31) and (28) enable us to determine the two unknown quantities Y_i and Z_i in terms of V_i and S_i by two simple equations. X_i thus occurs, as will be shown more satisfactorily presently, only when the surface forces are other than purely normal. In the case of purely normal surface forces (3) and (4), besides indicating that $X_i=0$, give only one equation. Hereafter when we speak of these giving only one independent equation this is what is meant.

Multiply (31) by i and subtract from (28) and so eliminating Z_i we get, after easy reduction,

$$Y_i=\frac{i+1}{m(2i^2+4i+3)-n(2i+1)}\left[(2i+3)\frac{S_i}{a^i}+\frac{\rho}{m+n}V_i\{m(2i+3)-n\}\right]\ \ldots\ldots(32),$$

whence, and from (31),

$$Z_i=i\left[\frac{S_ia^{-i+2}\{mi(i+2)-n\}+\rho a^2V_ii\{m(i+2)-n\}}{2n(i-1)\{m(2i^2+4i+3)-n(2i+1)\}}\right]\ldots\ldots\ldots\ldots\ldots\ldots(33).$$

§ 12. The coefficient of V_ir^{i+1} in the ith harmonic term of u is then from (21), substituting for Y_i from (32) and reducing,

$$\frac{-\rho i\{m(i+1)-n\}}{2n\{m(2i^2+4i+3)-n(2i+1)\}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(34).$$

The coefficient of $r^{i+1}\dfrac{dV_i}{d\theta}$ in v is from (25), substituting for Y_i from (32) and reducing,

$$-\rho\frac{m(i+3)-n}{2n\{m(2i^2+4i+3)-n(2i+1)\}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(35).$$

The same expressions occur in the equation for w, therefore the complete solution, depending on V_i and S_i, is

$$u=-r^{i+1}\left[\frac{i\rho V_i\{m(i+1)-n\}+S_ia^{-i}(i+1)(mi-2n)}{2n\{m(2i^2+4i+3)-n(2i+1)\}}\right]$$
$$+r^{i-1}i\left[\frac{\rho a^2V_ii\{m(i+2)-n\}+S_ia^{-i+2}\{mi(i+2)-n\}}{2n(i-1)\{m(2i^2+4i+3)-n(2i+1)\}}\right]\ldots\ldots\ldots\ldots\ldots\ldots(36),$$

$$v=-r^{i+1}\frac{d}{d\theta}\left[\frac{\rho V_i\{m(i+3)-n\}+S_ia^{-i}\{m(i+3)+2n\}}{2n\{m(2i^2+4i+3)-n(2i+1)\}}\right]$$
$$+r^{i-1}\frac{d}{d\theta}\left[\frac{a^2\rho V_ii\{m(i+2)-n\}+S_ia^{-i+2}\{mi(i+2)-n\}}{2n(i-1)\{m(2i^2+4i+3)-n(2i+1)\}}\right]\ldots\ldots\ldots\ldots\ldots(37),$$

Digitized by Google

$$w = -\frac{r^{i+1}d}{\sin\theta d\phi}\left[\frac{\rho V_i\{m(i+3)-n\}+S_i a^{-1}\{m(i+3)+2n\}}{2n\{m(2i^2+4i+3)-n(2i+1)\}}\right]$$

$$+\frac{r^{i-1}d}{\sin\theta d\phi}\left[\frac{a^2\rho V_i i\{m(i+2)-n\}+S_i a^{-i+2}\{mi(i+2)-n\}}{2n(i-1)\{m(2i^2+4i+3)-n(2i+1)\}}\right] \quad \text{.....(38)},$$

$$\delta = r^i\left[\frac{i\rho V_i+(i+1)(2i+3)S_i a^{-1}}{m(2i^2+4i+3)-n(2i+1)}\right] \quad \text{.....(39)}.$$

§ 13. One very curious relation between the displacements due to S_i and V_i when $\frac{n}{m}$ is negligible is obvious from these equations. Neglecting $\frac{n}{m}$ the coefficients of ρV_i and of $S_i a^{-1}$ in u, v, w are everywhere identical. Thus a surface *pressure* $-S_i$ has exactly the same effect in producing displacements as bodily forces whose potential, per unit volume, is $S_i\left(\frac{r}{a}\right)^i$ acting throughout the mass of the sphere.

This is noticed by Professor Darwin in a particular case which he treats in his paper "On the Bodily Tides of Viscous and Semi-elastic Spheroids," *Phil. Transactions*, 1879, Part I.

He considers there a gravitating nearly spherical body whose surface is $r = a + \sigma_i$, and considers the effect of the surface *pressure* $g\rho\sigma_i$, and of the gravitation potential due to the protuberance which is $\frac{3g}{2i+1}\left(\frac{r}{a}\right)^i\sigma_i$.

Calling α' the displacement in any direction due to the surface *pressure* $g\rho\sigma_i$, α'' the displacement in the same direction due to the bodily forces he finds $\alpha'' = -\frac{3}{2i+1}\alpha'$.

This agrees with the relation stated above, because in this case

$$-S_i : V_i a^i :: 1 : \frac{3}{2i+1}.$$

I find that the equations (10) in the same paper, noticing that $\rho \equiv u$, $\varpi \equiv v$, $\nu \equiv w$, etc., and putting $T_i = S_i a^{-1}$, and $\frac{n}{m} = 0$, give identically the same values as (36), (37), (38).

§ 14. Suppose now that the surface forces are no longer purely normal.

The condition (28) remains unchanged; but on the right hand of (29) we shall have terms $\frac{1}{n}\Sigma T_i$, where T_i is a surface harmonic of degree i, and on the right-hand side of (30) $\frac{1}{n}\Sigma U_i$, where U_i is a surface harmonic of degree i. Also on the left-hand sides of (29) and (30) we should write a Σ.

Call (29) and (30) when so modified (29 a) and (30 a).

Multiply both sides of (29 a) by $\sin\theta$, then differentiate with respect to θ, and add it to (30 a) differentiated with respect to ϕ. Then, noticing that

$$\frac{d}{d\theta}\cdot\sin\theta\frac{dV_i}{d\theta}+\frac{1}{\sin\theta}\frac{d^2V_i}{d\phi^2} = -i(i+1)\sin\theta V_i,$$

Digitized by Google

and similarly for Y_i and Z_i, we get

$$\Sigma i(i+1)\sin\theta\left[\frac{\rho a^i}{m+n}\frac{i+1}{2i+3}V_i+\frac{mi(i+2)-n}{n(i+1)(2i+3)}a^iY_i-\frac{2(i-1)}{i}a^{i-2}Z_i\right]$$
$$+\Sigma(i-1)a^{i-1}\left[\frac{dX_i}{d\theta}+\frac{dw_i}{d\phi}\right]=\frac{1}{n}\Sigma\left[\frac{d}{d\theta}.\sin\theta T_i+\frac{dU_i}{d\phi}\right];$$

but, by (27″),
$$\frac{dX_i}{d\theta}+\frac{dw_i}{d\phi}=0,$$

$$\therefore\ \Sigma i(i+1)\left[\frac{\rho}{m+n}\frac{i+1}{2i+3}a^iV_i+\frac{mi(i+2)-n}{n(i+1)(2i+3)}a^iY_i-\frac{2(i-1)}{i}a^{i-2}Z_i\right]$$
$$=\frac{1}{n}\Sigma\left[\frac{1}{\sin\theta}\frac{d}{d\theta}.\sin\theta T_i+\frac{1}{\sin\theta}\frac{dU_i}{d\phi}\right]\text{.....(31 a).}$$

Again differentiate both sides of (29 a) with respect to ϕ, and subtract (30 a) first multiplied by $\sin\theta$ and then differentiated with respect to θ. We thus get

$$\Sigma(i-1)\left[\frac{a^{i-1}}{\sin\theta}\frac{dX_i}{d\phi}-a^{-1}\frac{d}{d\theta}.\sin\theta w_i\right]=\frac{1}{n}\Sigma\left[\frac{dT_i}{d\phi}-\frac{d}{d\theta}.\sin\theta U_i\right].$$

Differentiate both sides of this with respect to ϕ; then noticing that $\frac{dw_i}{d\phi}=-\frac{dX_i}{d\theta}$, this becomes

$$\Sigma(i-1)a^{i-1}\left\{\frac{1}{\sin\theta}\frac{d^2X_i}{d\phi^2}+\frac{d}{d\theta}.\sin\theta\frac{dX_i}{d\theta}\right\}=\frac{1}{n}\Sigma\left[\frac{d^2T_i}{d\phi^2}-\frac{d}{d\theta}.\sin\theta\frac{dU_i}{d\phi}\right],$$

i. e.
$$\Sigma a^{i-1}i(i^2-1)X_i=\frac{1}{n\sin\theta}\Sigma\left[\frac{d}{d\theta}.\sin\theta\frac{dU_i}{d\phi}-\frac{d^2T_i}{d\phi^2}\right]\text{.....(40).}$$

T_i and U_i are known, and thus the right-hand sides of (40) and of (31 a) can be expanded in a known series of surface spherical harmonics of the various degrees.

Thus X_i is given by (40), while Y_i and Z_i are given by (28) and (31 a). It follows from the parallelism between (31 a) and (31) that the terms in Y_i and Z_i which depend on S_i are the same as before, and so are given by (32) and (33). Also X_i is independent of S_i. (40) shows clearly that $X_i=0$ if $U_i=0=T_i$, i.e. if the surface forces be purely normal; also that $X_i=0$ unless U_i and T_i are functions of ϕ, so for a series of forces expressed as Legendre's coefficients $X_i=0$.

§ 15. Let us now find the additional terms to be introduced when we have no longer a solid sphere but the material contained between two concentric spherical surfaces of radii a, a'. The forces are supposed similar in kind to those of the last case. Let us number the equations so as to show their connection with the preceding. Thus (13′) means that this is the additional term or terms to be added to the right-hand side of the equation (13) of a solid sphere.

Taking the ith harmonic as typical, suppose the body forces have a potential $\Sigma V_i'r^{-i-1}$;

then
$$\delta'=-\frac{\rho}{m+n}r^{-i-1}V_i'+r^{-i-1}Y_i'\text{.....(13′),}$$

where V_i', Y_i' are surface harmonics of degree i.

Digitized by Google

Also
$$\frac{d\mathfrak{C}}{d\theta} - \frac{d\mathfrak{B}}{d\phi} = \frac{m+n}{n} r^2 \sin\theta \frac{d}{dr} \cdot (r^{-i+1} Y_i') \quad \text{.....(14')}.$$

Substitute
$$\frac{d \cdot ur \sin\theta}{d\phi} = -\frac{\rho}{m+n} \sin^2\theta r^{-i+1} V_i' + \sin^2\theta r^{-i+1} Y_i' - \sin^2\theta \frac{d \cdot ur^2}{dr} - \sin\theta \frac{d \cdot vr \sin\theta}{d\theta},$$
and get
$$\frac{d^2}{dr^2} \cdot ur^2 + \frac{1}{\sin\theta} \frac{d}{d\theta} \cdot \sin\theta \frac{du}{d\theta} + \frac{1}{\sin^2\theta} \frac{d^2u}{d\phi^2} = (i-1) \frac{\rho}{m+n} r^{-i} V_i' + Y_i' r^{-i} \left\{\frac{m}{n}(i+1) + 2\right\}.$$

Assume $u = u_i' r^{-i}$.....(18'), and get
$$(i-2)(i-1) u_i' + \frac{1}{\sin\theta} \frac{d}{d\theta} \cdot \sin\theta \frac{du_i'}{d\theta} + \frac{1}{\sin^2\theta} \frac{d^2u_i'}{d\phi^2} = (i-1) \frac{\rho}{m+n} V_i' + Y_i' \left\{\frac{m}{n}(i+1) + 2\right\} \ldots (19').$$

The solution obviously is
$$u_i' = \frac{-\rho}{m+n} \frac{(i-1) V_i'}{2(2i-1)} - Y_i' \frac{m(i+1) + 2n}{2n(i-1)} + Z'_{i-2} \quad \text{.....(20')},$$
where Z'_{i-2} is an arbitrary surface harmonic of degree $i-2$. Thus the additional part of u, not included in (21), which involves a surface harmonic of the ith degree is
$$u = \frac{-\rho}{m+n} \frac{(i-1) r^{-i} V_i'}{2(2i-1)} - r^{-i} Y_i' \frac{m(i+1) + 2n}{2n(2i-1)} + r^{-i+2} Z_i' \quad \text{.....(21')},$$
where Z_i' is an arbitrary surface harmonic of degree i.

Treating in precisely the same way as we treated (15) the equation, which is here the equivalent of (15), viz.
$$\frac{d\mathfrak{A}}{d\phi} - \frac{d\mathfrak{C}}{dr} = \frac{m+n}{n} \sin\theta \frac{d}{d\theta} \cdot r^{-i-1} Y_i' \quad \text{.....(15')},$$
we get
$$\frac{d^2 \cdot vr \sin\theta}{dr^2} + \frac{1}{\sin\theta} \frac{d}{d\theta} \cdot \sin\theta \frac{d \cdot vr \sin\theta}{d\theta} + \frac{1}{\sin^2\theta} \frac{d^2 \cdot vr \sin\theta}{d\phi^2}$$
$$= \frac{-\rho}{m+n} \frac{ir^{-i+1}}{2i-1} \left\{\sin\theta \frac{dV_i'}{d\theta} + (i+1)\cos\theta V_i'\right\} - \frac{m(i-2) - 2n}{(2i-1)n} \left\{\sin\theta \frac{dY_i'}{d\theta} + (i+1)\cos\theta Y_i'\right\} r^{-i+1}$$
$$+ r^{-i+1} \left\{2(i-2)\cos\theta Z'_{i-2} - 2\sin\theta \frac{dZ'_{i-2}}{d\theta}\right\} \quad \text{.....(22')}.$$

Assume $v \sin\theta = r^{-i} v_i'$, and we get
$$(i-1) i v_i' + \frac{1}{\sin\theta} \frac{d}{d\theta} \cdot \sin\theta \frac{dv_i'}{d\theta} + \frac{1}{\sin^2\theta} \frac{d^2 v_i'}{d\phi^2} = \frac{-\rho i}{(m+n)(2i-1)} \left\{\sin\theta \frac{dV_i'}{d\theta} + (i+1)\cos\theta V_i'\right\}$$
$$- \frac{m(i-2) - 2n}{2i-1} \left\{\sin\theta \frac{dY_i'}{d\theta} + (i+1)\cos\theta Y_i'\right\} + 2\left\{(i-2)\cos\theta Z'_{i-2} - \sin\theta \frac{dZ'_{i-2}}{d\theta}\right\} \ldots (23');$$
and we find, precisely in the same way as we did in the case of (23), that the solution is (X'_{i-1} being an arbitrary surface harmonic of degree $i-1$)
$$v_i' = \frac{\rho}{(m+n)\, 2(2i-1)} \sin\theta \frac{dV_i'}{d\theta} + \frac{m(i-2) - 2n}{2ni(2i-1)} \sin\theta \frac{dY_i'}{d\theta} - \frac{\sin\theta}{i-1} \frac{dZ_{i-2}}{d\theta} + X'_{i-1} \ldots (24').$$

Digitized by Google

Therefore the additional part of v, not included in (25), which contains the surface harmonics of degree i which appear in u, adding the typical term X_i', is

$$v = r^{-i}\frac{d}{d\theta}\left[\frac{\rho}{m+n}\frac{V_i'}{2(2i-1)} + \frac{m(i-2)-2n}{2ni(2i-1)}Y_i'\right] - \frac{r^{-i-2}}{i+1}\frac{dZ_i'}{d\theta} + \frac{r^{-i-1}X_i'}{\sin\theta} \quad \text{.....} (25'),$$

where X_i' is an arbitrary surface harmonic of degree i. Similarly, to find the additional part of w proceed exactly as before, i.e. substitute in

$$\frac{d.wr\sin\theta}{d\phi} = r^2\sin^2\theta\delta - \sin^2\theta\frac{d.ur^2}{dr} - \sin\theta\frac{d.vr\sin\theta}{d\theta}$$

the values of δ, u, v as given by (13′), (21′) and (25′), and employ the surface harmonic differential equation. We thus get a differential equation similar to (26) whose solution presents no difficulty. This solution is

$$w = \frac{1}{\sin\theta}\frac{d}{d\phi}\left[\frac{\rho}{m+n}\frac{r^{-i}V_i'}{2(2i-1)} + \frac{m(i-2)-2n}{2ni(2i-1)}r^{-i}Y_i' - \frac{r^{-i-2}}{i+1}Z_i'\right] + w_i'r^{-i-1} \quad \text{.....} (27'),$$

where
$$\frac{dw_i'}{d\phi} = -\frac{dX_i'}{d\theta}.$$

It must be understood that the values of v and w given by (25′) and (27′) do not profess to be solid harmonics of the ith or any degree, or putting $r = a$ constant to be as they stand surface harmonics. The terms merely represent part of a series, a Σ being understood. This form is adopted as showing conveniently the parts of the solution for u, v and w depending on certain surface harmonics of the ith degree, i.e. on V_i', Y_i', Z_i', X_i'.

If we deal with purely normal forces or with Legendre's coefficients, $X_i' = 0 = w_i'$.

In the complete solution of a spherical shell the value of δ is given by (13) and (13′), that of u by (21) and (21′), that of v by (25) and (25′), and that of w by (27) and (27′).

Thus a type of the complete solution for a spherical shell is

$$u = -r^{i+1}\left\{\frac{\rho(i+2)V_i}{2(2i+3)(m+n)} + \frac{mi-2n}{2n(2i+3)}Y_i\right\} + r^{i-1}Z_i$$
$$-r^{-i}\left\{\frac{\rho}{m+n}\frac{(i-1)V_i'}{2(2i-1)} + \frac{m(i+1)+2n}{2n(2i-1)}Y_i'\right\} + r^{-i-2}Z_i' \quad \text{.....} (36\text{a});$$

$$v = \frac{-r^{i+1}}{2(2i+3)}\frac{d}{d\theta}\left\{\frac{\rho}{m+n}V_i + \frac{m(i+3)+2n}{n(i+1)}Y_i\right\} + \frac{r^{i-1}}{i}\frac{dZ_i}{d\theta} + \frac{r^i}{\sin\theta}X_i$$
$$+\frac{r^{-i}}{2(2i-1)}\frac{d}{d\theta}\left\{\frac{\rho}{m+n}V_i' + \frac{m(i-2)-2n}{ni}Y_i'\right\} - \frac{r^{-i-2}}{i+1}\frac{dZ_i'}{d\theta} + \frac{r^{-i-1}X_i'}{\sin\theta} \quad \text{...} (37\text{a});$$

$$w = \frac{-r^{i+1}}{2(2i+3)}\frac{1}{\sin\theta}\frac{d}{d\phi}\left\{\frac{\rho}{m+n}V_i + \frac{m(i+3)+2n}{n(i+1)}Y_i\right\} + \frac{r^{i-1}}{i\sin\theta}\frac{dZ_i}{d\phi} + r^iw_i$$
$$+\frac{r^{-i}}{2(2i-1)}\frac{1}{\sin\theta}\frac{d}{d\phi}\left\{\frac{\rho}{m+n}V_i' + \frac{m(i-2)-2n}{ni}Y_i'\right\} - \frac{r^{-i-2}}{i+1}\frac{dZ_i'}{\sin\theta\, d\phi} + r^{-i-1}w_i' \quad \text{...} (38\text{a}).$$

The expression for u gives all the terms which, when r is constant, are surface spherical harmonics of degree i, and in v and w are exhibited those terms which depend on the

Digitized by Google

above surface harmonics of degree i in u, with the addition of the typical term X_i. For the complete general solution we must write a Σ before each of the above expressions.

§ 16. In the most general case when the surface forces are not purely normal there are six series of quantities Y_i, Z_i, X_i, Y_i', Z_i', X_i' to be determined, and there are six series of equations to determine them, three at each surface. If the forces be purely normal exactly the same method as for a solid sphere shows that $X_i = 0 = X_i'$. In this case, as in a solid sphere, the two surface conditions (3) and (4), besides showing that $X_i = 0 = X_i'$, are of the form $\frac{d}{d\theta}(\Sigma L_i) = 0$ and $\frac{1}{\sin\theta}\frac{d}{d\phi}\Sigma(L_i) = 0$, where L_i stands for all the terms of the ith degree in surface harmonics. These relations hold all over the spherical surface, and thus the solution is $L_i = 0$. Thus (3) and (4) furnish at each surface only one complete equation for the surface harmonics of any one given degree.

The surface condition (2) gives one similar equation at each surface. Thus we can regard Y_i, Z_i, Y_i', Z_i' as four independent variables given by four simple equations, and so we can find them all in terms of V_i, V_i', S_i, S_i'; S_i' being the normal traction on the interior boundary.

§ 17. Let us then consider more fully the case when the surface pressures are purely normal. The additional terms for u, v, w in (21′), (25′) and (27′), putting $X_i' = 0 = w_i'$, supply towards the boundary condition (2) the terms

$$\frac{-r^{-i-1}}{2i-1}\left[\{m(2i-1) - n(i^2+i-1)\}\frac{\rho V_i'}{m+n} + \{m(i^2+3i-1)+n\}Y_i'\right] - 2n(i+2)r^{-i-3}Z_i' \ldots (41).$$

Towards the expression (3) they supply

$$\frac{d}{d\theta}\left[\frac{-r^{-i-1}}{2i-1}\left\{\frac{n\rho}{m+n}iV_i' + \frac{m(i^2-1)-n}{i}Y_i'\right\} + \frac{2(i+2)}{i+1}nr^{-i-3}Z_i'\right] \ldots\ldots\ldots\ldots (42),$$

and so obviously towards (4),

$$\frac{1}{\sin\theta}\frac{d}{d\phi}\left[-\frac{r^{-i-1}}{2i-1}\left\{\frac{n\rho}{m+n}iV_i' + \frac{m(i^2-1)-n}{i}Y_i'\right\} + \frac{2(i+2)}{i+1}nr^{-i-3}Z_i'\right] \ldots\ldots (43),$$

writing $r = a$ at the external boundary, and $r = a'$ at the internal.

It will be as well here to write out these surface conditions in full. They look cumbrous but in any special case are much simplified. The surface equations are, from (28) and (41), (29) and (42), (30) and (43),

$$\frac{-a^i}{2i+3}\left[\frac{\rho}{m+n}V_i\{m(2i+3)+n(i^2+i-1)\} + Y_i\{m(i^2-i-3)+n\}\right] + 2n(i-1)a^{i-2}Z_i$$
$$+\frac{a^{-i-1}}{2i-1}\left[\frac{-\rho}{m+n}V_i'\{m(2i-1)-n(i^2+i-1)\} + Y_i'\{m(i^2+3i-1)+n\}\right] - 2n(i+2)a^{-i-3}Z_i' = S_i \ldots (44),$$

$$\frac{-a^i}{2i+3}\left[\frac{\rho}{m+n}V_i n(i+1) + Y_i\frac{mi(i+2)-n}{i+1}\right] + 2n\frac{(i-1)}{i}a^{i-2}Z_i$$
$$-\frac{a^{-i-1}}{2i-1}\left[\frac{\rho}{m+n}V_i' ni + Y_i'\frac{m(i^2-1)-n}{i}\right] + 2n\frac{(i+2)}{i+1}a^{-i-3}Z_i' = 0 \ldots (45),$$

Digitized by Google

$$\frac{-a'^i}{2i+3}\left[\frac{\rho}{m+n}V_i\{m(2i+3)+n(i^2+i-1)\}+Y_i\{m(i^2-i-3)+n\}\right]+2n(i-1)a'^{i-2}Z_i$$

$$+\frac{a'^{-i-1}}{2i-1}\left[\frac{-\rho V_i'}{m+n}\{m(2i-1)-n(i^2+i-1)\}+Y_i'\{m(i^2+3i-1)+n\}\right]-2n(i+2)a'^{-i-3}Z_i'=S_i'\ldots(46),$$

$$\frac{-a'^i}{2i+3}\left[\frac{\rho}{m+n}V_in(i+1)+Y_i\frac{mi(i+2)-n}{i+1}\right]+2n\frac{(i-1)}{i}a'^{i-2}Z_i$$

$$-\frac{a'^{-i-1}}{2i-1}\left[\frac{\rho}{m+n}V_i'ni+Y_i'\frac{m(i^2-1)-n}{i}\right]+2n\frac{(i+2)}{i+1}a'^{-i-3}Z_i'=0\ldots\ldots(47).$$

These equations, supposing $i>1$*, determine Y_i, Y_i', Z_i, Z_i' without ambiguity of any kind.

§ 18. The following method gives the values of u, v and w very shortly. It is suggested by the form of (25) and (27). Supposing we have got (1 a), (13) and the solution (21), and have worked out the first side only of (22), we then get as before for the general solution of (22), selecting the term in X_i as a type,

$$v=\frac{r^iX_i}{\sin\theta}.$$

From (1 a), since this solution is independent of u or δ, we have

$$\frac{dw}{d\phi}=-\frac{d.v\sin\theta}{d\theta}=-r^i\frac{dX_i}{d\theta};$$

$$\therefore\ w=w_ir^i,\ \text{where}\ \frac{dw_i}{d\phi}=-\frac{dX_i}{d\theta}.$$

We then want particular solutions for v and w answering to

$$\delta=r^iY_i,\qquad u=-r^{i+1}\frac{mi-2n}{2n(2i+3)}Y_i;$$

$$\delta=\frac{-\rho}{m+n}r^iV_i,\quad u=-r^{i+1}\frac{\rho(i+2)V_i}{2(2i+3)(m+n)};$$

$$\delta=0,\qquad u=r^{i-1}Z_i.$$

For the first assume

$$v=f_1(r)\frac{dY_i}{d\theta},$$

$$w=f_1(r)\frac{1}{\sin\theta}\frac{dY_i}{d\phi}.$$

Then from (1 a)

$$f_1(r)\left\{\frac{1}{\sin\theta}\frac{d}{d\theta}.\sin\theta\frac{dY_i}{d\theta}+\frac{1}{\sin^2\theta}\frac{d^2Y_i}{d\phi^2}\right\}=r^{i+1}Y_i+\frac{1}{r}\frac{d}{dr}.r^{i+3}\frac{mi-2n}{2(2i+3)n}Y_i,$$

* A special case occurs when $i=1$ whether for solid sphere or shell. In the surface conditions Z_i occurs multiplied by $i-1$, and so its value would become infinite. For explanation, see paragraph r of § 737 of Thomson and Tait's *Nat. Phil.* The same remarks apply to the case of a cylinder.

Digitized by Google

i. e.
$$-i(i+1)f_1(r)\,Y_i = Y_i r^{i+1} i \frac{\frac{m}{n}(i+3)+2}{2(2i+3)},$$

$$\therefore\ f_1(r) = -r^{i+1}\frac{\frac{m}{n}(i+3)+2}{2(i+1)(2i+3)}.$$

For the second assume
$$u = f_2(r)\frac{dV_i}{d\theta},$$

$$w = f_2(r)\frac{1}{\sin\theta}\frac{dV_i}{d\phi},$$

and precisely as before we get from (1 a)
$$f_2(r) = \frac{-\rho}{m+n}r^{i+1}\frac{1}{2(m+n)(2i+3)}.$$

For the third assume
$$v = f_3(r)\frac{dZ_i}{d\theta},$$

$$w = f_3(r)\frac{1}{\sin\theta}\frac{dZ_i}{d\phi},$$

and from (1 a) we get
$$f_3(r) = \frac{1}{i}r^{i-1}.$$

But
$$v = f_1(r)\frac{dY_i}{d\theta} + f_2(r)\frac{dV_i}{d\theta} + f_3(r)\frac{dZ_i}{d\theta} + \frac{r^i X_i}{\sin\theta},$$

which, on substituting for the functions of r, we see is identical with (25). Similarly (27) is given by the above values for w. The same method applies equally shortly in the case of a shell.

The forms assumed for v and w in this method would scarcely suggest themselves unless we had already found them by some direct method, so that the process is from one point of view unsatisfactory though practically serviceable.

§ 19. The case $i = 0$ seems slightly different from that of the general solution, but is really included in it.

It differs mainly in the number of arbitrary constants, but, as we shall see presently, there are enough to satisfy the surface conditions.

To save space let us consider first a shell the radii of whose surfaces are a and a'. From this we can pass at once to a solid sphere.

From (13) we have $\delta = \text{constant} = Y_0$ .....(48).

Digitized by Google

The surface forces are purely normal, call them F and F'. Supposing no bodily forces to act, we get from (21) and (21′), leaving out V_0 and V_0',

$$u = \frac{1}{3} r Y_0 + \frac{1}{r^2} Z_0' \quad \text{.....(49).}$$

Z_0 is zero, for as it is a constant there are no corresponding terms in v or w, and in (1) it is impossible that δ should vanish while $u = \frac{Z_0}{r}$. Also Y_0' is zero, for $\delta = \frac{Y_0'}{r}$ is inconsistent with (5).

[Of course the above value of u follows directly from $\delta = \frac{1}{r^2} \frac{d.ur^2}{dr} = Y_0$; but it is desirable to show that it is contained in the general solution.]

Noticing that now $\quad (m-n)\,\delta + 2n \frac{du}{dr} = (m+n)\,\delta - 4n \frac{u}{r}$,

it will be seen from (30), Sect. I., that the surface conditions are

$$(m+n)\,\delta - 4n \frac{u}{r} = F, \text{ when } r = a, \quad \text{.....(50),}$$

$$= F', \text{ when } r = a', \quad \text{.....(51).}$$

We get thence

$$Y_0 = \frac{a^3 F - a'^3 F'}{\left(m - \frac{n}{3}\right)(a^3 - a'^3)},$$

$$Z_0' = \frac{a^3 a'^3 (F - F')}{4n\,(a^3 - a'^3)};$$

therefore

$$u = \frac{1}{a^3 - a'^3} \left[\frac{a^3 F - a'^3 F'}{3m - n} r + \frac{a^3 a'^3 (F - F')}{4nr^2} \right] \quad \text{.....(52),}$$

$$\delta = \frac{a^3 F - a'^3 F'}{\left(m - \frac{n}{3}\right)(a^3 - a'^3)} \quad \text{.....(53).}$$

For a solid sphere put $a' = 0$; F' goes out as it should, and

$$u = \frac{Fr}{3m - n} \quad \text{.....(54),}$$

$$\delta = \frac{F}{m - \frac{n}{3}} \quad \text{.....(55).}$$

Suppose next that the shell is very thin, having $a' = a\,(1 - \epsilon)$, where ϵ^2 is negligible. Then,

$$u = \left(\frac{F - F'}{3\epsilon} + F'\right) \frac{r}{3m - n} + (F - F') \left(\frac{1}{3\epsilon} - 1\right) \frac{1}{4nr^2} \quad \text{.....(56),}$$

$$\delta = \left(\frac{F - F'}{3\epsilon} + F'\right) \frac{1}{m - \frac{n}{3}} \quad \text{.....(57).}$$

Thus unless $F - F'$ be very small, u is very large.

Digitized by Google

If $F = F'$, $u = \dfrac{Fr}{3m - n}$, $\delta = \dfrac{F}{m - \dfrac{n}{3}}$. This is exactly the same expression as for a solid sphere, as is obvious, since in the solid sphere $(m+n)\,\delta - 4m\dfrac{u}{r}$ is constant throughout.

[*April* 20, 1888. If the tangential surface forces instead of themselves being surface harmonics be derived from a potential expressed in harmonics, so that $T_i = \dfrac{d\Omega_i}{d\theta}$ and $U_i = \dfrac{1}{\sin\theta}\dfrac{d\Omega_i}{d\phi}$, where Ω_i represents the ith harmonic terms, then by (40) we have $X_i = 0$; also the Σ can be dropped on both sides of (31 a), the right-hand side reducing to $-i(i+1)\,\Omega_i/n$. In this important case the solution is thus obtainable explicitly. The additional terms to be added to (36), (37) and (38) being respectively

$$u = M_i i(i+1)\,\Omega_i\left[(i-1)(mi-2n)\,r^{i+1}a^{-i} - \{m(i^2-i-3)+n\}\,r^{i-1}a^{-i+2}\right],$$

$$v = M_i N_i \frac{d\Omega_i}{d\theta}, \qquad w = \frac{M_i N_i}{\sin\theta}\frac{d\Omega_i}{d\phi};$$

where for shortness

$$1/M_i \equiv 2n(i-1)\{m(2i^2+4i+3) - n(2i+1)\},$$

$$N_i \equiv i(i-1)\{m(i+3)+2n\}\,r^{i+1}a^{-i} - (i+1)\{m(i^2-i-3)+n\}\,r^{i-1}a^{-i+2}.$$

With similar surface forces the case of a shell can also be solved explicitly.]

SECTION III.

TENDENCY TO RUPTURE.

§ 20. One of the most interesting questions in the problems we are to be engaged in is the tendency of the body to rupture which we shall now consider. It is fairly obvious that the tendency to rupture at a point inside a solid will depend only on the state of stress or of strain in the material surrounding that point, and not at all on the nature of the external forces which produce the strain, nor on the form of the external surface of the solid.

Various measures of the tendency to rupture have been advanced. Lamé considers that a body will rupture when the greatest stress rises to a certain value, obtained experimentally for each substance. On this hypothesis the tendency to rupture would be measured by the greatest stress. By others, e.g. Professor Darwin, the difference between the greatest and least principal stresses, called the "*maximum stress-difference*," has been taken as giving the tendency to rupture. On this hypothesis the solid would rupture when the maximum stress-difference at any point reached a critical value determined experimentally for each substance. By others, e.g. Saint-Venant, the greatest strain, provided it be positive, has been taken as the measure. On this hypothesis if the strain exceed a certain value, obtained experimentally, the solid will rupture. In all three methods the actual critical value will be modified according to the different ends aimed

Digitized by Google

at. If the stress is to act only for a short time a higher limit may be taken, but if the stress is to be long continued it must never reach a point at which it may begin to cause deterioration of the substance of the solid.

In the first and third methods the limit might be obtained by experiments with the traction of a bar of uniform cross section. In the second method the limit would be best obtained by experiments on shearing force.

In some cases, for instance in the simple case of longitudinal traction of a bar where there is only one stress, the different methods coalesce.

I consider the last two views much more probable than the first, but do not venture to assign the palm to either. In fact I am doubtful if certain cases do not throw doubt on the universal application of any such measure.

§ 21. It may seem premature to consider the question at all while holding these views, but so long as the physical basis is clearly stated no possible harm can ensue from the mathematical developments. The results we obtain may possibly even be of use in showing how to discriminate experimentally between the different theories. Further any one of these different methods requires the greatest principal stress to be known, or is most easily applied when a knowledge of this stress is obtained, so our results will be at once applicable to any of the methods.

To show this for the third method, suppose e, f, g the three principal strains, P, Q, R the three principal stresses, and $\delta \equiv e+f+g$.

Then if P be the algebraically greatest stress, e is the algebraically greatest strain, and the relation between them is

$$P = (m-n)\,\delta + 2ne,$$

or

$$e = \frac{P-(m-n)\,\delta}{2n} \quad \text{.....(1).}$$

If then we know δ and P we at once know e.

For our future applications we require a knowledge of the *stress-quadric*. For the properties of this quadric see Thomson and Tait's *Natural Philosophy*, Vol. I. Part II. §§ 662—665.

The greatest and least principal stresses act along the least and greatest principal axes of this quadric, the properties of which are of course independent of the system of axes employed. In the cases we are to consider one of the principal stresses Φ is perpendicular to the meridian plane, the other principal stresses R_1 and Θ_1 lie in the meridian plane. If the equation of the stress quadric referred to our usual system of axes be

$$Rx^2 + \Theta y^2 + 2R_\theta xy + \Phi z^2 = 1 \quad \text{.....(2),}$$

then referred to the principal stress axes it is

$$R_1 x^2 + \Theta_1 y^2 + \Phi z^2 = 1 \quad \text{.....(3);}$$

Digitized by Google

and by ordinary geometry

$$\left.\begin{aligned} R_1 &= \frac{R+\Theta+S}{2} \\ \Theta_1 &= \frac{R+\Theta-S}{2} \end{aligned}\right\} \quad \text{.....(4)},$$

where

$$S = \sqrt{(R-\Theta)^2 + 4R_\theta^2} \quad \text{.....(5)}.$$

We shall suppose this square root always taken positively, so that R_1 is the algebraically greater principal stress in the meridian plane.

It is easily seen from (33), (36) and (37) of Section I. that

$$S^2 = 4n^2\left\{\left(r\frac{d\frac{u}{r}}{dr} - \frac{1}{r}\frac{dv}{d\theta}\right)^2 + \left(r\frac{d\frac{v}{r}}{dr} + \frac{1}{r}\frac{du}{d\theta}\right)^2\right\} \quad \text{.....(6)}.$$

If α be the angle which the axis of R_1 makes with the radius vector drawn outwards,

$$\frac{\cos\alpha}{R_\theta} = \frac{2\sin\alpha}{S+\Theta-R}.$$

The greatest stress algebraically is thus either R_1 or Φ, and the maximum stress-difference is S, $R_1 - \Phi$, or $\Phi - \Theta_1$, according as Φ is the mean, the algebraically least, or the algebraically greatest principal stress.

We have

$$\Phi = (m-n)\,\delta + 2n\left(\frac{u}{r} + \frac{v}{r}\cot\theta\right) \quad \text{.....(7)},$$

$$R_1 + \Theta_1 = R + \Theta = (3m-n)\,\delta - \Phi \quad \text{.....(8)},$$

and

$$R_1 - \Theta_1 = S \quad \text{.....(9)};$$

whence we get

$$\left.\begin{aligned} 2(\Phi - R_1) &= 2n\left\{3\left(\frac{u}{r} + \frac{v}{r}\cot\theta\right) - \delta\right\} - S \\ 2(\Phi - \Theta_1) &= 2n\left\{3\left(\frac{u}{r} + \frac{v}{r}\cot\theta\right) - \delta\right\} + S \end{aligned}\right\} \quad \text{.....(10)}.$$

We can divide our results into three cases.

Case (i). $3\left(\frac{u}{r} + \frac{v}{r}\cot\theta\right) - \delta$ positive, and at the same time $\left\{3\left(\frac{u}{r} + \frac{v}{r}\cot\theta\right) - \delta\right\}^2$ numerically greater than $\left(\frac{S}{2n}\right)^2$. In this case $\Phi - R_1$ is positive, thus Φ is the algebraically greatest stress and the maximum stress-difference is

$$\Phi - \Theta_1 = n\left\{3\left(\frac{u}{r} + \frac{v}{r}\cot\theta\right) - \delta\right\} + \frac{S}{2} \quad \text{.....(11)}.$$

Case (ii). $3\left(\frac{u}{r} + \frac{v}{r}\cot\theta\right) - \delta$ negative, and at the same time $\left\{3\left(\frac{u}{r} + \frac{v}{r}\cot\theta\right) - \delta\right\}^2$ numerically greater than $\left(\frac{S}{2n}\right)^2$. In this case Φ is algebraically the least principal stress,

Digitized by Google

and the maximum stress-difference is

$$R_1 - \Phi = \frac{S}{2} - n\left\{3\left(\frac{u}{r} + \frac{v}{r}\cot\theta\right) - \delta\right\} \quad \text{.....(12).}$$

Case (iii). $\left\{3\left(\frac{u}{r} + \frac{v}{r}\cot\theta\right) - \delta\right\}^2$ numerically less than $\left(\frac{S}{2n}\right)^2$. In this case Φ is the mean principal stress and the maximum stress-difference is S.

In what we have called the third, or Saint-Venant's, method the tendency to rupture is in Case (i) measured by $\frac{\Phi - (m-n)\delta}{2n}$, if positive, i.e. by

$$\frac{u}{r} + \frac{v}{r}\cot\theta \quad \text{.....(13);}$$

in Cases (ii) and (iii) by $\dfrac{\frac{R+\Theta+S}{2} - (m-n)\delta}{2n}$, if positive, i.e. by

$$\frac{S}{4n} + \frac{1}{2}\left\{\delta - \left(\frac{u}{r} + \frac{v}{r}\cot\theta\right)\right\} \quad \text{.....(14).}$$

It will be found that in general there is a surface or series of surfaces over which $\Phi = R_1$ or $= \Theta_1$. These surfaces are of importance on any of the three theories, as they indicate the limits within which the several cases mentioned above apply. The surfaces themselves may be regarded as loci where Case (iii) applies.

§ 22. As an easy illustration of the hypotheses as to the rupture of a solid we may consider the solutions (52) and (54) of Section II. In the case of (54), giving the displacement for a solid homogeneous sphere, $\frac{du}{dr}$ obviously $= \frac{u}{r}$, and so $R = \Theta = \Phi$; thus no stress-difference exists, and according to the second hypothesis no amount of uniform normal traction or pressure would have the least tendency to cause rupture anywhere.

The greatest strain is in either case $\frac{u}{r}$ or $\frac{du}{dr}$ indifferently, and is positive or negative according as F is a traction or a pressure. Thus on the third hypothesis no amount of uniform normal pressure tends to produce rupture, but uniform normal traction on the other hand will eventually produce rupture if $\frac{u}{r}$ exceed a certain experimental limit.

Thus both theories agree in stating that no amount of uniform normal pressure will rupture a solid sphere; but they differ completely as to the effects of uniform normal traction. There is no very obvious experimental method of testing the results of a uniform traction, but it certainly seems probable that rupture would follow from a very severe traction.

Greater interest attaches to the result (52), Sect. II., which is for the case of a shell, and is

$$u = \frac{1}{a^3 - a'^3}\left[\frac{a^3F - a'^3F'}{3m-n}\,r + \frac{a^3a'^3(F-F')}{4nr^2}\right].$$

Digitized by Google

Here
$$\Theta = \Phi = (m-n)\,\delta + 2n\frac{u}{r},$$
and
$$R = (m-n)\,\delta + 2n\frac{du}{dr};$$
therefore R is algebraically greater or less than Θ according as $F - F'$ is negative or positive.

The greatest value of $R \sim \Theta$, i.e. the maximum stress-difference, occurs when $r = a'$, or at the inner surface of the shell, and is
$$\frac{3}{2}a^3\frac{F \sim F'}{a^3 - a'^3} \quad\text{.....(15)},$$
taken always positive.

As being of greater practical interest we shall suppose $F = -p$, $F' = -p'$, so that p and p' represent uniform pressures, e.g. of a gas, on the outside and inside boundaries. Then the greatest strain is $\frac{du}{dr}$ or $\frac{u}{r}$ according as p is greater or less than p'.

If $p > p'$ the greatest value of $\frac{du}{dr}$ occurs when $r = a'$, and is
$$\frac{3(m-n)a^3(p-p') - 2np'(a^3 - a'^3)}{4n(3m-n)(a^3 - a'^3)} \quad\text{.....(16)}.$$

On the third hypothesis no tendency to rupture will exist unless this be positive, i.e. unless
$$\frac{p-p'}{p'} > \frac{2n}{3(m-n)}\left(1 - \frac{a'^3}{a^3}\right) \quad\text{.....(17)}.$$

Thus no rupture can ensue however great p may be unless the ratio of $p : p'$ exceed a certain value depending on the ratio of the limiting radii of the shell.

If $p < p'$ the greatest strain is $\frac{u}{r}$, and its greatest value occurs when $r = a'$, and is
$$\frac{3(m+n)a^3(p'-p) - 4np'(a^3 - a'^3)}{4n(3m-n)(a^3 - a'^3)} \quad\text{.....(18)}.$$

This will be positive only provided
$$\frac{p}{p'} < 1 - \frac{4n}{3(m+n)}\left(1 - \frac{a'^3}{a^3}\right) \quad\text{.....(19)}.$$

Thus however great p' may be there will be no tendency to rupture unless the ratio $p : p'$ be less than a certain proper fraction depending on the ratio of the limiting radii of the shell.

Thus what we have called the second and third hypotheses agree in stating that the tendency to rupture, if existent, is always greatest at the inner surface of the shell. The hypothesis of maximum stress-difference states that in all cases rupture will ensue if the difference between the forces on the outside and inside surfaces be sufficiently

Digitized by Google

increased, while the hypothesis of maximum strain requires certain relations between the ratios of these surface forces.

This seems to afford means of experimentally deciding between the two theories.

SECTION IV.

GRAVITATING NEARLY SPHERICAL MASS.

§ 23. The results of previous sections admit of an easy application to the case of a nearly spherical solid gravitating mass the equation to whose surface is

$$r = a + \Sigma a_i \sigma_i \text{.....} (1),$$

where σ_i represents any spherical harmonic terms of the ith degree, and a_i is a type of constants which are small compared to a. The earth regarded as isotropic exemplifies the kind of body meant.

Owing to the mutual attraction of the material, bodily forces exist given by a potential

$$-\tfrac{1}{2}g\frac{r^2}{a} + \Sigma \frac{3ga_i\sigma_i r^i a^{-i}}{2i+1},$$

where g represents the mean value of "gravity" over the surface.

Since $\frac{a_i}{a}$, etc. are small the direction cosines of the normal, referred to the fundamental directions at each point of the surface, may be taken as

$$\lambda = 1, \quad \mu = -\Sigma \frac{a_i}{a}\frac{d\sigma_i}{d\theta}, \quad \nu = -\Sigma \frac{a_i}{a}\frac{1}{\sin\theta}\frac{d\sigma_i}{d\phi} \text{.....} (2).$$

In what follows the surface is supposed free of force, so the surface conditions will be got from (27), (28) and (29) of Sect. I. by using the values (2) and putting

$$F = G = H = 0.$$

We may without increasing the difficulty of the problem suppose the bodily forces of a more general kind than those following from the potential given above. Suppose them to be given by the potential

$$-\tfrac{1}{2}g\frac{r^2}{a} + \Sigma a_i V_i \sigma_i r^i;$$

then as a type of the strains depending on spherical harmonics of the ith degree we have (21), (25) and (27) of Sect. II., writing $a_i V_i \sigma_i$ for V_i. It will also be convenient to write $a_i Y_i \sigma_i$ and $a_i Z_i \sigma_i$ in those expressions for Y_i and Z_i respectively, so that V_i, Y_i and Z_i in what follows represent constants merely. Also the X_i and w_i of (25) and (27) Sect. II. are not required.

Digitized by Google

§ 24. Answering to the part $-\frac{1}{2}g\frac{r^2}{a}$ of the potential, we find from (5) Sect. II., for the particular solution

$$\delta = \frac{1}{2}\frac{g\rho r^2}{a(m+n)}.$$

Answering to which, as will be seen from (1) Sect. II., we get

$$u = \frac{g\rho r^2}{10a(m+n)}.$$

We have also from the part of the general solution considered in (48) and (49) Sect. II.,

$$\delta = Y_0,$$
$$u = \frac{1}{3}rY_0.$$

Thus the complete solution is

$$u = \frac{g\rho r^2}{10a(m+n)} + \frac{1}{3}rY_0 - \Sigma a_i\sigma_i\left[\frac{r^{i+1}}{2(2i+3)}\left\{\frac{\rho(i+2)}{m+n}V_i + \frac{mi-2n}{n}Y_i\right\} - r^{i-1}Z_i\right] \ldots\ldots (3),$$

$$\left.\begin{aligned} v &= \Sigma\frac{d\psi_i}{d\theta} \\ w &= \Sigma\frac{1}{\sin\theta}\frac{d\psi_i}{d\phi} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

where

$$\psi_i = a_i\sigma_i\left[\frac{-r^{i+1}}{2(2i+3)}\left\{\frac{\rho}{m+n}V_i + \frac{m(i+3)+2n}{n(i+1)}Y_i\right\} + \frac{r^{i-1}}{i}Z_i\right] \ldots\ldots\ldots\ldots (5),$$

$$\delta = \frac{1}{2}\frac{g\rho r^2}{a(m+n)} + Y_0 + \Sigma r^i a_i\sigma_i\left[\frac{-\rho}{m+n}V_i + Y_i\right] \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

§ 25. Neglecting terms of order $\left(\frac{a_i}{a}\right)^2$, it will be seen from the above expressions for the strains and for the direction cosines of the normals to the surface, that the first surface condition (27) Sect. I. reduces to

$$(m-n)\delta + 2n\frac{du}{dr} = 0, \text{ where } r = a + \Sigma a_i\sigma_i.$$

From (3) and (6) we thence obtain, consulting the left-hand side of (28) Sect. II.,

$$\frac{g\rho a(5m+n)}{10(m+n)}\left\{1 + 2\Sigma\frac{a_i}{a}\sigma_i\right\} + \frac{3m-n}{3}Y_0$$
$$-\Sigma a_i\sigma_i\left[\frac{m(2i+3)+n(i^2+i-1)}{(m+n)(2i+3)}\rho a^i V_i + \frac{m(i^2-i-3)+n}{2i+3}a^i Y_i - 2n(i-1)a^{i-2}Z_i\right] = 0.$$

Equating to zero the constant part we get

$$Y_0 = -\frac{3g\rho a(5m+n)}{10(m+n)(3m-n)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7),$$

and equating separately to zero the coefficients of the harmonics of different degrees, taking the ith as a type, we find

$$\frac{m(2i+3)+n(i^2+i-1)}{(m+n)(2i+3)}\rho a^i V_i + \frac{m(i^2-i-3)+n}{2i+3}a^i Y_i - 2n(i-1)a^{i-2}Z_i - \frac{g\rho(5m+n)}{5(m+n)} = 0 \ldots (8).$$

Digitized by Google

In the second surface condition, (28) Sect. I., all the terms are of the order $\frac{a_i}{a}$ at least, thus we may write $r=a$ over the free surface.

Substituting for Y_0 from (7), and making the terms depending on the differential with respect to θ of each surface harmonic separately vanish we get, with the aid of the left-hand side of (31) Sect. II., equations of which a type is

$$\frac{n}{m+n}\frac{i+1}{2i+3}\rho a^i V_i+\frac{mi(i+2)-n}{(i+1)(2i+3)}a^i Y_i-\frac{2n(i-1)}{i}a^{i-2}Z_i-\frac{2g\rho n}{5(m+n)}=0 \quad\ldots\ldots\ldots (9).$$

Treating the third surface condition, (29) Sect. I., similarly we find that the coefficient of $\frac{-a_i}{\sin\theta}\frac{d\sigma_i}{d\phi}$ is simply the left-hand side of (9). Thus by making the expressions of which this is a type vanish we satisfy the third surface condition as well as the second, and this is the only way we can satisfy both conditions.

Putting for shortness

$$5(m+n)\{m(2i^2+4i+3)-n(2i+1)\}=D_i \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10),$$

we find from (8) and (9),

$$Y_i=\frac{\rho(i+1)}{D_i}[5\{m(2i+3)-n\}V_i-(2i+3)\{5m-n(2i-1)\}ga^{-i}] \quad\ldots\ldots\ldots\ldots (11),$$

$$Z_i=\frac{\rho i}{2n(i-1)D_i}[5(m+n)i\{m(i+2)-n\}a^2V_i$$
$$-\{5m^2i(i+2)-mn(2i^3-i^2-10i-1)-n^2(2i+3)\}ga^{-i+2}] \ldots (12).$$

For the case we are more specially interested in when $V_i=\frac{3g}{2i+1}a^{-i}$, these expressions become

$$Y_i=\frac{-2g\rho a^{-i}(i^2-1)}{(2i+1)D_i}\{5m(2i+3)-n(4i^2+10i+9)\},$$

$$Z_i=\frac{-ig\rho a^{-i+2}}{2n(2i+1)D_i}\{10m^2i(i+2)-mn(4i^3+4i^2-2i+1)-n^2(4i-3)\}.$$

Substituting these values in (3), (4), (5) and (6) we obtain for the strains in the nearly spherical body (1) acted on only by its gravitational forces

$$u=\frac{g\rho r}{10a(m+n)}\left\{r^2-\frac{a^2(5m+n)}{3m-n}\right\}$$

$$+g\rho\Sigma\frac{a_i\sigma_i}{2n(2i+1)D_i}[r^{i+1}a^{-i}\{10m^2i(i^2-1)-mn(4i^4+4i^3+34i^2+29i+10)+n^2(8i^3+8i^2+13i-2)\}$$
$$-r^{i-1}a^{-i+2}i\{10m^2i(i+2)-mn(4i^3+4i^2-2i+1)-n^2(4i-3)\}] \ldots\ldots\ldots\ldots (13),$$

$$\left.\begin{aligned} v&=\Sigma\frac{d\psi_i}{d\theta}\\ w&=\Sigma\frac{1}{\sin\theta}\frac{d\psi_i}{d\phi}\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14),$$

Digitized by Google

where

$$\psi_i = \frac{g\rho a_i \sigma_i}{2n(2i+1)D_i}\left[r^{i+1}a^{-i}\{10m^2(i-1)(i+3) - mn(4i^3+12i^2-6i+17) - n^2(8i^2-17)\}\right.$$
$$\left. - r^{i-1}a^{-i+2}\{10m^2 i(i+2) - mn(4i^3+4i^2-2i+1) - n^2(4i-3)\}\right] \ldots\ldots(15),$$

$$\delta = \frac{g\rho}{10a(m+n)}\left\{5r^2 - \frac{3a^2(5m+n)}{3m-n}\right\}$$
$$- g\rho\Sigma\frac{a_i\sigma_i r^i a^{-i}}{(2i+1)D_i}\left[5m(4i^3+12i^2+8i+3) - n(8i^4+20i^3+10i^2+10i-3)\right] \ldots\ldots(16).$$

In every case Σ denotes summation with respect to i, and σ_i represents all the surface harmonics of the ith degree that occur in the equation to the free surface.

The fact that the magnitude of the strain depends only on the degree of the surface harmonic and not on its special form seems worthy of notice.

§ 26. This problem has been treated by an entirely different method by Professor Darwin*, mainly for the special case $\frac{n}{m} = 0$. He has applied his results to the determination of the tendency to rupture, on the stress-difference theory, in the earth regarded as an isotropic body. Except in this case and on this theory, with an exception presently to be noticed, the tendency to rupture will depend only to a trifling extent on the divergence of the earth from a truly spherical form. It will usually depend mainly on the strain

$$u = \frac{g\rho r}{10a(m+n)}\left\{r^2 - \frac{a^2(5m+n)}{3m-n}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17),$$

which is very large compared to the other terms.

Neglecting for a moment the other terms in the strain, we should get for our principal stresses

$$R = \frac{g\rho(5m+n)(r^2-a^2)}{10(m+n)a},$$

$$\Theta = \Phi = \frac{g\rho\{(5m-3n)r^2 - (5m+n)a^2\}}{10(m+n)a}.$$

Thus the maximum stress-difference is

$$R - \Theta = \frac{2g\rho n r^2}{5(m+n)a} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18),$$

and the greatest strain

$$\frac{du}{dr} = \frac{g\rho}{10(m+n)a}\left\{3r^2 - a^2\frac{5m+n}{3m-n}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(19).$$

Both these expressions have their greatest values at the surface. These values are respectively

$$\frac{2g\rho n a}{5(m+n)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18\text{ a}),$$

and

$$\frac{2g\rho(m-n)a}{5(m+n)(3m-n)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19\text{ a}).$$

* *Philosophical Transactions*, 1882, pp. 187—230.

Digitized by Google

§ 27. In the special case $\frac{n}{m}=0$ it is thus correct to neglect the stress-difference arising from these terms, but in an ordinary elastic solid this stress-difference would have in the circumstances of the earth an enormous value. For instance, if following most foreign elasticians we suppose $m=2n$, these greatest values of the maximum stress-difference and greatest strain would be the same as in wires of the earth's mean structure exposed to tractions equal to the weights at the earth's surface of lengths of their material exceeding 500 and 250 miles respectively. In fact on neither theory could an earth of ordinary elastic solid structure exist for an instant. On the stress-difference theory the earth regarded as isotropic could remain a solid only if $\frac{n}{m}$ were extremely small.

From (17) we see that $\frac{u}{r}$ is everywhere negative, and from (19) $\frac{du}{dr}$ is positive only at distances from the centre exceeding r_1, where

$$r_1=a\left\{\frac{5m+n}{3(3m-n)}\right\}^{\frac{1}{2}} \qquad (20).$$

So on the maximum strain theory it is only at distances from the centre exceeding r_1 that there is any tendency to rupture. As $\frac{n}{m}$ increases from 0 to 1, r_1 increases from $\cdot 74a$ to a, while for the mean value $m=2n$ we have $r_1=\cdot 86a$.

So on this theory there would be in the earth, regarded as isotropic, no tendency to rupture except at a distance from the centre exceeding a certain fraction of the radius depending on the material. For the mean case $m=2n$ there would be no tendency to rupture at greater depths than 600 miles from the surface. For the limiting case $m=n$ there would, but for the deviation from a truly spherical form, be no tendency to rupture anywhere, and the smaller $\frac{m-n}{m}$ the smaller the region exposed to rupture and the less the tendency to rupture. Since $\frac{m-n}{2m}$ measures the ratio of lateral contraction to longitudinal expansion for longitudinal traction, it can scarcely vanish much less be negative; but it is at least conceivable that there are materials for which it is small. It is thus on the maximum strain theory by no means impossible that a gravitating isotropic sphere of the same size and density of the earth might remain solid throughout. It would however if composed of such a material as iron be incapable of remaining as a connected solid within distances not exceeding a few hundred miles from the surface.

§ 28. From the previous remarks it follows that there is no use in considering the tendency to rupture due to the terms represented by those in σ_i except on the stress-difference theory for the case $\frac{n}{m}$ very small and on the maximum strain theory for the case $\frac{m-n}{m}$ very small. In thus further considering the matter it will be as well to limit ourselves to the most important special case when $\sigma_i=P_2$.

Digitized by Google

Supposing then $i=2$ and neglecting terms in n compared to terms in m, disregarding for reasons already stated terms independent of P_2, we get from (13)—(16),

$$\left.\begin{aligned} u &= g\rho a_2 P_2 \frac{6r^3 - 16a^2 r}{95na^2} \\ v &= g\rho a_2 \frac{dP_2}{d\theta} \frac{5r^3 - 8a^2 r}{95na^2} \\ w &= \delta = 0 \end{aligned}\right\} \quad \text{.....(21).}$$

Whence we find
$$r\frac{d\frac{v}{r}}{dr} + \frac{1}{r}\frac{du}{d\theta} = \frac{24g\rho a_2}{95na^2}(a^2 - r^2)\sin 2\theta,$$

$$r\frac{d\frac{u}{r}}{dr} - \frac{1}{r}\frac{dv}{d\theta} = \frac{3g\rho a_2}{95na^2}\{r^2 - 8(a^2 - r^2)\cos 2\theta\}.$$

From (6) of Sect. III. it easily follows that

$$\left(\frac{95a^2 S}{6g\rho a_2}\right)^2 = (8a^2 - 9r^2)^2 + 32r^2(a^2 - r^2)\sin^2\theta \quad \text{.....(22).}$$

Thus for a given value of r, S is a maximum in the equator and a minimum in the polar axis. Further, as depending on the sum of two squares, S can vanish only at the two points in the polar axis at the distance from the centre

$$r_0 = a\left(\tfrac{8}{9}\right)^{\frac{1}{2}} \quad \text{.....(23).}$$

Again from (21) we find, noticing that δ vanishes,

$$3\left(\frac{u}{r} + \frac{v}{r}\cot\theta\right) - \delta = \frac{3g\rho a_2}{95na^2}\{8a^2 - 9r^2 + 6r^2\sin^2\theta\} \quad \text{.....(24).}$$

This expression is finite and continuous, being of the same sign as a_2 inside and of the reverse sign outside the spheroid

$$8a^2 - 9r^2 + 6r^2\sin^2\theta = 0 \quad \text{.....(25).}$$

The polar, or minor, semi-axis of this spheroid is r_0 and the equatorial is $a\left(\frac{8}{3}\right)^{\frac{1}{2}}$. The spheroid cuts the sphere, $r=a$, in points whose polar distance is

$$\theta_0 = \sin^{-1}\left(\tfrac{1}{6}\right)^{\frac{1}{2}}.$$

Since r_0 is nearly equal to a and θ_0 is small, $3\left(\frac{u}{r} + \frac{v}{r}\cot\theta\right) - \delta$ is positive throughout the whole of the sphere excepting a small superficial portion round the poles.

We have next to find the locus

$$\left\{3\left(\frac{u}{r} + \frac{v}{r}\cot\theta\right) - \delta\right\}^2 - \left(\frac{S}{2n}\right)^2 = 0,$$

which determines the surfaces over which Φ is equal either to R_1 or to Θ_1. From (22) and (24) the locus is easily found to be

$$4r^2\sin^2\theta\{16a^2 - 19r^2 + 9r^2\sin^2\theta\} = 0 \quad \text{.....(26).}$$

Digitized by Google

It thus consists of the polar axis, and the spheroid

$$16a^2 - 19r^2 + 9r^2 \sin^2\theta = 0 \quad\text{.....(27).}$$

This spheroid cuts the sphere at points whose polar distance is $\sin^{-1}(\frac{1}{3})^{\frac{1}{2}}$, and its polar and equatorial semi-axes are respectively $\frac{4a}{\sqrt{19}}$ and $\frac{4a}{\sqrt{10}}$. Again

$$\left\{3\left(\frac{u}{r}+\frac{v}{r}\cot\theta\right)-\delta\right\}^2-\left(\frac{S}{2n}\right)^2,$$

having the same sign as the left-hand side of (26), is positive within and negative without the spheroid (27).

We have now sufficient data to apply the criteria of Section III. The spheroid (25) within which $3\left(\frac{u}{r}+\frac{v}{r}\cot\theta\right)-\delta$ has the same sign as a_2 completely incloses the spheroid (27) limiting the region within which $\left\{3\left(\frac{u}{r}+\frac{v}{r}\cot\theta\right)-\delta\right\}^2-\left(\frac{S}{2n}\right)^2$ is positive. Thus inside the spheroid (27) Case (i) of Sect. III. applies when a_2 is positive and Case (ii) applies when, as in the case of the earth, a_2 is negative. Thus inside the spheroid (27) the maximum stress-difference is $\frac{S}{2}\pm n\left\{3\left(\frac{u}{r}+\frac{v}{r}\cot\theta\right)-\delta\right\}$, the upper sign being taken when a_2 is positive, the lower when it is negative. Since S is always taken to be positive, we see from (22) and (24) that in either case the maximum stress-difference is

$$\frac{3g\rho a_2}{95na^2}\left[+\{(8a^2-9r^2)^2+32r^2(a^2-r^2)\sin^2\theta\}^{\frac{1}{2}}+8a^2-9r^2+6r^2\sin^2\theta\right] \quad\text{.....(28),}$$

taken positively.

Outside the spheroid (27), whether a_2 be positive or negative, Case (iii) of Sect. III. applies, S being the maximum stress-difference.

S is also a correct measure, as well as (28), of the maximum stress-difference along the whole polar axis. The circumpolar regions in which Case (iii) of Sect. III. applies are very limited, and are further the precise localities where the stress-difference is least.

The maximum stress-difference has its absolutely greatest value at the centre, where by (22) taking a_2 positively

$$S = \tfrac{48}{95}g\rho a_2 \quad\text{.....(29).}$$

In passing outwards we see from (28) that it is greatest in the equator and there continually decreases as the distance from the centre increases till it reaches the value $\frac{18}{95}g\rho a_2$ at the surface $r=a$.

Noticing that a_2 is equivalent to Professor Darwin's $-\frac{2}{3}ae$, it will be seen that our maximum value for S agrees with that given in his paper.

In it he assumed S to be the maximum stress-difference everywhere, but he has since added a correction which agrees with the above conclusions.

Digitized by Google

§ 29. It remains to consider the case $m = n$, supposing still $\sigma_1 = P_2$.

From (13), (14), (15) and (16), leaving out the terms independent of P_2, we get

$$\left.\begin{aligned} u &= -3g\rho a_2 P_2 \frac{2r^3 + a^2 r}{70na^2} \\ v &= -g\rho a_2 \frac{dP_2}{d\theta} \frac{5r^3 + 3a^2 r}{140na^2} \\ w &= 0 \\ \delta &= -\frac{3g\rho a_2 P_2 r^2}{14na^2} \end{aligned}\right\} \quad \text{.....(30).}$$

Using these values in (6) of Sect. III., we find

$$\left(\frac{70a^2 S}{3g\rho a_2}\right)^2 = (3a^2 + 13r^2)^2 - 8r^2 (3a^2 + 11r^2) \sin^2\theta \quad \text{.....(31),}$$

$$\frac{140na^2}{3g\rho a_2}\left\{3\left(\frac{u}{r} + \frac{v}{r}\cot\theta\right) - \delta\right\} = 3a^2 + 13r^2 - 12r^2 \sin^2\theta \quad \text{.....(32).}$$

The last expression is obviously positive everywhere throughout the nearly spherical mass. Thus $3\left(\frac{u}{r} + \frac{v}{r}\cot\theta\right) - \delta$ is everywhere of the same sign as a_2.

From (31) and (32) we find

$$\left(\frac{140na^2}{3g\rho a_2}\right)^2 \left[\left\{3\left(\frac{u}{r} + \frac{v}{r}\cot\theta\right) - \delta\right\}^2 - \left(\frac{S}{2n}\right)^2\right] = -16r^2 \sin^2\theta\,(3a^2 + 14r^2 - 9r^2 \sin^2\theta).$$

This expression vanishes along the polar axis and everywhere else is negative. Thus Case (iii) of Sect. III. applies everywhere, the maximum strain being

$$\frac{S}{4n} + \frac{1}{2}\left(\delta - \frac{u}{r} - \frac{v}{r}\cot\theta\right).$$

By means of this expression we find from (31) and (30) that the greatest strain is

$$\frac{3g\rho a_2}{280na^2}\left[\pm\left\{(3a^2 + 13r^2)^2 - 8r^2(3a^2 + 11r^2)\sin^2\theta\right\}^{\frac{1}{2}} - a^2 - 11r^2 + 14r^2 \sin^2\theta\right].$$

The positive sign is to be taken when a_2 is positive, the negative sign when a_2 is negative, for in the formula S is supposed to be always taken so as to be positive.

When a_2 is positive this strain is greatest in the equator, where its value is simply

$$3g\rho a_2 \frac{a^2 + 6r^2}{140na^2},$$

which increases from $\frac{3g\rho a_2}{140n}$ at the centre to $\frac{3g\rho a_2}{20n}$ at the surface.

When, as in the case of the earth, a_2 is negative, the strain is greatest along the polar axis, where it increases continually from $\frac{-3g\rho a_2}{70n}$ at the centre to $\frac{-3g\rho a_2}{10n}$ at the surface.

Digitized by Google

§ 30. When $m = n$, Young's modulus is $2n$, thus the greatest strain in the case when a_2 is negative would be the same as in a wire of this material exposed to a traction equal to the weight, at the surface of the nearly spherical body, of a length $-\frac{3}{5}a_2$ of itself. In the case $\frac{n}{m} = 0$, we see from (29) that the greatest value of the stress-difference would be the same as in a wire of the material under a traction equal to the weight of a length $\frac{48}{55}a_2$ of itself taken positively.

In a spheroid whose polar and equatorial axes equalled those of the earth $-a_2$ would be about 8·8 miles. In such a spheroid, supposed isotropic of either of the above kinds and of a uniform density equal to the earth's mean density, the tendency to rupture, though very small relatively to the tendency if the material were of an average isotropic kind, would thus be absolutely large.

§ 31. Professor Darwin also considered the case $\frac{3m-n}{m}$ very small, and arrived at the conclusion that "infinite compressibility largely relieves the stress-difference due to ellipticity of figure."* He also concluded that "except for the case of the second harmonic inequality, compressibility makes very little difference."† In the light of the results (18 a) and (19 a) the practical importance of these conclusions may be questioned. It should also be noticed that in a material of this sort longitudinal expansion under traction would be accompanied by a nearly equal lateral *expansion*.

Professor Darwin has pointed out that the maximum stress-difference arising from the simultaneous application of two sets of forces is not necessarily the sum of the maximum stress-differences arising from their separate action. In like manner the greatest strains are not in general obtainable by superimposition. Thus any conclusions as to the strength of the earth, regarded as isotropic, which are based on its slight deviations from a truly spherical form are *a priori* inconclusive, if the effects of rotation and especially of the attraction of the mean sphere are neglected.

It should also be noticed that in so far as the surface strata are neither isotropic nor of the earth's mean density, the actual problem certainly differs from that treated here. Further if the earth be solid there seems no very conclusive reason for supposing that æotropoly occurs only in the surface strata.

SECTION V.

SOLID ROTATING SPHERE.

§ 32. One of the most interesting applications of the general solution is to the case of a solid sphere which rotates with uniform angular velocity ω about an axis (polar).

We may regard the sphere as being at rest under the action of a "centrifugal" force of potential $V = \frac{\omega^2 r^2}{2}\sin^2\theta$; i.e.

$$V = \tfrac{1}{3}\omega^2 r^2 - \frac{\omega^2 r^2}{3} P_2 \quad \text{.....(1)},$$

where P_2 is Legendre's second coefficient.

* l. c. p. 217. † l. c. p. 218.

Digitized by Google

Before applying the general solution, consider first the solution for

$$V = \tfrac{1}{3}\omega^2 r^2 \quad \text{.....(2).}$$

From the general equations, Section II., we see that

$$\delta = -\frac{\omega^2 \rho r^2}{3(m+n)} + C \text{ (a constant)} \quad \text{.....(3),}$$

$$v = 0 = w;$$

also
$$\delta = \frac{1}{r^2}\frac{d.ur^2}{dr}, \quad \therefore \frac{d.ur^2}{dr} = Cr^2 - \frac{1}{3}\frac{\omega^2 \rho r^4}{m+n}.$$

Thus
$$ur^2 = -\frac{1}{15}\frac{\omega^2 \rho r^5}{m+n} + \frac{Cr^3}{3} + D;$$

where D is a constant which for a solid sphere is obviously zero, and so

$$u = -\frac{1}{15}\frac{\omega^2 \rho r^3}{m+n} + \frac{Cr}{3} \quad \text{.....(4).}$$

From $(m-n)\delta + 2n\dfrac{du}{dr} = 0$ at the surface $r = a$, we get

$$C = \frac{\omega^2 \rho a^2 (5m+n)}{5(m+n)(3m-n)},$$

$$\therefore \; u = -\frac{1}{15}\frac{\omega^2 \rho r^3}{m+n} + \frac{\omega^2 \rho a^2 r (5m+n)}{15(m+n)(3m-n)} \quad \text{.....(4 a).}$$

From the general solution (36) and (37), Section II., we get for

$$V = -\frac{\omega^2 r^2 P_2}{3},$$

$$u = \frac{\omega^2 \rho r^3 P_2 (3m-n)}{3n(19m-5n)} - \frac{2\omega^2 \rho a^2 r P_2 (4m-n)}{3n(19m-5n)},$$

$$v = \omega^2 \rho \frac{dP_2}{d\theta}\left\{\frac{r^3(5m-n) - 2a^2 r(4m-n)}{6n(19m-5n)}\right\}.$$

At present leaving out the common factor $\omega^2\rho$, we get for the complete solution

$$u = P_2\left\{\frac{r^3(3m-n) - 2a^2 r(4m-n)}{3n(19m-5n)}\right\} - \frac{1}{15}\frac{r^3}{m+n} + \frac{a^2 r(5m+n)}{15(m+n)(3m-n)} \quad \text{.....(5),}$$

$$v = \frac{dP_2}{d\theta}\left\{\frac{r^3(5m-n) - 2a^2 r(4m-n)}{6n(19m-5n)}\right\} \quad \text{.....(6),}$$

$$\delta = -\frac{2r^2 P_2}{3(19m-5n)} - \frac{r^2}{3(m+n)} + \frac{a^2(5m+n)}{5(m+n)(3m-n)} \quad \text{.....(7).}$$

Noticing that $P_2 = \dfrac{3\cos^2\theta - 1}{2} = 1 - \dfrac{3}{2}\sin^2\theta$, we can replace (7) by a more convenient formula, viz.

$$\delta = \left\{\frac{(a^2 - r^2)(5m+n)(19m-5n) - 2r^2(5m^2 - 22mn + 5n^2)}{5(m+n)(3m-n)(19m-5n)} + \frac{r^2 \sin^2\theta}{19m-5n}\right\} \quad \text{.....(7 a).}$$

Digitized by Google

§ 33. The most interesting application of the solution is to the determination of the tendency to rupture. Using the previous notation, we have first to find

$$S \equiv 2n\sqrt{\left(r\frac{d\frac{u}{r}}{dr}-\frac{1}{r}\frac{dv}{d\theta}\right)^2+\left(r\frac{d\frac{v}{r}}{dr}+\frac{1}{r}\frac{du}{d\theta}\right)^2} \quad\text{.....(8).}$$

From above
$$r\frac{d\frac{u}{r}}{dr}=-\frac{2r^2}{15(m+n)}+\frac{2P_2r^2(3m-n)}{3n(19m-5n)},$$

$$\frac{1}{r}\frac{dv}{d\theta}=\frac{\cos 2\theta\left\{a^2(4m-n)-\frac{r^2}{2}(5m-n)\right\}}{n(19m-5n)}.$$

Noticing that
$$\frac{d^2P_2}{d\theta^2}=-3\cos 2\theta,$$

and substituting $2P_2=\frac{3}{2}\cos 2\theta+\frac{1}{2}$, we get at once

$$r\frac{d\frac{u}{r}}{dr}-\frac{1}{r}\frac{dv}{d\theta}=\frac{-(4m-n)(a^2-r^2)\cos 2\theta+r^2\frac{5m^2-22mn+5n^2}{10(m+n)}}{n(19m-5n)} \quad\text{.....(9).}$$

Also it is at once obvious that

$$r\frac{d\frac{v}{r}}{dr}+\frac{1}{r}\frac{du}{d\theta}=\frac{\sin 2\theta(4m-n)(a^2-r^2)}{n(19m-5n)} \quad\text{.....(10);}$$

$$\therefore\left(\frac{S}{2}\right)^2=\frac{(a^2-r^2)^2(4m-n)^2}{(19m-5n)^2}-2r^2\frac{(a^2-r^2)\cos 2\theta(4m-n)(5m^2-22mn+5n^2)}{10(m+n)(19m-5n)^2}$$
$$+r^4\frac{(5m^2-22mn+5n^2)^2}{100(m+n)^2(19m-5n)^2} \quad\text{.....(11).}$$

A convenient form of (11) is

$$\left\{\frac{S}{2}(19m-5n)\right\}^2=\left\{(a^2-r^2)(4m-n)-r^2\frac{(5m^2-22mn+5n^2)}{10(m+n)}\right\}^2$$
$$+4r^2(a^2-r^2)\sin^2\theta\frac{(4m-n)(5m^2-22mn+5n^2)}{10(m+n)} \quad\text{.....(11 a).}$$

Again S can vanish only when both (9) and (10) vanish simultaneously. Unless in the special case $5m^2-22mn+5n^2=0$, when S vanishes all over the surface of the sphere, this happens only when $\sin 2\theta=0$,

$$\text{and } r^2\frac{5m^2-22mn+5n^2}{10(m+n)}-(4m-n)(a^2-r^2)\cos 2\theta=0.$$

The first gives $\theta=(0,\pi)$ or $\frac{\pi}{2}$; i.e. S vanishes along the polar axis, or in the equator. Putting $\theta=0$ in the second equation we get

$$\frac{r_0^2}{a^2}=\frac{1}{1+\frac{5m^2-22mn+5n^2}{10(m+n)(4m-n)}} \quad\text{.....(12).}$$

This gives r_0 less or greater than a, according as $5m^2-22mn+5n^2$ is positive or negative.

Digitized by Google

Putting $\theta = \frac{\pi}{2}$ in the second equation we get

$$\frac{r^2}{a^2} = \frac{1}{1 - \dfrac{5m^2 - 22mn + 5n^2}{10(m+n)(4m-n)}} \quad \text{.....} (13).$$

This gives r greater or less than a, according as $5m^2 - 22mn + 5n^2$ is positive or negative.

In both these cases we have assumed that $10(m+n)(4m-n) - (5m^2 - 22mn + 5n^2)$ is positive, which it certainly is if $\frac{m}{n} > \cdot 09$.

If then $5m^2 - 22mn + 5n^2$ be positive, S vanishes inside the sphere at two points in the polar axis at a distance from the centre given by

$$\frac{r_0^2}{a^2} = \frac{1}{1 + \dfrac{5m^2 - 22mn + 5n^2}{10(m+n)(4m-n)}} \quad \text{.....} (12).$$

While if $5m^2 - 22mn + 5n^2$ be negative, S vanishes inside the sphere in a ring in the equator, whose radius is given by

$$\frac{r^2}{a^2} = \frac{1}{1 - \dfrac{5m^2 - 22mn + 5n^2}{10(m+n)(4m-n)}} \quad \text{.....} (13).$$

For $5m^2 - 22mn + 5n^2 = 0$, S vanishes all over the surface of the sphere.

§ 34. We have next to find $3\left(\frac{u}{r} + \frac{v}{r}\cot\theta\right) - \delta$.

From (5) and (6), noticing that $\cot\theta \frac{dP}{d\theta}^2 = 3(\sin^2\theta - 1)$, we obtain

$$\frac{u}{r} + \frac{v}{r}\cot\theta = \frac{mr^2\sin^2\theta}{n(19m-5n)} + \frac{(a^2 - r^2)}{3}\left\{\frac{5m+n}{5(m+n)(3m-n)} + \frac{4m-n}{n(19m-5n)}\right\}$$
$$- r^2\frac{(5m^2 - 22mn + 5n^2)}{10n(3m-n)(19m-5n)} \quad \dots (14).$$

Thence and from (7 a) we get

$$\left\{3\left(\frac{u}{r} + \frac{v}{r}\cot\theta\right) - \delta\right\} n(19m-5n) = r^2\sin^2\theta(3m-n) + (a^2 - r^2)(4m-n)$$
$$- r^2\frac{(5m^2 - 22mn + 5n^2)}{10(m+n)} \quad \dots\dots (15).$$

Thus, supposing of course $3m - n$ positive, if $5m^2 - 22mn + 5n^2$ be negative $3\left(\frac{u}{r} + \frac{v}{r}\cot\theta\right) - \delta$ is positive throughout the whole sphere. If however $5m^2 - 22mn + 5n^2$ be positive there is a spheroidal surface

$$r^2\sin^2\theta(3m-n) + (a^2 - r^2)(4m-n) - r^2\frac{(5m^2 - 22mn + 5n^2)}{10(m+n)} = 0 \quad \dots\dots\dots (16),$$

over which
$$3\left(\frac{u}{r} + \frac{v}{r}\cot\theta\right) - \delta = 0.$$

Digitized by Google

It is easily seen that $3\left(\frac{u}{r}+\frac{v}{r}\cot\theta\right)-\delta$ is positive inside this surface and negative outside.

The polar axis of the spheroid is given from (16) by

$$r_0^2 = a^2 \frac{1}{1+\dfrac{5m^2-22mn+5n^2}{10(m+n)(4m-n)}} \text{ ..... , see (12);}$$

and so the ends of the polar axis of this spheroid are the curious points where $S=0$. These lie inside the sphere only if $5m^2-22mn+5n^2$ be positive, and at them $R_1=\Theta_1=\Phi$, and the stress quadric is a sphere.

The equatorial semi-axis is given by

$$\frac{r^2}{a^2} = \frac{1}{1+\dfrac{5m^2-22mn+5n^2}{10(m+n)(4m-n)}-\dfrac{3m-n}{4m-n}} \text{ ..... (17).}$$

This gives $r>a$ if $5m^2-22mn+5n^2$ be negative while $m>\frac{n}{3}$. Also $r>a$ if $5m^2-22mn+5n^2$ be positive while $25m^2+42mn-15n^2$ is positive, and if we assume $m>n$ this is certainly the case. We shall see afterwards that $5m^2-22mn+5n^2$ is positive if $\frac{m}{n}>4{\cdot}159$.

Thus in all practical cases the equatorial semi-axis of the above spheroid is $>a$. The spheroid entirely encloses the sphere provided $5m^2-22mn+5n^2$ be negative; while if $5m^2-22mn+5n^2$ be positive the spheroid cuts the surface of the sphere at a polar distance given by

$$\theta_1 = \sin^{-1}\sqrt{\frac{5m^2-22mn+5n^2}{10(m+n)(3m-n)}} \text{ ..... (18).}$$

This value is certainly real supposing $\frac{m}{n}>\frac{1}{3}$.

§ 35. We have next to find the surface $\left\{3\left(\frac{u}{r}+\frac{v}{r}\cot\theta\right)-\delta\right\}^2-\left(\frac{S}{2n}\right)^2=0$.

From (11 a) and (15) the surface becomes after reduction

$$r^2\sin^2\theta\left[(3m-n)^2r^2\sin^2\theta+4(a^2-r^2)(4m-n)\left\{\frac{5m^2+16mn-5n^2}{5(m+n)}\right\}\right.$$
$$\left.-r^2(3m-n)\frac{5m^2-22mn+5n^2}{5(m+n)}\right]=0 \text{ ..... (19).}$$

So the surface consists of the polar axis $r^2\sin^2\theta=0$, and the spheroid

$$(3m-n)^2r^2\sin^2\theta+4(a^2-r^2)(4m-n)\frac{(5m^2+16mn-5n^2)}{5(m+n)}$$
$$-r^2(3m-n)\frac{5m^2-22mn+5n^2}{5(m+n)}=0 \text{ ..... (20).}$$

Digitized by Google

The polar semi-axis is given by

$$\frac{r_2^2}{a^2} = \frac{1}{1 + \frac{1}{4} \cdot \frac{3m-n}{4m-n} \frac{5m^2 - 22mn + 5n^2}{5m^2 + 16mn - 5n^2}} \quad \text{.....(21).}$$

For $m > \frac{n}{3}$, r_2 is $>$, $=$, or $< a$, according as $5m^2 - 22mn + 5n^2$ is negative, zero, or positive.

It is easy to show directly that the equatorial semi-axis is for any value of $m : n$, which is $> \frac{1}{3}$, greater than a. This is very easily seen however from the fact that the spheroid cuts the surface of the sphere at a polar distance

$$\theta_2 = \sin^{-1} \sqrt{\frac{5m^2 - 22mn + 5n^2}{5(m+n)(3m-n)}} \quad \text{.....(22);}$$

thus θ_2 is imaginary if $5m^2 - 22mn + 5n^2$ be negative. In this case the polar semi-axis is $> a$, thus the spheroid, as it never cuts the sphere, lies entirely outside it.

If $5m^2 - 22mn + 5n^2$ be positive θ_2 is real, because $5(m+n)(3m-n)$ is certainly $> 5m^2 - 22mn + 5n^2$ for any value of $m : n > 4{\cdot}159$, thus the spheroid cuts the sphere. Also in this case the polar semi-axis is $< a$, $\therefore$ the equatorial must be $> a$.

If $5m^2 - 22mn + 5n^2$ be negative, $\left\{3\left(\frac{u}{r} + \frac{v}{r}\cot\theta\right) - \delta\right\}^2 - \left(\frac{S}{2n}\right)^2$ cannot pass through the value zero except along the polar axis inside the sphere, and so being finite and continuous must have the same sign throughout. This sign, being the same as that of the left side of (20), is obviously positive for any value of $\frac{m}{n} > \frac{1}{3}$.

If $5m^2 - 22mn + 5n^2$ de positive, the surface $\left\{3\left(\frac{u}{r} + \frac{v}{r}\cot\theta\right) - \delta\right\}^2 - \left(\frac{S}{2n}\right)^2 = 0$, besides the polar axis, consists of a spheroid part of which lies inside the sphere. Inside this spheroid the sign is the same as that of the left side of (20) and so is positive.

Lastly for $5m^2 - 22mn + 5n^2 = 0$, the surface $\left\{3\left(\frac{u}{r} + \frac{v}{r}\cot\theta\right) - \delta\right\}^2 - \left(\frac{S}{2n}\right)^2 = 0$ consists of the polar axis, and of a spheroid whose least (polar) semi-axis is a. Thus the whole surface so far as concerns us is the polar axis alone.

The problem resolves itself into three parts, according as $5m^2 - 22mn + 5n^2$ is (i) negative, (ii) positive, (iii) zero. The equation $5m^2 - 22mn + 5n^2 = 0$ is a reciprocal equation whose roots we shall denote by α and $\frac{1}{\alpha}$, where $\alpha = 4{\cdot}1596$ approximately. In case (i) $\frac{m}{n}$ must lie between α and $\frac{1}{\alpha}$, in case (ii) $\frac{m}{n}$ must be greater than α or else less than $\frac{1}{\alpha}$, while in case (iii) $\frac{m}{n}$ equals either α or $\frac{1}{\alpha}$. In what follows we shall suppose $m > n$, which is certainly true of all ordinary materials. Thus we shall have in case (i), $\frac{m}{n} > 1$ and $< \alpha$; in case (ii), $\frac{m}{n} > \alpha$; and in case (iii), $\frac{m}{n} = \alpha$.

Digitized by Google

§ 36. Case (i) $\frac{m}{n} < \alpha$ and > 1.

Throughout the whole sphere $3\left(\frac{u}{r} + \frac{v}{r}\cot\theta\right) - \delta$ is positive; and the same is true of $\left\{3\left(\frac{u}{r} + \frac{v}{r}\cot\theta\right) - \delta\right\}^2 - \left(\frac{S}{2n}\right)^2$, excluding the polar axis along which this last expression is zero. Along the whole polar axis $\Phi = R$, so that either may be looked on as the greatest stress, and the maximum stress-difference is correctly given, writing in $\omega^2\rho$ again, by

$$S = \frac{2\omega^2\rho}{19m - 5n}\left\{(a^2 - r^2)(4m - n) - r^2\frac{5m^2 - 22mn + 5n^2}{10(m+n)}\right\} \quad \text{.....(23)},$$

and the greatest strain by

$$\frac{u}{r} + \frac{v}{r}\cot\theta = \omega^2\rho\left[\frac{a^2 - r^2}{3}\left\{\frac{5m+n}{5(m+n)(3m-n)} + \frac{4m-n}{n(19m-5n)}\right\} - r^2\frac{5m^2 - 22mn + 5n^2}{10n(3m-n)(19m-5n)}\right] \quad \text{.....(24)}$$

Throughout the rest of the sphere, the maximum stress-difference is

$$n\left\{3\left(\frac{u}{r} + \frac{v}{r}\cot\theta\right) - \delta + \frac{S}{2n}\right\},$$

and the greatest strain $\qquad \frac{u}{r} + \frac{v}{r}\cot\theta.$

These may be got at once from (11a), (14) and (15) writing in the factor $\omega^2\rho$.

It is easily seen also that $3\left(\frac{u}{r} + \frac{v}{r}\cot\theta\right) - \delta$, S, and $\frac{u}{r} + \frac{v}{r}\cot\theta$ are all algebraically greatest when $r = 0$, and diminish continuously as r increases. Thus on either of the two last theories the tendency to rupture is greatest at the centre. On the one theory it is measured by

$$S = \frac{2\omega^2\rho a^2(4m - n)}{19m - 5n} \quad \text{.....(25)},$$

and on the other by

$$\frac{u}{r} + \frac{v}{r}\cot\theta = \frac{\omega^2\rho a^2}{3}\left\{\frac{4m-n}{n(19m-5n)} + \frac{5m+n}{5(m+n)(3m-n)}\right\} \quad \text{.....(26)}.$$

§ 37. Case (ii) $\frac{m}{n} > \alpha$.

$3\left(\frac{u}{r} + \frac{v}{r}\cot\theta\right) - \delta$ is positive inside and negative outside the spheroid (16);

$\left\{3\left(\frac{u}{r} + \frac{v}{r}\cot\theta\right) - \delta\right\}^2 - \left(\frac{S}{2n}\right)^2$ is positive inside and negative outside the spheroid (20), excluding the polar axis along which it vanishes. The first of these spheroids is that whose polar semi-axis is r_0, and which cuts the surface of the sphere at a polar distance θ_1; the second has for its polar semi-axis r_2, and cuts the surface of the sphere at a polar distance θ_2. From (18) and (22)

$$\sin^2\theta_1 = \tfrac{1}{2}\sin^2\theta_2 \quad \text{.....(27)},$$

Digitized by Google

also from (12) and (21)

$$a^2\left(\frac{1}{r_2^2}-\frac{1}{r_0^2}\right)=\frac{5m^2-22mn+5n^2}{4m-n}\left\{\frac{3m-n}{5m^2+16mn-5n^2}-\frac{1}{10(m+n)}\right\}\text{.....}(28);$$

$$\therefore\ \theta_1<\theta_2,\text{ and for }\frac{m}{n}>\alpha\text{ obviously }r_0>r_2.$$

So far then as concerns the sphere, the region throughout which $3\left(\frac{u}{r}+\frac{v}{r}\cot\theta\right)-\delta$ is positive is greater than and includes the whole region throughout which

$$\left\{3\left(\frac{u}{r}+\frac{v}{r}\cot\theta\right)-\delta\right\}^2-\left(\frac{S}{2n}\right)^2$$

is positive.

Thus inside the spheroid generated by (20) Case (i) of Sect. III. applies, Φ is the algebraically greatest stress, $\Phi-\Theta_1$ the maximum stress-difference, and $\frac{u}{r}+\frac{v}{r}\cot\theta$ the greatest strain. The extremities of the polar axis of the spheroid (16) are the points where $R=\Theta=\Phi$; along the polar axis between these points $\Phi=R_1$, and beyond them $\Phi=\Theta_1$. Between the surface of the spheroid (20) and that of the sphere Case (iii) of Sect. III. applies, S being the maximum stress-difference and $\frac{S}{4n}+\frac{1}{2}\left\{\delta-\left(\frac{u}{r}+\frac{v}{r}\cot\theta\right)\right\}$ the greatest strain.

Since the least value of r_2 is $\frac{4a}{\sqrt{19}}$ and the greatest value of θ_2 is $\sin^{-1}\sqrt{\frac{1}{3}}$, these values occurring for $\frac{n}{m}=0$, the region throughout which Case (iii) of Sect. III. applies is limited to a comparatively small superficial portion round the poles. In this region a^2-r^2 is very small, and it is easily seen that the maximum stress-difference and the greatest strain are both very small compared to their values near the centre. At a given distance from the centre the maximum stress-difference and strain occur when $\theta=\frac{\pi}{2}$, i.e. in the equator; they both diminish as r increases. Their absolutely greatest values occur at the centre being as in the former case given by (25) and (26).

§ 38. Case (iii), $5m^2-22mn+5n^2=0$, or $\frac{m}{n}=4{\cdot}1596\ldots$. All the expressions become very simple;

$$S=\frac{2(a^2-r^2)(4m-n)}{19m-5n}\text{ ..... }(29),$$

and so vanishes over the surface of the sphere, at every point of which the stress quadric is thus a surface of revolution. Along the whole polar axis $\Phi=R_1$, the extremities being the points $R=\Theta=\Phi$. Throughout the whole of the sphere correct measures are for the maximum stress-difference

$$\Phi-\Theta_1=\frac{2(a^2-r^2)(4m-n)+r^2\sin^2\theta\,(3m-n)}{19m-5n}\text{ .....}(30),$$

Digitized by Google

for the greatest strain

$$\frac{u}{r}+\frac{v}{r}\cot\theta=\frac{mr^2\sin^2\theta}{n(19m-5n)}+\frac{a^2-r^2}{3}\left\{\frac{5m+n}{5(m+n)(3m-n)}+\frac{4m-n}{n(19m-5n)}\right\}\text{.....(31).}$$

These have their absolutely greatest values given as in the previous cases by (25) and (26), writing in $\omega^2\rho$.

The above value of $\frac{m}{n}$ gives $\frac{m-n}{2m}=\cdot3793....$ Thomson and Tait, *Nat. Phil.* Part II. § 684, mention that for brass Kirchhoff found $\frac{m-n}{2m}=\cdot387$. Thus Case (iii) may represent very closely the phenomena attending the rotation of a solid sphere of brass.

§ 39. The value (25) for the greatest value of the maximum stress-difference may be put in the form

$$S=2\omega^2\rho a^2\frac{1}{5-\dfrac{m}{4m-n}}\text{.....(32);}$$

thus it obviously decreases as $\frac{m}{n}$ increases from $\frac{1}{3}$ to ∞. This decrease is extremely small however, the values being

$$\left.\begin{aligned}\frac{m}{n}&=\tfrac{1}{3},\quad S=\cdot5\omega^2\rho a^2\\ &=2\qquad\;\;=\cdot424\omega^2\rho a^2\\ &=\infty\qquad\;\;=\cdot421\omega^2\rho a^2\end{aligned}\right\}\text{.....(33).}$$

Thus for any ordinary substance, $\frac{17}{40}\omega^2\rho a^2$ is a very close approximation to the maximum value of the stress-difference. If we suppose the sphere to have the same radius, mean density, and angular velocity as the earth, we know that $\omega^2 a=\frac{g}{289}$ very approximately, g being the acceleration due to gravity at the earth's surface. This sphere would thus have a maximum stress-difference given by $S=\frac{g\rho a}{680}$.

This would represent the pressure, per unit area, on the base of a structure on the earth's surface and of the earth's mean density whose height was $\frac{a}{680}$, or about 5·9 miles.

The value (26) for the greatest strain is capable of greater though still very limited variation with the ratio $m:n$. For $m=2n$ it is very approximately $\frac{\cdot1196\omega^2\rho a^2}{n}$. If $E=\frac{5}{2}n$ denote Young's modulus, this is very nearly $\frac{\cdot3\omega^2\rho a^2}{E}$.

Taking the earth again as an illustration, this would represent the strain at the end of a wire of the same mean density as the earth and of a length slightly exceeding 4 miles; supposing the point of suspension the required height above the ground.

On either theory the effect of the rotation of the earth in producing rupture is very considerable, which seriously affects any conclusions drawn from the results of Sect. IV. alone.

Digitized by Google

SECTION VI.

ROTATING SPHERICAL SHELL.

§ 40. As an illustration of the general formulæ for shells let us consider, what is in itself an interesting problem, a spherical shell rotating with uniform angular velocity ω about a "polar" axis, having special reference to the case when the shell is thin.

The potential is
$$V=\frac{1}{3}\omega^2r^2-\frac{\omega^2r^2}{3}P_2 \quad\text{.....(1).}$$

As in the case of the solid sphere consider the parts of the solution separately.

First
$$V=\frac{1}{3}\omega^2r^2 \quad\text{.....(2).}$$

As in the solid sphere, this gives
$$\delta=\frac{-\rho}{m+n}\,\frac{\omega^2r^2}{3}+C \quad\text{.....(3),}$$
$$u=-\tfrac{1}{15}\,\frac{\omega^2\rho r^3}{m+n}+\frac{Cr}{3}+Er^{-2} \quad\text{.....(4);}$$

where C and E are constants, determined from the surface conditions $(m-n)\,\delta+2n\dfrac{du}{dr}=0$ where $r=a$ or $r=a'$.

Thus
$$C=\frac{\omega^2\rho}{m+n}\,\frac{a^5-a'^5}{a^3-a'^3}\,\frac{5m+n}{5(3m-n)} \quad\text{.....(5),}$$
$$E=\frac{\omega^2\rho}{m+n}(aa')^2\,\frac{a^2-a'^2}{a^3-a'^3}\,\frac{5m+n}{15n} \quad\text{.....(6).}$$

Next
$$V=\frac{-\omega^2r^2}{3}P_2 \quad\text{.....(7).}$$

Employing the general solution (36 a), (37 a) Sect. II., writing $Y_2\equiv A.P_2$, $Z_2\equiv B.P_2$, $Y'_2\equiv A'.P_2$, $Z'_2=B'.P_2$; we have from the general surface conditions, (44)—(47), Sect. II.

$$-a^2A\,\frac{-m+n}{7}+2nB+A'a^{-3}\,\frac{9m+n}{3}-8na^{-5}B'=\frac{-\rho}{m+n}\,\frac{\omega^2a^2}{3}\,\frac{7m+5n}{7} \quad\text{.....(8),}$$
$$-a'^2A\,\frac{-m+n}{7}+2nB+A'a'^{-3}\,\frac{9m+n}{3}-8na'^{-5}B'=\frac{-\rho}{m+n}\,\frac{\omega^2a'^2}{3}\,\frac{7m+5n}{7} \quad\text{.....(9),}$$
$$-a^2A\,\frac{8m-n}{21}+nB-A'a^{-3}\,\frac{3m-n}{6}+\frac{8n}{3}a^{-5}B'=\frac{-\rho}{m+n}\,\frac{\omega^2a^2}{3}\,\frac{3n}{7} \quad\text{.....(10),}$$
$$-a'^2A\,\frac{8m-n}{21}+nB-A'a'^{-3}\,\frac{3m-n}{6}+\frac{8n}{3}a'^{-5}B'=\frac{-\rho}{m+n}\,\frac{\omega^2a'^2}{3}\,\frac{3n}{7} \quad\text{.....(11).}$$

Digitized by Google

Eliminate B from (8) and (10), and from (9) and (11); thus

$$a^2A\,\frac{19m-5n}{21}+4mA'a^{-3}-\frac{40}{3}nB'a^{-5}=\frac{-\rho}{m+n}\,\omega^2a^2\,\frac{7m-n}{21}\quad\text{.....(12)},$$

$$a'^2A\,\frac{19m-5n}{21}+4mA'a'^{-3}-\frac{40}{3}nB'a'^{-5}=\frac{-\rho}{m+n}\,\omega^2a'^2\,\frac{7m-n}{21}\quad\text{.....(13)}.$$

Eliminate B' from (8) and (10), and from (9) and (11); thus

$$-a^2Am+5nB+a^{-3}A'\,\frac{9m+5n}{6}=\frac{-\rho}{m+n}\,\omega^2a^2\,\frac{m+2n}{3}\quad\text{.....(14)},$$

$$-a'^2Am+5nB+a'^{-3}A'\,\frac{9m+5n}{6}=\frac{-\rho}{m+n}\,\omega^2a'^2\,\frac{m+2n}{3}\quad\text{.....(15)}.$$

Eliminate B' from (12) and (13), and B from (14) and 15); thus

$$A(a^7-a'^7)\,\frac{19m-5n}{21}+A'(a^2-a'^2)\,4m=\frac{-\rho}{m+n}\,\omega^2(a^7-a'^7)\,\frac{7m-n}{21}\quad\text{.....(16)},$$

$$A(a^2-a'^2)\,m+A'\,\frac{a^3-a'^3}{(aa')^3}\,\frac{9m+5n}{6}=\frac{\rho\omega^2}{m+n}\,(a^2-a'^2)\,\frac{m+2n}{3}\quad\text{.....(17)}.$$

We thus find

$$A=\frac{\rho\omega^2}{m+n}\,\frac{1}{D}\,\{168m(m+2n)(aa')^3(a^2-a'^2)^2+(7m-n)(9m+5n)(a^3-a'^3)(a^7-a'^7)\}\text{...(18)},$$

$$A'=\frac{-\rho\omega^2}{D}\,\{20(4m-n)(aa')^3(a^2-a'^2)(a^7-a'^7)\}\text{.....(19)},$$

where for shortness

$$D\equiv 504m^2\,(aa')^3(a^2-a'^2)^2-(19m-5n)\,(9m+5n)(a^3-a'^3)(a^7-a'^7)\text{.....(20)}.$$

Multiplying (14) by a'^2, (15) by a^2 and subtracting we get

$$B=-\frac{(9m+5n)}{30n}\,\frac{(a^5-a'^5)A'}{(aa')^3(a^2-a'^2)};$$

$\therefore$, from (19),
$$B=\rho\omega^2\,\frac{9m+5n}{3n}\,\frac{2(4m-n)(a^5-a'^5)(a^7-a'^7)}{D}\quad\text{.....(21)}.$$

Similarly from (12), (13) and (19) we find

$$B'=-\rho\omega^2\,\frac{m}{n}\,\frac{6\,(4m-n)\,(aa')^3(a^2-a'^2)\,(a^5-a'^5)}{D}\text{.....(22)}.$$

The expressions for A', B, B' are comparatively short; and in a given case, when $m:n$ is given and also $a:a'$, they would be quite simple.

Digitized by Google

From the general solution, (36 a) Sect. II., the coefficient of r^3P_2 in u is

$$\left\{\frac{2\omega^2\rho}{21(m+n)} - \frac{m-n}{7n}A\right\},$$

and so is

$$-\frac{\rho\omega^2}{nD}\left\{24m(m-2n)(aa')^3(a^2-a'^2)^2 + (9m+5n)\left(m-\frac{n}{3}\right)(a^3-a'^3)(a^7-a'^7)\right\} \text{.....(23).}$$

As here all we have to do is to write the values (18)—(22) for Y_i, Y_i', Z_i, Z_i' in the general solution (36 a), (37 a) Sect. II.

§ 41. The expressions for u, v, and δ are of course long, and a determination of the stresses and tendency to rupture would be a tedious process, so we shall confine ourselves to the case when the shell is thin, say

$$a' = a(1-\epsilon) \text{.....(24).}$$

The method we shall follow is this. First determine the parts of A, B, A', B' that do not contain ϵ in their expression. The physical side of this step is that we assume the shell so thin that the displacements at any point are the same as at the point on the surface which lies on the same radius vector. We thus trace merely differences at different points of the surface both for the displacements and stresses, and so do not attempt to find how the stress rises inside the surface of the envelope. We next shall find the additional terms introduced in the values of A, B, A', B' when we keep terms in ϵ neglecting those in ϵ^2, and thence find the variations in the displacements and stresses along the same radius vector.

§ 42. So first neglecting terms in ϵ, and dropping $\omega^2\rho$ for the present, we get from (18) and (20) after arithmetical reduction,

$$A = -\frac{1}{m+n}\,\frac{19m-n}{5(3m-n)} \text{.....(25).}$$

Similarly

$$A' = \frac{8}{15}\,\frac{a^5}{m+n}\,\frac{4m-n}{3m-n} \text{.....(26),}$$

$$B = -\frac{2}{45}\,\frac{a^3}{m+n}\,\frac{(4m-n)(9m+5n)}{n(3m-n)} \text{.....(27),}$$

$$B' = \frac{4}{35}\,\frac{a^7}{m+n}\,\frac{m(4m-n)}{n(3m-n)} \text{.....(28),}$$

$$C = \frac{1}{3}\,\frac{a^2}{m+n}\,\frac{5m+n}{(3m-n)} \text{.....(29),}$$

$$E = \frac{2a^5}{m+n}\,\frac{5m+n}{45n} \text{.....(30).}$$

The coefficient of r^3P_2 in u is, by (23),

$$\frac{1}{m+n}\,\frac{57m^2-30mn-7n^2}{105n(3m-n)} \text{.....(31).}$$

Digitized by Google

Substituting the values (25)—(30) in the general expressions for δ, u, v, (13 and 13′), (36 a), (37 a) Sect. II., and in (3) and (4), we get

$$\delta = P_2 \left\{ \frac{-2r^2(21m+n) + \frac{8a^5}{r^3}(4m-n)}{15(m+n)(3m-n)} \right\} + \frac{a^2(5m+n) - r^2(3m-n)}{3(m+n)(3m-n)} \quad \text{.....(32)},$$

$$u = \frac{P_2}{5(m+n)n(3m-n)} \left\{ r^3 \frac{57m^2 - 30mn - 7n^2}{21} - 2a^2 r \frac{(9m+5n)(4m-n)}{9} \right.$$

$$\left. - \frac{4a^5}{r^2} \frac{(3m+2n)(4m-n)}{9} + \frac{4a^7}{r^4} \frac{m(4m-n)}{7} \right\}$$

$$+ \frac{1}{45(m+n)} \left\{ -3r^3 + 5a^2 r \frac{5m+n}{3m-n} + \frac{2a^5}{r^2} \frac{5m+n}{n} \right\} \quad \text{.....(33)},$$

$$v = \frac{dP_2}{d\theta} \frac{r^3 \frac{95m^2 + 48mn - 7n^2}{14} - \frac{a^2 r}{3}(9m+5n)(4m-n) - \frac{4a^5}{r^2} \frac{n(4m-n)}{3} - \frac{4a^7}{r^4} \frac{m(4m-n)}{7}}{15n(m+n)(3m-n)} \quad \text{...(34)}.$$

§ 43. From the assumption we have made as to neglecting terms in ϵ we might assert at once that $r\frac{d\frac{v}{r}}{dr} + \frac{1}{r}\frac{du}{d\theta} = 0$, for it is so at the surface, this being in fact the second surface condition.

But as the expression will be useful afterwards when we retain ϵ, and is a very strict test of the accuracy of the results (33) and (34), we shall find its value at present.

Noticing that $$\frac{dP_2}{d\theta} = -\tfrac{3}{2}\sin 2\theta,$$

we have, after addition and reduction,

$$r\frac{d\frac{v}{r}}{dr} + \frac{1}{r}\frac{du}{d\theta} = -\frac{\sin 2\theta(4m-n)}{10n(m+n)(3m-n)} \left[\frac{2}{7} r^3(19m+7n) - \frac{2}{3}a^2(9m+5n) - \frac{4}{3}\frac{a^5}{r^2}(3m-n) + \frac{32}{7}\frac{a^7}{r^4}m \right]$$

.....(35).

Now put $r = a$, and the expression inside the square bracket in (35) vanishes. The principal axes at any point are thus of course the radius vector and the perpendiculars to it in the meridian and perpendicular to the meridian. Calling the three principal stresses R, Θ, Φ, in this case

$$R - \Theta = 2n\left(r\frac{d\frac{u}{r}}{dr} - \frac{1}{r}\frac{dv}{d\theta} \right) \quad \text{.....(36)}.$$

Digitized by Google

From (33) and (34) putting $\frac{3\cos 2\theta + 1}{4}$ for P_2, and reducing, we get

$$15n(m+n)(3m-n)\left\{r\frac{d.\frac{u}{r}}{dr} - \frac{1}{r}\frac{dv}{d\theta}\right\} =$$

$$3\cos 2\theta\left[\frac{r^2}{14}(152m^2+18mn-14n^2) - (4m-n)\left\{\frac{a^2}{3}(9m+5n) - \frac{a^5}{3r^3}(9m+2n) + \frac{19}{7}\frac{a^7}{r^5}m\right\}\right]$$

$$+\frac{r^2}{14}(57m^2-30mn-7n^2) + (4m-n)\left\{\frac{a^5}{r^3}(3m+2n) - \frac{15}{7}\frac{a^7}{r^5}m\right\}$$

$$-2n(3m-n)\left\{r^2 + \frac{1}{4}\frac{a^5}{r^3}\frac{5m+n}{n}\right\} \quad \text{.....(37).}$$

Putting $r=a$ we easily find that the right-hand side of (37) becomes identically zero, and therefore

$$\Theta \sim R \equiv S = 0 \quad \text{.....(38).}$$

Again from (32) and (33), remembering that

$$R = (m-n)\delta + 2n\frac{du}{dr},$$

we find, after reduction,

$$15(m+n)(3m-n)R$$

$$= 4P_2\left[\frac{r^2}{7}(12m^2+25mn-7n^2) - (4m-n)\left\{\frac{a^2}{3}(9m+5n) - \frac{2}{3}\frac{a^5}{r^3}(9m+n) + \frac{24}{7}\frac{a^7}{r^5}m\right\}\right]$$

$$+\tfrac{1}{3}(3m-n)(5m+n)\{5a^2-3r^2-2a^5r^{-3}\} \quad \text{.....(39).}$$

Putting $r=a$ we find that the right-hand side of (39) becomes identically zero, and thus $R=0$.

This is in fact the first surface condition, and might have been assumed without proof, but (39) will be found useful when we cease to neglect the thickness of the shell.

From (38) it follows that $\Theta = 0$; and so the only stress which does not vanish is Φ, which we proceed to calculate.

§ 44. From (33) and (34) we obtain, on reduction,

$$15n(m+n)(3m-n)\left(\frac{u}{r}+\frac{v}{r}\cot\theta\right) = 3m\cos^2\theta\left\{-\frac{r^2}{7}(19m+39n) - \frac{2a^5}{r^3}(4m-n) + \frac{10}{7}\frac{a^7}{r^5}(4m-n)\right\}$$

$$-\frac{r^2}{14}(57m^2+12mn-21n^2) + 12a^2m(m+n) + \frac{3}{2}\frac{a^5}{r^3}(7m^2+2mn-n^2) - \frac{6}{7}\frac{a^7}{r^5}m(4m-n) \ldots(40).$$

Put now $r=a$, and write $1-\sin^2\theta$ for $\cos^2\theta$; then, denoting Young's modulus by E, and reintroducing the factor $\omega^2\rho$, we find after an easy reduction

$$\frac{u}{r} + \frac{v}{r}\cot\theta = \frac{\omega^2\rho a^2}{E}\sin^2\theta \quad \text{.....(41).}$$

Digitized by Google

From (32), putting $r=a$, and reintroducing $\omega^2\rho$, we easily find

$$\delta=\frac{\omega^2\rho a^2}{3m-n}\sin^2\theta \qquad (42),$$

therefore from (41) and (42), remembering that

$$\Phi=(m-n)\,\delta+2n\left(\frac{u}{r}+\frac{v}{r}\cot\theta\right),$$

we obtain

$$\Phi=\omega^2\rho a^2\sin^2\theta \qquad (43).$$

Since Φ is the only stress which does not vanish, the maximum stress-difference is simply the right-hand side of (43).

The maximum stress-difference thus vanishes at the poles, and has in the equator its maximum value

$$\omega^2\rho a^2 \qquad (44).$$

Comparing this with the results (32) and (33) of Sect. v., we see that according to the stress-difference theory the tendency to rupture in a thin shell bears to that in a solid sphere, of the same radius and rotating with the same velocity, a ratio which in any ordinary isotropic substance is very approximately 40/17.

Since Φ, the only existent stress, is by (43) necessarily positive, it follows that the corresponding strain, viz. $\frac{u}{r}+\frac{v}{r}\cot\theta$, is everywhere the greatest strain. Thus the greatest strain is given by (41), and so has its maximum value

$$\omega^2\rho a^2/E \qquad (45)$$

in the equator.

Comparing (45) with (26) Sect. v., we see that according to the greatest strain theory the tendency to rupture in a thin shell bears to that in a solid sphere, of the same radius and rotating with the same velocity, a ratio which in all ordinary isotropic materials lies between 3 : 1 and 5 : 1.

It should be noticed that the longitudinal tractions in a standard bar which the two theories would indicate as producing the same tendency to rupture as rotation in a thin shell are absolutely identical.

§ 45. We shall now consider what change is introduced when we retain terms in ϵ, neglecting terms in ϵ^2. We have first to find the change introduced in the values of A, A', etc. Returning to (18) and (20), and retaining those powers of ϵ which are required, we get

$$A\,(m+n)=\frac{168\,m\,(m+2n)\,(1-3\epsilon)\,4\epsilon^2\,(1-\epsilon)+(7m-n)\,(9m+5n)\,3\,(1-\epsilon)\,\epsilon^2\,.\,7\,(1-3\epsilon)}{4\times504\,m^2\,(1-3\epsilon)\,(1-\epsilon)\,\epsilon^2-(19m-5n)\,(9m+5n)\,3\,(1-\epsilon)\,\epsilon^2\,.\,7\,(1-3\epsilon)}.$$

We see that ϵ^2 cuts out above and below, and so also does $1-4\epsilon$, neglecting terms in ϵ^2. The value of A is thus unchanged if we retain only terms in ϵ.

From (19),

$$A'\,(m+n)=-\frac{20\,(4m-n)\,(m+n)\,(2\epsilon-\epsilon^2)\,(1-3\epsilon)\,7\epsilon\,(1-3\epsilon)}{\text{same denominator as for } A}.$$

Digitized by Google

Dividing out by ϵ^2, we have

$$\text{new value of } A' = \text{old value} \times \frac{\left(1-\frac{\epsilon}{2}\right)(1-3\epsilon)(1-3\epsilon)}{1-4\epsilon},$$

therefore
$$A' = \frac{8}{15}\,\frac{a^5}{m+n}\,\frac{4m-n}{3m-n}\,(1-\tfrac{5}{2}\epsilon).$$

So from (21), new value of $B = \text{old value} \times \dfrac{(1-2\epsilon)(1-3\epsilon)}{1-4\epsilon}$,

therefore
$$B = -\frac{2}{45}\,\frac{a^3}{m+n}\,\frac{(4m-n)(9m+5n)}{n(3m-n)}\,(1-\epsilon).$$

So from (22), new value of $B' = \text{old value} \times \dfrac{(1-5\epsilon)\left(1-\frac{\epsilon}{2}\right)(1-2\epsilon)}{1-4\epsilon}$,

therefore
$$B' = \frac{4}{35}\,\frac{a^7}{m+n}\,\frac{m(4m-n)}{n(3m-n)}\,(1-\tfrac{7}{2}\epsilon).$$

From (5) $\qquad$ new value of $C = \text{old value} \times \dfrac{1-2\epsilon}{1-\epsilon}$,

therefore
$$C = \frac{a^3}{m+n}\,\frac{5m+n}{3(3m-n)}\,(1-\epsilon).$$

From (6)*, $\qquad$ new value of $E = \text{old value} \times \dfrac{(1-3\epsilon)\left(1-\frac{\epsilon}{2}\right)}{1-\epsilon}$,

therefore
$$E = \frac{a^5}{m+n}\,\frac{5m+n}{90n}\,(1-\tfrac{5}{2}\epsilon).$$

Thus we get, reintroducing the factor $\omega^2\rho$,

$$u = \frac{\omega^2\rho P_2}{5(m+n)}\left[r^3\frac{57m^2-30mn-7n^2}{21n(3m-n)} - (4m-n)\left\{2a^3r\frac{(9m+5n)(1-\epsilon)}{9n(3m-n)}\right.\right.$$
$$\left.\left. + \frac{4a^5}{r^2}\,\frac{(3m+2n)(1-\frac{5}{2}\epsilon)}{9n(3m-n)} - \frac{4a^7}{r^4}\,\frac{m(1-\frac{7}{2}\epsilon)}{7n(3m-n)}\right\}\right]$$
$$+ \frac{\omega^2\rho}{45(m+n)}\left[-3r^3 + 5a^3r\frac{5m+n}{3m-n}(1-\epsilon) + \frac{a^5}{2r^3}\,\frac{5m+n}{n}(1-\tfrac{5}{2}\epsilon)\right] \text{.....(46)†},$$

$$v = \frac{\omega^2\rho}{15n(m+n)(3m-n)}\,\frac{dP_2}{d\theta}\left[r^3\frac{95m^2+48mn-7n^2}{14} - (4m-n)\left\{\frac{a^3r}{3}(9m+5n)(1-\epsilon)\right.\right.$$
$$\left.\left. + \frac{4}{3}\,\frac{a^5}{r^2}\,n(1-\tfrac{5}{2}\epsilon) + \frac{4}{7}\,\frac{a^7}{r^4}\,m(1-\tfrac{7}{2}\epsilon)\right\}\right] \text{.....(47)}.$$

* The values given for E in (6) and (30) are each 4 times too great.

† The value given the coefficient of the last term in (33) is 4 times too great.

Digitized by Google

It will be found useful to notice that in (46) and (47) we invariably find $(1-\epsilon)$ associated with a^3r, $(1-\frac{5}{2}\epsilon)$ with a^5r^{-2}, and $(1-\frac{7}{2}\epsilon)$ with a^7r^{-4}, while ϵ does not appear in the terms in r^3.

§ 46. Let us now find the maximum stress-difference and greatest strain at points at a distance from the centre $r=a(1-x)$, where we neglect x^2.

First let us find the value of S. We can do this at once from the expressions (35) and (37), by noticing from what terms in (46) and (47) the terms of these expressions are derived, and modifying the coefficients by the introduction of ϵ after the scheme indicated above. Doing this and writing $r=a(1-x)$, we get, from (35),

$$-\left\{r\frac{d\cdot\frac{v}{r}}{dr}+\frac{1}{r}\frac{du}{d\theta}\right\}\frac{10n(m+n)(3m-n)}{\omega^2\rho a^2\sin 2\theta(4m-n)}$$

$$=\frac{2(19m+7n)}{7}(1-2x)-\tfrac{2}{3}(9m+5n)(1-\epsilon)-\tfrac{4}{3}(3m-n)(1-\tfrac{5}{2}\epsilon)(1+3x)+\frac{32m}{7}(1-\tfrac{7}{2}\epsilon)(1+5x).$$

When terms in ϵ^2, x^2, and ϵx are neglected this vanishes identically; and thus, even when terms in ϵ and x are retained, the three principal axes of stress at any point coincide with the radius vector, the perpendicular to it in the meridian, and the perpendicular to the meridian.

Similarly, from (37), we find, after reduction,

$$\left\{r\frac{d\cdot\frac{u}{r}}{dr}-\frac{1}{r}\frac{dv}{d\theta}\right\}\frac{m+n}{\omega^2\rho a^2}=\frac{\epsilon-2x}{4n(3m-n)}\left[(3m-n)(7m-n)-8m(4m-n)\sin^2\theta\right]\quad\ldots(48).$$

Remembering that

$$R-\Theta\equiv\pm S=2n\left\{r\frac{d\cdot\frac{u}{r}}{dr}-\frac{1}{r}\frac{dv}{d\theta}\right\},$$

we accordingly have

$$\Theta-R=\frac{\omega^2\rho a^2(2x-\epsilon)}{2(m+n)(3m-n)}\{(3m-n)(7m-n)-8m(4m-n)\sin^2\theta\}\quad\ldots\ldots\ldots(49).$$

By noticing from what terms in (46) and (47) the several terms of (39) are derived, and introducing the modified coefficients in ϵ, we in like manner find

$$15(m+n)(3m-n)R/\omega^2\rho a^2=4P_2\left[\frac{1-2x}{7}(12m^2+25mn-7n^2)\right.$$

$$\left.-(4m-n)\left\{\frac{1-\epsilon}{3}(9m+5n)-\tfrac{2}{3}(1-\tfrac{5}{2}\epsilon)(1+3x)(9m+n)+\tfrac{24}{7}(1-\tfrac{7}{2}\epsilon)(1+5x)m\right\}\right]$$

$$+\tfrac{1}{3}(3m-n)(5m+n)\{5(1-\epsilon)-3(1-2x)-2(1-\tfrac{5}{2}\epsilon)(1+3x)\}.$$

When terms in ϵ^2, x^2, and ϵx are neglected this vanishes identically. Thus R vanishes throughout the entire thickness of the shell, and so (49) is simply an equation giving Θ. The only other stress which does not vanish is Φ, which we proceed to calculate.

Digitized by Google

§ 47. Returning to the expression (40), and introducing the modifications necessary in consequence of the retention of ϵ and the coefficient $\omega^2\rho$, we get

$$15n(m+n)(3m-n)\left(\frac{u}{r}+\frac{v}{r}\cot\theta\right)/\omega^2\rho =$$

$$3\cos^2\theta\left[-r^2 m\frac{19m+39n}{7}-\frac{2a^5}{r^3}(1-\tfrac{5}{2}\epsilon)\,m(4m-n)+\frac{10}{7}\frac{a^7}{r^5}(1-\tfrac{7}{2}\epsilon)\,m(4m-n)\right]$$

$$-r^2\frac{57m^2+12mn-21n^2}{14}+12a^2(1-\epsilon)\,m(m+n)$$

$$+\frac{3}{2}\frac{a^5}{r^3}(1-\tfrac{5}{2}\epsilon)(7m^2+2mn-n^2)-\frac{6}{7}\frac{a^7}{r^5}(1-\tfrac{7}{2}\epsilon)\,m(4m-n)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(50).$$

Write now $r=a(1-x)$ and, neglecting x^2, the right-hand side becomes

$$3a^2\cos^2\theta\left[-(1-2x)\frac{m(19m+39n)}{7}-2(1-\tfrac{5}{2}\epsilon)(1+3x)\,m(4m-n)+\tfrac{10}{7}(1-\tfrac{7}{2}\epsilon)(1+5x)m(4m-n)\right]$$

$$-a^2(1-2x)\frac{57m^2+12mn-21n^2}{14}+12a^2(1-\epsilon)\,m(m+n)+\tfrac{3}{2}a^2(1-\tfrac{5}{2}\epsilon)(1+3x)(7m^2+2mn-n^2)$$

$$-\tfrac{6}{7}a^2(1-\tfrac{7}{2}\epsilon)(1+5x)\,m(4m-n).$$

Reducing, we easily find

$$\frac{u}{r}+\frac{v}{r}\cot\theta=\frac{\omega^2\rho a^2}{E}\left\{(1-2x)\sin^2\theta+(2x-\epsilon)\frac{7m-n}{4m}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(51).$$

Similarly from (32) modified we obtain

$$5(m+n)(3m-n)\,\delta/\omega^2\rho a^2=$$

$$-\cos^2\theta\,\{(1-2x)(21m+n)-4(1-\tfrac{5}{2}\epsilon)(1+3x)(4m-n)\}$$

$$+2(1-2x)(m+n)+\tfrac{5}{3}(1-\epsilon)(5m+n)-\tfrac{4}{3}(1-\tfrac{5}{2}\epsilon)(1+3x)(4m-n);$$

reducing, we find

$$\delta=\frac{\omega^2\rho a^2}{(m+n)(3m-n)}\left[(m+n)\sin^2\theta+(2x-\epsilon)(7m-n)+2\,\{\epsilon(4m-n)-x(9m-n)\}\sin^2\theta\right]\ldots(52).$$

Thus

$$\Phi/\omega^2\rho a^2=$$

$$\left[\sin^2\theta+(2x-\epsilon)\frac{7m}{2(m+n)}+\frac{2\sin^2\theta}{(m+n)(3m-n)}\{\epsilon(m-n)(4m-n)-x(11m^2-8mn+n^2)\}\right]\ldots(53).$$

Of the three principal stresses we thus know Φ from (53) and Θ from (49), while R is zero.

From (53) and (49) we have

$$\Phi-\Theta=\omega^2\rho a^2\sin^2\theta\left\{1+2\frac{x(5m-n)-\epsilon(4m-n)}{3m-n}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(54).$$

Digitized by Google

If θ_1 be the angle given by

$$\sin\theta_1 = \sqrt{\frac{(3m-n)(7m-n)}{8m(4m-n)}} \quad \text{.....(55)},$$

then at polar distances not exceeding θ_1 the maximum stress-difference is given by (54) when x is less than $\epsilon/2$, and by (53) when x is greater than $\epsilon/2$; at polar distances exceeding θ_1 the reverse relation holds.

The absolutely greatest value of the maximum stress-difference occurs on the inner surface in the equator, and has the value

$$\Phi - \Theta = \omega^2\rho a^2\left(1 + \frac{2\epsilon n}{E}\right) \quad \text{.....(56)}.$$

Since R and Θ, two principal stresses, vanish the corresponding principal strains are each equal to $-\delta(m-n)/2n$, where δ is given by (52).

These two strains are clearly negative except in the immediate neighbourhood of the polar axis where $\sin^2\theta$ is of the order ϵ. Since in this extremely limited region all the strains are of the order ϵ, we may for all practical purposes, in treating of the tendency to rupture on the greatest strain theory, confine our attention to the third principal strain, given by (51).

This increases with θ, and has its absolutely greatest value on the inner surface in the equator, where the measure of the tendency to rupture is given by

$$\frac{u}{r} + \frac{v}{r}\cot\theta = \frac{\omega^2\rho a^2}{E}\left(1 - \epsilon\frac{m+n}{4m}\right) \quad \text{.....(57)}.$$

Thus on either theory the tendency to rupture in a thin rotating spherical shell is, for all isotropic substances, greatest in the equatorial regions of the inner surface.

Since ϵ is supposed small the remarks made at the end of § 44 apply with very little modification.

SECTION VII.

Sphere or spherical shell with given surface displacements.

§ 48. Let us briefly consider the application of the general solution (36 a), (37 a), (38 a), Section II., to the case of given surface displacements, say $u = \Sigma U_i$, $v = \Sigma V_i$, $w = \Sigma W_i$ at the surface $r = a$; $u = \Sigma U_i'$, $v = \Sigma V_i'$, $w = \Sigma W_i'$ at the surface $r = a'$; where U_i, V_i, W_i, U_i', V_i', W_i' are surface harmonics of degree i. As in the case of given surface forces this form of solution is especially suitable for purely normal effects, so we shall consider this case specially; afterwards we shall briefly consider the change to be made when v and w are not zero at the surface. Using the solutions mentioned above, our surface conditions shew us that $X_i = w_i = X_i' = w_i' = 0$. Thus the equations $v = 0$, $w = 0$, when $r = a$, give us one complete equation between Y_i, Z_i, Y_i', Z_i'.

Digitized by Google

For $v=0$ gives an equation $\quad \dfrac{d}{d\theta}\cdot f(\theta, \phi)=0,$

and $\quad w=0$ the equation $\quad \dfrac{1}{\sin\theta}\dfrac{d}{d\phi}\cdot f(\theta, \phi)=0,$

all over the surface; therefore the complete equation is $f(\theta, \phi)=0$.

Thus the surface conditions are

$$-\frac{\frac{mi}{n}-2}{2(2i+3)}a^{i+1}Y_i+a^{i-1}Z_i-\frac{\frac{m}{n}(i+1)+2}{2(2i-1)}a^{-i}Y_i'+a^{-i-2}Z_i'=U_i \quad\text{.....(1)},$$

$$-\frac{\frac{m}{n}(i+3)+2}{2(i+1)(2i+3)}a^{i+1}Y_i+\frac{a^{i-1}}{i}Z_i+\frac{\frac{m}{n}(i-2)-2}{2i(2i-1)}a^{-i}Y_i'-\frac{a^{-i-2}}{i+1}Z_i'=0 \quad\text{.....(2)},$$

$$-\frac{\frac{mi}{n}-2}{2(2i+3)}a'^{i+1}Y_i+a'^{i-1}Z_i-\frac{\frac{m}{n}(i+1)+2}{2(2i-1)}a'^{-i}Y_i'+a'^{-i-2}Z_i'=U_i' \quad\text{.....(3)},$$

$$-\frac{\frac{m}{n}(i+3)+2}{2(i+1)(2i+3)}a'^{i+1}Y_i+\frac{a'^{i-1}}{i}Z_i+\frac{\frac{m}{n}(i-2)-2}{2i(2i-1)}a'^{-i}Y_i'-\frac{a'^{-i-2}}{i+1}Z_i'=0 \quad\text{.....(4)}.$$

Multiply (2) by $i+1$ and add it to (1), also multiply (4) by $i+1$ and add it to (3); then we get

$$-\frac{m}{2n}a^{i+1}Y_i+\frac{2i+1}{i}a^{i-1}Z_i-\frac{m(i+1)+n(2i+1)}{ni(2i-1)}a^{-i}Y_i'=U_i \quad\text{.....(5)},$$

$$-\frac{m}{2n}a'^{i+1}Y_i+\frac{2i+1}{i}a'^{i-1}Z_i-\frac{m(i+1)+n(2i+1)}{ni(2i-1)}a'^{-i}Y_i'=U_i' \quad\text{.....(6)};$$

whence, eliminating Z_i,

$$\frac{m}{2n}(a^2-a'^2)Y_i-\frac{m(i+1)+n(2i+1)}{ni(2i-1)}\frac{a^{2i-1}-a'^{2i-1}}{(aa')^{2i-1}}Y_i'=-U_ia^{-i+1}+U_i'a'^{-i+1} \quad\text{.....(7)}.$$

Again, multiply (2) by i and subtract it from (1), also multiply (4) by i and subtract it from (3), then we get

$$\frac{mi+n(2i+1)}{n(i+1)(2i+3)}a^{i+1}Y_i-\frac{m}{2n}a^{-i}Y_i'+\frac{2i+1}{i+1}a^{-i-2}Z_i'=U_i \quad\text{.....(8)},$$

$$\frac{mi+n(2i+1)}{n(i+1)(2i+3)}a'^{i+1}Y_i-\frac{m}{2n}a'^{-i}Y_i'+\frac{2i+1}{i+1}a'^{-i-2}Z_i'=U_i' \quad\text{.....(9)};$$

whence, eliminating Z_i',

$$\frac{mi+n(2i+1)}{n(i+1)(2i+3)}(a^{2i+3}-a'^{2i+3})Y_i-\frac{m}{2n}(a^2-a'^2)Y_i'=U_ia^{i+2}-U_i'a'^{i+2} \quad\text{.....(10)}.$$

(7) and (10) give Y_i and Y_i', whence we may obtain Z_i from (6) or (5), and Z_i' from (8) or (9).

Digitized by Google

If for shortness

$$D = \begin{vmatrix} \dfrac{mi+n(2i+1)}{n(i+1)(2i+3)}(a^{2i+3}-a'^{2i+3}) & \dfrac{m}{2n}(a^2-a'^2) \\ \dfrac{m}{2n}(a^2-a'^2) & \dfrac{m(i+1)+n(2i+1)}{ni(2i-1)}(a'^{-2i+1}-a^{-2i+1}) \end{vmatrix};$$

then

$$DY_i = \begin{vmatrix} U_i a^{i+2} - U_i' a'^{i+2} & \dfrac{m}{2n}(a^2-a'^2) \\ U_i' a'^{-i+1} - U_i a^{-i+1} & \dfrac{m(i+1)+n(2i+1)}{ni(2i-1)}(a'^{-2i+1}-a^{-2i+1}) \end{vmatrix} \quad \text{.....(11)},$$

$$DY_i' = \begin{vmatrix} U_i a^{i+2} - U_i' a'^{i+2} & \dfrac{mi+n(2i+1)}{n(i+1)(2i+3)}(a^{2i+3}-a'^{2i+3}) \\ U_i' a'^{-i+1} - U_i a^{-i+1} & \dfrac{m}{2n}(a^2-a'^2) \end{vmatrix} \quad \text{.....(12)}.$$

§ 49. If the sphere were solid $Y_i' = Z_i' = 0$, and $a' = 0$; therefore from (1) and (2), which are now the only conditions,

$$Y_i = U_i a^{-i-1} \frac{n(i+1)(2i+3)}{mi+n(2i+1)} \quad \text{.....(13)},$$

$$Z_i = iU_i a^{-i+1} \frac{m(i+3)+2n}{2\{mi+n(2i+1)\}} \quad \text{.....(14)}.$$

Substituting these values in the general equations we get

$$\delta = U_i r^i a^{-i-1} \frac{n(i+1)(2i+3)}{mi+n(2i+1)} \quad \text{.....(15)},$$

$$u = \frac{U_i}{2\{mi+n(2i+1)\}}\left[\left(\frac{r}{a}\right)^{i-1} i\{m(i+3)+2n\} - \left(\frac{r}{a}\right)^{i+1}(i+1)(mi-2n)\right] \quad \text{.....(16)},$$

$$v = \frac{m(i+3)+2n}{2\{mi+n(2i+1)\}}\frac{dU_i}{d\theta}\left\{\left(\frac{r}{a}\right)^{i-1} - \left(\frac{r}{a}\right)^{i+1}\right\} \quad \text{.....(17)},$$

$$w = \frac{m(i+3)+2n}{2\{mi+n(2i+1)\}}\frac{1}{\sin\theta}\frac{dU_i}{d\phi}\left\{\left(\frac{r}{a}\right)^{i-1} - \left(\frac{r}{a}\right)^{i+1}\right\} \quad \text{.....(18)}.$$

§ 50. The method to be adopted when the surface displacements are not purely normal is exactly the same as in the case of surface forces which are not purely normal, so we shall merely glance at the case of a solid sphere under surface displacements

$$u = \Sigma U_i; \quad v = \Sigma V_i; \quad w = \Sigma W_i.$$

Digitized by Google

Instead of (1) and (2) we get from the general equations of Section II.

$$-\left(\frac{\frac{mi}{n}-2}{2(2i+3)}\right)a^{i+1}Y_i+a^{i-1}Z_i=U_i \quad\text{.....(19),}$$

$$\frac{d}{d\theta}\Sigma\left[-\frac{\frac{m}{n}(i+3)+2}{2(i+1)(2i+3)}a^{i+1}Y_i+\frac{a^{i-1}}{i}Z_i\right]+\Sigma\frac{a^iX_i}{\sin\theta}=\Sigma V_i \quad\text{.....(20),}$$

$$\frac{1}{\sin\theta}\frac{d}{d\phi}\Sigma\left[-\frac{\frac{m}{n}(i+3)+2}{2(i+1)(2i+3)}a^{i+1}Y_i+\frac{a^{i-1}}{i}Z_i\right]+\Sigma a^iw_i=\Sigma W_i \quad\text{.....(21),}$$

where

$$\frac{dw_i}{d\phi}=-\frac{dX_i}{d\theta} \quad\text{.....(22).}$$

Multiply both sides of (20) by $\sin\theta$, then differentiate with respect to θ, and differentiate both sides of (21) with respect to ϕ. Adding these two results we eliminate X_i and w_i by means of (22) and get, using the surface harmonic equation,

$$\Sigma\left[\frac{m(i+3)+2n}{2n(2i+3)}ia^{i+1}Y_i-(i+1)a^{i-1}Z_i\right]=\frac{1}{\sin\theta}\Sigma\left[\frac{d}{d\theta}.\sin\theta V_i+\frac{dW_i}{d\phi}\right] \quad\text{.....(23).}$$

Multiply both sides of (21) by $\sin\theta$ and differentiate with respect to θ, then differentiate (20) with respect to ϕ and subtract from (21) so modified. This eliminates Y_i and Z_i. Finally using (22) and the surface harmonic differential equation, and then differentiating with respect to ϕ, we get

$$\Sigma i(i+1)a^iX_i=\frac{1}{\sin\theta}\Sigma\left(\frac{d}{d\theta}.\sin\theta\frac{dW_i}{d\phi}-\frac{d^2V_i}{d\phi^2}\right) \quad\text{.....(24).}$$

[September 18, 1888. If over the surface $r=a$ of a solid sphere the displacements be derivable from a potential expressible in spherical harmonics—i.e. if $u=\Sigma\frac{1}{i}\frac{d}{dr}\left(S_ir^i/a^{i-1}\right)$, $v=\Sigma\frac{1}{ir}\frac{d}{d\theta}\left(S_ir^i/a^{i-1}\right)$, and $w=\Sigma\frac{1}{ir\sin\theta}\frac{d}{d\phi}\left(S_ir^i/a^{i-1}\right)$, when $r=a$, representing by S_i the typical surface harmonic — then throughout the sphere $u=\Sigma S_i(r/a)^{i-1}$, $v=\Sigma\frac{1}{i}\frac{dS_i}{d\theta}\left(\frac{r}{a}\right)^{i-1}$, and $w=\Sigma\frac{1}{i\sin\theta}\frac{dS_i}{d\phi}\left(\frac{r}{a}\right)^{i-1}$. If the displacements over the surfaces $r=a$ and $r=a'$ of a spherical shell be similarly derivable from potentials involving typical surface harmonics S_i and S_i' respectively, then X_i, X_i', w_i, w_i' etc. vanish, and an explicit solution in terms of S_i and S_i' is in like manner easily obtainable. Since in these cases $\delta=0$ the solutions are applicable to the case of bodies bounded by spherical surfaces which are immersed in or contain incompressible fluid.]

Digitized by Google

SECTION VIII.

Vibrations of a sphere or spherical shell.

§ 51. Supposing no internal non-elastic forces to act, the equations of motion are, see (24), (25) and (26) Section I.,

$$(m+n)\, r^2 \sin\theta \frac{d\delta}{dr} - n\frac{d\mathfrak{C}}{d\theta} + n\frac{d\mathfrak{B}}{d\phi} = \rho r^2 \sin\theta \frac{d^2u}{dt^2} \quad \text{.....(1)},$$

$$(m+n)\sin\theta \frac{d\delta}{d\theta} - n\frac{d\mathfrak{A}}{d\phi} + n\frac{d\mathfrak{C}}{dr} = \rho r \sin\theta \frac{d^2v}{dt^2} \quad \text{.....(2)},$$

$$(m+n)\operatorname{cosec}\theta \frac{d\delta}{d\phi} - n\frac{d\mathfrak{B}}{dr} + n\frac{d\mathfrak{A}}{d\theta} = \rho r \frac{d^2w}{dt^2} \quad \text{.....(3)};$$

where u, v, w, δ, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ have the same signification as before, viz.

$$\delta = \frac{1}{r^2}\left(\frac{d\,.\,ur^2}{dr} + \frac{1}{\sin\theta}\frac{d\,.\,vr\sin\theta}{d\theta} + \frac{1}{\sin^2\theta}\frac{d\,.\,wr\sin\theta}{d\phi}\right) \quad \text{.....(4)},$$

&c.

Differentiate (1) with respect to r,
(2) ..... θ,
and (3) ..... ϕ;

adding we get

$$(m+n)\,\nabla^2\delta = \rho\frac{d^2\delta}{dt^2} \quad \text{.....(5)},$$

where ∇^2 has its usual meaning.

In (5) we may at once assume $\delta \propto \cos kt$ and then determine it as a function of r, θ and ϕ. This gives

$$\frac{d^2\delta}{dr^2} + \frac{2}{r}\frac{d\delta}{dr} + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}\cdot\sin\theta\frac{d\delta}{d\theta} + \frac{1}{r^2\sin^2\theta}\frac{d^2\delta}{d\phi^2} + \frac{k^2\rho\delta}{m+n} = 0 \quad \text{.....(6)}.$$

Regarding δ as a function of θ and ϕ, we may suppose it expanded in a series of surface spherical harmonics of different degrees. Then to determine the function of r forming the coefficient of Y_i, the surface spherical harmonic of degree i in this expansion, we have

$$\frac{d^2\delta}{dr^2} + \frac{2}{r}\frac{d\delta}{dr} + \delta\left\{k^2\alpha^2 - \frac{i(i+1)}{r^2}\right\} = 0 \quad \text{.....(7)},$$

where for shortness

$$\alpha^2 \equiv \frac{\rho}{m+n} \quad \text{.....(8)}.$$

Put also

$$\beta^2 \equiv \frac{\rho}{n} \quad \text{.....(9)}.$$

We may write (7) in the form

$$\frac{d^2\,.\,r^{\frac{1}{2}}\delta}{dr^2} + \frac{1}{r}\frac{d\,.\,r^{\frac{1}{2}}\delta}{dr} + \left\{k^2\alpha^2 - \frac{(i+\frac{1}{2})^2}{r^2}\right\} r^{\frac{1}{2}}\delta = 0 \quad \text{.....(10)}.$$

Digitized by Google

Equation (10) is the well-known Bessel's equation having two solutions $J_{i+\frac{1}{2}}(k\alpha r)$ and $Y_{i+\frac{1}{2}}(k\alpha r)$ in the usual notation. Only the former occurs if the sphere be solid, because $Y_{i+\frac{1}{2}}(k\alpha r)$ is infinite when $r=0$. But for a spherical shell both occur. Of course the Y here has no connection whatever with the Y used above in the expression for a spherical harmonic.

For the present confining ourselves to a solid sphere we have

$$\delta = \cos kt Y_i r^{-\frac{1}{2}} J_{i+\frac{1}{2}}(k\alpha r) \qquad (11).$$

§ 52. The method to be followed in determining u, v, and w is exactly the same as in the case of equilibrium. Thus in (1) substitute the values of $\mathfrak{B}$ and $\mathfrak{C}$, confining ourselves to a single surface spherical harmonic, we thus have

$$\frac{d}{d\theta}.\sin\theta\left(\frac{d.vr}{dr}-\frac{du}{d\theta}\right)-\frac{1}{\sin\theta}\left(\frac{d^2u}{d\phi^2}-\frac{d}{dr}.\frac{d.wr\sin\theta}{d\phi}\right)=-\frac{\rho}{n}r^2\sin\theta\frac{d^2u}{dt^2}+\frac{m+n}{n}r^2\sin\theta\frac{d\delta}{dr} \qquad (12).$$

Substitute in (12) from (4),

$$\frac{d.wr\sin\theta}{d\phi}=-\sin^2\theta\left(\frac{d.ur^2}{dr}+\frac{1}{\sin\theta}\frac{d.vr\sin\theta}{d\theta}-r^2\delta\right);$$

we thus get

$$-\frac{d}{d\theta}.\sin\theta\frac{du}{d\theta}-\frac{1}{\sin\theta}\frac{d^2u}{d\phi^2}+\frac{d}{d\theta}.\sin\theta\frac{d.vr}{dr}-\frac{1}{\sin\theta}\frac{d}{dr}.\sin^2\theta\left\{\frac{d.ur^2}{dr}+\frac{1}{\sin\theta}\frac{d.vr\sin\theta}{d\theta}-\delta r^2\right\}$$
$$=-\frac{\rho}{n}r^2\sin\theta\frac{d^2u}{dt^2}+\frac{m+n}{n}r^2\sin\theta\frac{d\delta}{dr}.$$

The terms in v cut out leaving a differential equation containing u and δ. Divide out by $\sin\theta$ and, confining ourselves at present to the spherical harmonics of the ith degree, substitute the value (11) for δ. Assuming

$$u \propto U_i \cos kt \qquad (13),$$

where U_i represents spherical surface harmonics of degree i, we get to determine u as a function of r the equation

$$\frac{1}{r^2}\frac{d^2}{dr^2}.ur^2-\frac{i(i+1)}{r^2}u+k^2\beta^2u=\frac{1}{r^2}\left[\frac{d}{dr}.r^{\frac{3}{2}}J_{i+\frac{1}{2}}(k\alpha r)-\frac{m+n}{n}r^2\frac{d}{dr}.r^{-\frac{1}{2}}J_{i+\frac{1}{2}}(k\alpha r)\right]Y_i\cos kt \qquad (14).$$

We may write (14) in the more convenient form

$$\frac{d^2}{dr^2}.ur^{\frac{3}{2}}+\frac{1}{r}\frac{d}{dr}.ur^{\frac{3}{2}}+\left\{k^2\beta^2-\frac{(i+\frac{1}{2})^2}{r^2}\right\}ur^{\frac{3}{2}}$$
$$=-\left[\frac{m}{n}r\frac{d}{dr}J_{i+\frac{1}{2}}(k\alpha r)-\left(\frac{m}{2n}+2\right)J_{i+\frac{1}{2}}(k\alpha r)\right]Y_i\cos kt \qquad (15).$$

The general solution of (15) is obviously

$$ur^{\frac{3}{2}}=BJ_{i+\frac{1}{2}}(k\beta r) \qquad (16),$$

Digitized by Google

where B is independent of r and is, in accordance with (13), of the form $Z_i \cos kt$, where Z_i is an undetermined surface harmonic of degree i. Thus for the general solution of (15) we have

$$u = Z_i \cos kt\, r^{-\frac{1}{2}} J_{i+\frac{1}{2}}(k\beta r) \qquad (17).$$

The particular solution is clearly of the form $u = Y_i \cos kt f(r)$, where $f(r)$ is a function of r. Now from the Bessel's equation it is easy to show that

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \left\{k^2\beta^2 - \frac{(i+\frac{1}{2})^2}{r^2}\right\}\right] r \frac{d}{dr} J_{i+\frac{1}{2}}(k\alpha r) = k^2(\beta^2 - \alpha^2) r \frac{d}{dr} J_{i+\frac{1}{2}}(k\alpha r) - 2k^2\alpha^2 J_{i+\frac{1}{2}}(k\alpha r),$$

i.e. by (8) and (9)

$$= \frac{k^2\rho}{m+n}\left[\frac{m}{n} r \frac{d}{dr} J_{i+\frac{1}{2}}(k\alpha r) - 2J_{i+\frac{1}{2}}(k\alpha r)\right] \qquad (18).$$

Also
$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \left\{k^2\beta^2 - \frac{(i+\frac{1}{2})^2}{r^2}\right\}\right] J_{i+\frac{1}{2}}(k\alpha r) = k^2(\beta^2 - \alpha^2) J_{i+\frac{1}{2}}(k\alpha r)$$
$$= \frac{k^2\rho}{m+n}\frac{m}{n} J_{i+\frac{1}{2}}(k\alpha r).$$

Thus the particular solution of (15) is obviously

$$ur^{\frac{1}{2}} = \frac{m+n}{k^2\rho}\left[-r\frac{d}{dr} J_{i+\frac{1}{2}}(k\alpha r) + \tfrac{1}{2} J_{i+\frac{1}{2}}(k\alpha r)\right] Y_i \cos kt.$$

Thus the complete value of u, so far as depends on $\cos kt$ and surface spherical harmonics of degree i, is

$$u = \cos kt\left[\frac{1}{k^2\alpha^2} Y_i\left\{\frac{r^{-\frac{1}{2}}}{2} J_{i+\frac{1}{2}}(k\alpha r) - r^{-\frac{1}{2}}\frac{d}{dr} J_{i+\frac{1}{2}}(k\alpha r)\right\} + Z_i r^{-\frac{1}{2}} J_{i+\frac{1}{2}}(k\beta r)\right] \qquad (19).$$

§ 53. In the case of v let us follow the same plan as in the case of equilibrium, and find the portion of the solution for v which depends on Y_i and Z_i and on the general solution. The method is the same exactly.

In (2) substitute their values for $\mathfrak{A}$ and $\mathfrak{C}$ and for $\frac{d.\,vr\sin\theta}{d\phi}$ in terms of u, v, and δ, also put in its value for δ. We thus get

$$-\frac{1}{r^2\sin\theta}\frac{d^2.\,vr}{d\phi^2} + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}.\sin^2\theta\left[r^{\frac{1}{2}} Y_i J_{i+\frac{1}{2}}(k\alpha r) - \frac{d.\,ur^2}{dr} - \frac{1}{\sin\theta}\frac{d.\,vr\sin\theta}{d\theta}\right]$$
$$- \sin\theta\frac{d^2.\,vr}{dr^2} + \sin\theta\frac{d^2u}{dr\,d\theta} + \frac{\rho}{n}\sin\theta\, r\frac{d^2v}{dt^2} = \frac{m+n}{n}\sin\theta\frac{dY_i}{d\theta} r^{-\frac{1}{2}} J_{i+\frac{1}{2}}(k\alpha r)\cos kt.$$

Here v is assumed $\propto \cos kt$, so put $\frac{d^2v}{dt^2} = -k^2 v$, and the above becomes

$$\frac{1}{r^2\sin^2\theta}\frac{d^2.\,vr\sin\theta}{d\phi^2} + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}.\sin\theta\frac{d.\,vr\sin\theta}{d\theta} + \frac{d^2.\,vr\sin\theta}{dr^2} + k^2\beta^2 vr\sin\theta$$
$$= r^{-\frac{1}{2}}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}.\sin^2\theta Y_i - \frac{m+n}{n}\sin\theta\frac{dY_i}{d\theta}\right]\cos kt\, J_{i+\frac{1}{2}}(k\alpha r)$$
$$+ \sin\theta\frac{d^2u}{dr\,d\theta} - \frac{1}{r^2\sin\theta}\frac{d}{d\theta}.\sin^2\theta\frac{d.\,ur^2}{dr}.$$

Digitized by Google

Substitute for u from (19), and this gives

$$\frac{1}{\cos kt}\left\{\frac{d^2 . vr \sin\theta}{dr^2}+\frac{1}{r^2\sin\theta}\frac{d}{d\theta}.\sin\theta\frac{d.vr\sin\theta}{d\theta}+\frac{1}{r^2\sin^2\theta}\frac{d^2.vr\sin\theta}{d\phi^2}+k^2\beta^2 vr\sin\theta\right\}$$

$$=\left(\sin\theta\frac{dY_i}{d\theta}+2\cos\theta Y_i\right)\left[r^{-\frac{1}{2}}J_{i+\frac{1}{2}}(kar)-\frac{m+n}{k^2\rho r^2}\frac{d}{dr}\left\{\frac{r^{\frac{1}{2}}}{2}J_{i+\frac{1}{2}}(kar)-r^{\frac{3}{2}}\frac{d}{dr}J_{i+\frac{1}{2}}(kar)\right\}\right]$$

$$+\sin\theta\frac{dY_i}{d\theta}\left[-\frac{m+n}{n}r^{-\frac{1}{2}}J_{i+\frac{1}{2}}(kar)+\frac{m+n}{k^2\rho}\frac{d}{dr}\left\{\frac{r^{-\frac{3}{2}}}{2}J_{i+\frac{1}{2}}(kar)-r^{-\frac{1}{2}}\frac{d}{dr}J_{i+\frac{1}{2}}(kar)\right\}\right]$$

$$+\sin\theta\frac{dZ_i}{d\theta}\frac{d}{dr}.r^{-\frac{1}{2}}J_{i+\frac{1}{2}}(k\beta r)-\frac{1}{r^2}\left(\sin\theta\frac{dZ_i}{d\theta}+2\cos\theta Z_i\right)\frac{d}{dr}.r^{\frac{1}{2}}J_{i+\frac{1}{2}}(k\beta r)\quad\text{.....(20).}$$

To simplify this we must employ Bessel's equation, and carry out the differentiations when required. We thus find

$$\frac{d}{dr}\left\{\frac{r^{\frac{1}{2}}}{2}J_{i+\frac{1}{2}}(kar)-r^{\frac{3}{2}}\frac{d}{dr}J_{i+\frac{1}{2}}(kar)\right\}=r\left\{k^2\alpha^2-\frac{i(i+1)}{r^2}\right\}J_{i+\frac{1}{2}}(kar)\quad\text{.....(21).}$$

Thus, noticing that $\frac{\rho}{m+n}=\alpha^2$, we reduce the coefficient of $\sin\theta\frac{dY_i}{d\theta}+2\cos\theta Y_i$ in the first line of the second side of (20) to

$$\frac{1}{k^2\alpha^2}\frac{i(i+1)}{r^{\frac{3}{2}}}J_{i+\frac{1}{2}}(kar)\quad\text{.....(22).}$$

So again the coefficient of $\sin\theta\frac{dY_i}{d\theta}$ in the second line is reducible to

$$\frac{2}{k^2\alpha^2}r^{-\frac{3}{2}}\frac{d}{dr}J_{i+\frac{1}{2}}(kar)-r^{-\frac{1}{2}}J_{i+\frac{1}{2}}(kar)\left\{\frac{m}{n}+\frac{i(i+1)+1}{k^2\alpha^2r^2}\right\}\quad\text{.....(23).}$$

Thus equation (20) becomes

$$\frac{1}{\cos kt}\left[\frac{d^2.vr\sin\theta}{dr^2}+\frac{1}{r^2\sin\theta}\frac{d}{d\theta}.\sin\theta\frac{d.vr\sin\theta}{d\theta}+\frac{1}{r^2\sin^2\theta}\frac{d^2.vr\sin\theta}{d\phi^2}+k^2\beta^2 vr\sin\theta\right]$$

$$=\frac{2\cos\theta Y_i}{k^2\alpha^2}\frac{i(i+1)}{r^{\frac{3}{2}}}J_{i+\frac{1}{2}}(kar)+\sin\theta\frac{dY_i}{d\theta}\left\{\frac{2r^{-\frac{3}{2}}}{k^2\alpha^2}\frac{d}{dr}J_{i+\frac{1}{2}}(kar)-\left(\frac{m}{n}r^{-\frac{1}{2}}+\frac{r^{-\frac{3}{2}}}{k^2\alpha^2}\right)J_{i+\frac{1}{2}}(kar)\right\}$$

$$-2\cos\theta\frac{Z_i}{r^2}\frac{d}{dr}.r^{\frac{1}{2}}J_{i+\frac{1}{2}}(k\beta r)+\sin\theta\frac{dZ_i}{d\theta}\{-2r^{-\frac{3}{2}}J_{i+\frac{1}{2}}(k\beta r)\}\quad\text{.....(24).}$$

From the form of the first side of (24), for the general solution assume $vr\sin\theta\propto X_i$, a surface spherical harmonic of degree i; whence, to determine vr as a function of r, we have

$$\frac{d^2.vr}{dr^2}+\left\{k^2\beta^2-\frac{i(i+1)}{r^2}\right\}vr=0,$$

i.e.

$$\frac{d^2.vr^{\frac{1}{2}}}{dr^2}+\frac{1}{r}\frac{d}{dr}.vr^{\frac{1}{2}}+\left\{k^2\beta^2-\frac{(i+\frac{1}{2})^2}{r^2}\right\}vr^{\frac{1}{2}}=0,$$

or

$$vr^{\frac{1}{2}}\propto J_{i+\frac{1}{2}}(k\beta r).$$

Digitized by Google

So, writing in the coefficients $\cos kt$ and X_i, we get

$$v = \cos kt \frac{X_i}{\sin\theta} r^{-\frac{1}{2}} J_{i+\frac{1}{2}}(k\beta r) \quad \text{.....(25)},$$

as a typical term of this part of the solution for v.

We have next to find the particular solution of (24). The analogy of the solution obtained for the case of equilibrium suggests the trial of $\sin\theta \frac{dY_i}{d\theta}$ as the function of θ and ϕ which will give the part of the particular solution depending on Y_i. Thus substituting $\cos kt f(r) \sin\theta \frac{dY_i}{d\theta}$ in the left-hand side of (24) for $vr \sin\theta$, we get

$$\sin\theta \frac{dY_i}{d\theta} \frac{d^2 f(r)}{dr^2} + \frac{f(r)}{r^2 \sin\theta} \frac{d}{d\theta} \cdot \sin\theta \frac{d}{d\theta} \cdot \sin\theta \frac{dY_i}{d\theta} + \frac{f(r)}{r^2 \sin^2\theta} \frac{d^2}{d\phi^2} \cdot \sin\theta \frac{dY_i}{d\theta} + k^2\beta^2 f(r) \sin\theta \frac{dY_i}{d\theta}.$$

After using $$\frac{1}{\sin^2\theta} \frac{d^2 Y_i}{d\phi^2} = -i(i+1) Y_i - \frac{1}{\sin\theta} \frac{d}{d\theta} \cdot \sin\theta \frac{dY_i}{d\theta},$$

this gives for the first side of (24) the expression

$$\sin\theta \frac{dY_i}{d\theta} \left\{ \frac{d^2 f(r)}{dr^2} + \left(k^2\beta^2 - \frac{i(i+1)}{r^2}\right) f(r) \right\} - \frac{2i(i+1)}{r^2} f(r) \cos\theta Y_i \quad \text{.....(26)}.$$

If $\cos kt f(r) \sin\theta \frac{dY_i}{d\theta}$ is really the particular solution of (24), then the expression (26) must be identical with that part of the second side of (24) which contains Y_i. Equating the coefficients of $\cos\theta Y_i$ and of $\sin\theta \frac{dY_i}{d\theta}$ in (26) and (24) we get two equations which ought to be identities. The former requires

$$f(r) = -\frac{r^{-\frac{1}{2}}}{k^2\alpha^2} J_{i+\frac{1}{2}}(k\alpha r) \quad \text{.....(27)};$$

and thus in accordance with the second we must have

$$\left[\frac{d^2}{dr^2} + k^2\beta^2 - \frac{i(i+1)}{r^2}\right] \left(-\frac{r^{-\frac{1}{2}}}{k^2\alpha^2} J_{i+\frac{1}{2}}(k\alpha r)\right) = \text{coefficient of } \sin\theta \frac{dY_i}{d\theta} \text{ in (24)}.$$

Substituting for $\frac{d^2}{dr^2} J_{i+\frac{1}{2}}(k\alpha r)$ from the Bessel's equation we reduce this expression to

$$-\frac{r^{-\frac{1}{2}}}{k^2\alpha^2} \left[-\frac{2}{r} \frac{d}{dr} J_{i+\frac{1}{2}}(k\alpha r) + \left\{k^2(\beta^2 - \alpha^2) + \frac{1}{r^2}\right\} J_{i+\frac{1}{2}}(k\alpha r)\right].$$

Noticing that $\frac{\beta^2 - \alpha^2}{\alpha^2} \equiv \frac{m}{n}$, it is obvious that this is actually identical with the coefficient of $\sin\theta \frac{dY_i}{d\theta}$ in (24). Thus our hypothesis was correct, and the particular solution of (24) depending on Y_i is

$$v = -\frac{\cos kt}{k^2\alpha^2} r^{-\frac{1}{2}} J_{i+\frac{1}{2}}(k\alpha r) \frac{dY_i}{d\theta} \quad \text{.....(28)}.$$

Digitized by Google

The same analogy also suggests $\cos kt \sin\theta \frac{dZ_i}{d\theta}\chi(r)$ as the form of particular solution of (24) giving the terms depending on Z_i. Writing in Z_i for Y_i and $\chi(r)$ for $f(r)$ in (26), we see from (24) that if this hypothesis be correct $\chi(r)$ must satisfy both the equations

$$-\frac{2i(i+1)}{r^2}\chi(r) = -\frac{2}{r^2}\frac{d}{dr}\cdot r^{\frac{1}{2}} J_{i+\frac{1}{2}}(k\beta r) \quad\text{.....(29)},$$

and

$$\frac{d^2}{dr^2}\chi(r) + k^2\beta^2\chi(r) - \frac{i(i+1)}{r^2}\chi(r) = -2r^{-\frac{1}{2}} J_{i+\frac{1}{2}}(k\beta r) \quad\text{.....(30)}.$$

These come from equating the coefficients of $\cos\theta Z_i$ and of $\sin\theta\frac{dZ_i}{d\theta}$ respectively.

Now (29) gives

$$\chi(r) = \frac{1}{i(i+1)}\left\{r^{\frac{1}{2}}\frac{d}{dr}J_{i+\frac{1}{2}}(k\beta r) + \frac{r^{-\frac{1}{2}}}{2}J_{i+\frac{1}{2}}(k\beta r)\right\} \quad\text{.....(31)},$$

and using the Bessel's equation it will be found that this satisfies (30) as it ought.

Thus

$$\chi(r) = \frac{1}{i(i+1)}\frac{d}{dr}r^{\frac{1}{2}}J_{i+\frac{1}{2}}(k\beta r) \quad\text{.....(32)},$$

and the particular solution of (24) depending on Z_i is

$$v = \frac{\cos kt}{i(i+1)}\frac{dZ_i}{d\theta}r^{-1}\frac{d}{dr}\cdot r^{\frac{1}{2}}J_{i+\frac{1}{2}}(k\beta r) \quad\text{.....(33)}.$$

Thus the complete value of v so far as depends on Y_i, Z_i and the additional typical term X_i, is

$$v = \cos kt\left\{-\frac{1}{k^2\alpha^2}\frac{dY_i}{d\theta}r^{-\frac{1}{2}}J_{i+\frac{1}{2}}(k\alpha r) + \frac{1}{i(i+1)}\frac{dZ_i}{d\theta}r^{-1}\frac{d}{dr}\cdot r^{\frac{1}{2}}J_{i+\frac{1}{2}}(k\beta r) + \frac{X_i}{\sin\theta}r^{-\frac{1}{2}}J_{i+\frac{1}{2}}(k\beta r)\right\} \quad\text{...(34)}.$$

§ 54. Just as in the case of equilibrium* we need not form the differential equation in w given from (3); for we know now u, v, and δ, and so get w from the expression (4) for δ. The plan followed before applies here in its minutest detail, as it was independent of the functions of r occurring in u and v, and depended only on the differential equation satisfied by Y_i, Z_i, and X_i. Thus

$$w = \frac{\cos kt}{\sin\theta}\frac{d}{d\phi}\left[-\frac{Y_i}{k^2\alpha^2}r^{-\frac{1}{2}}J_{i+\frac{1}{2}}(k\alpha r) + \frac{Z_i r^{-1}}{i(i+1)}\frac{d}{dr}\cdot r^{\frac{1}{2}}J_{i+\frac{1}{2}}(k\beta r)\right] + \cos kt\, w_i r^{-\frac{1}{2}}J_{i+\frac{1}{2}}(k\beta r) \quad\text{...(35)},$$

where

$$\frac{dw_i}{d\phi} = -\frac{dX_i}{d\theta} \quad\text{.....(36)},$$

(34) and (35) give that part of the solution of v and w which depends on the typical term for u in (19) containing surface harmonics of degree i, and also the typical independent term X_i.

§ 55. The foregoing proof is rather long, but the result, now that its form is known, can be deduced quite shortly as follows. Proceed as before till equation (19) is obtained; then, to find the corresponding terms for v and w, assume

$$vr = \cos kt\left\{f(r)\frac{dY_i}{d\theta} + \chi(r)\frac{dZ_i}{d\theta}\right\} \quad\text{.....(37)},$$

* See § 10, Section II.

Digitized by Google

and
$$wr = \frac{\cos kt}{\sin\theta}\frac{d}{d\phi}\{f(r)\,Y_i + \chi(r)\,Z_i\} \qquad (38).$$

First to determine $f(r)$ use (4), substituting from (37) and (38) for v and w. This gives
$$\cos kt\, f(r)\left\{\frac{1}{\sin\theta}\frac{d}{d\theta}.\sin\theta\frac{dY_i}{d\theta} + \frac{1}{\sin^2\theta}\frac{d^2Y_i}{d\phi^2}\right\} = r^2\delta - \frac{d.ur^2}{dr}.$$

Substitute for δ from (11) and for u from (19); we then get
$$-i(i+1)f(r)\,Y_i = Y_i\left[r^{\frac{1}{2}}J_{i+\frac{1}{2}}(kar) - \frac{1}{k^2a^2}\frac{d}{dr}\left\{\frac{r^{\frac{1}{2}}}{2}J_{i+\frac{1}{2}}(kar) - r^{\frac{3}{2}}\frac{d}{dr}J_{i+\frac{1}{2}}(kar)\right\}\right] \qquad (39).$$

But the coefficient of Y_i on the second side of (39) is equal to the coefficient of $\left(\sin\theta\frac{dY_i}{d\theta} + 2\cos\theta Y_i\right)$ in (20) multiplied by r^2; thus from the form we got for this coefficient in (22), it follows at once that
$$f(r) = -\frac{r^{-\frac{1}{2}}}{k^2a^2}J_{i+\frac{1}{2}}(kar) \qquad (40),$$

and the corresponding term in v is $-\frac{\cos kt}{k^2a^2}\frac{dY_i}{d\theta}r^{-\frac{3}{2}}J_{i+\frac{1}{2}}(kar)$.

The term in Z_i may be got in the same way with even less difficulty. Then to obtain the typical term X_i we require only the first side of (24), which it is comparatively simple to obtain.

§ 56. We have now to determine X_i, Y_i, Z_i, &c. from the surface conditions. Suppose the surface forces depending on $\cos kt$ are $\Sigma S_i\cos kt$ normally, $\Sigma T_i\cos kt$ tangentially in the meridian, and $\Sigma U_i\cos kt$ perpendicular to the meridian; where S_i, T_i and U_i are spherical surface harmonics of degree i. Then so far as depends on $\cos kt$ the surface conditions require
$$(m-n)\,\delta + 2n\frac{du}{dr} = \Sigma S_i\cos kt \qquad (41),$$
$$n\left\{r\frac{d.\frac{v}{r}}{dr} + \frac{1}{r}\frac{du}{d\theta}\right\} = \Sigma T_i\cos kt \qquad (42),$$
$$n\left\{r\frac{d.\frac{w}{r}}{dr} + \frac{1}{r\sin\theta}\frac{du}{d\phi}\right\} = \Sigma U_i\cos kt \qquad (43).$$

We shall find, as in the case of equilibrium, that if the surface forces be purely normal X_i and w_i are zero so far as depends on the surface forces, and so exist only as "free" vibrations. In this case we can drop the Σ in (41), (42) and (43); and in every case we can drop it in (41). Thus from (41), (11), and (19), we have, putting $r = a$ after differentiation,

Digitized by Google

$$(m-n)\,Y_i a^{-\frac{1}{2}} J_{i+\frac{1}{2}}(k\alpha a) + \frac{2n}{k^2\alpha^2}\,Y_i \frac{d}{dr}\left\{\frac{1}{2} r^{-\frac{3}{2}} J_{i+\frac{1}{2}}(k\alpha r) - r^{-\frac{1}{2}} \frac{d}{dr} J_{i+\frac{1}{2}}(k\alpha r)\right\}$$
$$+ 2nZ_i \frac{d}{dr}\left\{r^{-\frac{3}{2}} J_{i+\frac{1}{2}}(k\beta r)\right\} = S_i \ldots\ldots(44).$$

Further, we see that

$$\frac{d}{dr}\left\{\frac{1}{2} r^{-\frac{3}{2}} J_{i+\frac{1}{2}}(k\alpha r) - r^{-\frac{1}{2}} \frac{d}{dr} J_{i+\frac{1}{2}}(k\alpha r)\right\} = 2r^{-\frac{3}{2}} \frac{d}{dr} J_{i+\frac{1}{2}}(k\alpha r) + r^{-\frac{1}{2}} J_{i+\frac{1}{2}}(k\alpha r)\left\{k^2\alpha^2 - \frac{i(i+1)+1}{r^2}\right\}.$$

Using the notation $\frac{1}{k\alpha}\frac{d}{dr} J_{i+\frac{1}{2}}(k\alpha r) = J'_{i+\frac{1}{2}}(k\alpha r)$ etc., we thus reduce (44) to

$$a^{-\frac{1}{2}} Y_i \left[J_{i+\frac{1}{2}}(k\alpha a)\left\{m+n-\frac{2n(i^2+i+1)}{k^2\alpha^2 a^2}\right\} + \frac{4n}{k\alpha a} J'_{i+\frac{1}{2}}(k\alpha a)\right]$$
$$+ \beta n k a^{-\frac{3}{2}} Z_i \left\{2J'_{i+\frac{1}{2}}(k\beta a) - \frac{3}{k\beta a} J_{i+\frac{1}{2}}(k\beta a)\right\} = S_i \ldots\ldots(45).$$

So from (42) it is easy to find

$$\Sigma \frac{d}{d\theta}\left[\frac{Y_i a^{-\frac{1}{2}}}{k\alpha}\left\{\frac{3}{k\alpha a} J_{i+\frac{1}{2}}(k\alpha a) - 2J'_{i+\frac{1}{2}}(k\alpha a)\right\} - \frac{Z_i a^{-\frac{3}{2}} k\beta}{i(i+1)}\left\{2J'_{i+\frac{1}{2}}(k\beta a) + (k^2\beta^2 a^2 - 2i.\overline{i+1}+1)\frac{J_{i+\frac{1}{2}}(k\beta a)}{k\beta a}\right\}\right]$$
$$+ \Sigma \frac{X_i}{\sin\theta} k\beta a^{-\frac{1}{2}}\left\{J'_{i+\frac{1}{2}}(k\beta a) - \frac{3}{2}\frac{J_{i+\frac{1}{2}}(k\beta a)}{k\beta a}\right\} = \frac{1}{n}\Sigma T_i \ldots\ldots(46).$$

So from (43)

$$\Sigma \frac{1}{\sin\theta}\frac{d}{d\phi}\left[\frac{Y_i a^{-\frac{1}{2}}}{k\alpha}\left\{\frac{3}{k\alpha a} J_{i+\frac{1}{2}}(k\alpha a) - 2J'_{i+\frac{1}{2}}(k\alpha a)\right\} - \frac{Z_i a^{-\frac{3}{2}} k\beta}{i(i+1)}\left\{2J'_{i+\frac{1}{2}}(k\beta a) + (k^2\beta^2 a^2 - 2i.\overline{i+1}+1)\frac{J_{i+\frac{1}{2}}(k\beta a)}{k\beta a}\right\}\right]$$
$$+ \Sigma w_i k\beta a^{-\frac{1}{2}}\left\{J'_{i+\frac{1}{2}}(k\beta a) - \frac{3}{2}\frac{J_{i+\frac{1}{2}}(k\beta a)}{k\beta a}\right\} = \frac{1}{n}\Sigma U_i \ldots\ldots\ldots\ldots(47),$$

where
$$\frac{dw_i}{d\phi} = -\frac{dX_i}{d\theta}.$$

As in the case of equilibrium we can from (46) and (47) get two more convenient equations.

Multiply both sides of (46) by $\sin\theta$ and differentiate with respect to θ, differentiate both sides of (47) with respect to ϕ, then add together (46) and (47) thus modified; we obviously eliminate X_i and w_i, and noticing

$$\left[\frac{d}{d\theta}.\sin\theta\frac{d}{d\theta} + \frac{1}{\sin\theta}\frac{d^2}{d\phi^2}\right]\begin{matrix}Y_i\\Z_i\end{matrix} = -i(i+1)\sin\theta\begin{matrix}Y_i\\Z_i\end{matrix},$$

we get

$$\Sigma i(i+1)n\left[\frac{Y_i a^{-\frac{1}{2}}}{k\alpha}\left\{\frac{3}{k\alpha a} J_{i+\frac{1}{2}}(k\alpha a) - 2J'_{i+\frac{1}{2}}(k\alpha a)\right\} - \frac{Z_i a^{-\frac{3}{2}} k\beta}{i(i+1)}\left\{2J'_{i+\frac{1}{2}}(k\beta a) + (k^2\beta^2 a^2 - 2i.\overline{i+1}+1)\frac{J_{i+\frac{1}{2}}(k\beta a)}{k\beta a}\right\}\right]$$
$$= -\frac{1}{\sin\theta}\Sigma\left[\frac{d}{d\theta}.\sin\theta T_i + \frac{dU_i}{d\phi}\right] \ldots\ldots\ldots\ldots\ldots(46\text{ a}).$$

In the special case $T_i = 0 = U_i$, we may drop the Σ on the first side of (46 a).

Digitized by Google

Next multiply both sides of (47) by $\sin\theta$, then differentiate with respect to θ, and subtract (47) so modified from (46) modified by differentiating both sides with respect to ϕ. We thus eliminate Y_i, Z_i, etc., and get

$$\Sigma\left[\left\{\frac{1}{\sin\theta}\frac{dX_i}{d\phi}-\frac{d}{d\theta}.\sin\theta w_i\right\}nk\beta a^{-\frac{1}{2}}\left\{J'_{i+\frac{1}{2}}(k\beta a)-\frac{3}{2}\frac{J_{i+\frac{1}{2}}(k\beta a)}{k\beta a}\right\}\right]=\Sigma\left[\frac{dT_i}{d\phi}-\frac{d}{d\theta}.\sin\theta U_i\right],$$

all over the surface.

Differentiate both sides of this last expression with respect to ϕ; then, noticing that $\frac{dw_i}{d\phi}=-\frac{dX_i}{d\theta}$, we find that the first factor on the left-hand side becomes

$$\frac{1}{\sin\theta}\frac{d^2X_i}{d\phi^2}+\frac{d}{d\theta}.\sin\theta\frac{dX_i}{d\theta},$$

i.e. is equal to $$-i(i+1)\sin\theta X_i.$$

Thus

$$nk\beta a^{-\frac{1}{2}}\Sigma i(i+1)X_i\left\{J'_{i+\frac{1}{2}}(k\beta a)-\frac{3}{2}\frac{J_{i+\frac{1}{2}}(k\beta a)}{k\beta a}\right\}=\Sigma\frac{1}{\sin\theta}\left[\frac{d}{d\theta}.\sin\theta U_i-\frac{dT_i}{d\phi}\right]\quad\ldots(47\text{ a})^*.$$

From (45), (46 a), and (47 a) we can determine all the quantities of which Y_i, Z_i, X_i are representatives.

The coefficient of X_i in (47 a) equated to zero, gives the frequency of those free vibrations in which there is no radial motion.

§ 57. Let us now consider more particularly that part of the solution depending on purely normal forces.

Since in this case we can separate the spherical harmonics of different degrees in the surface equations, we may treat them separately. Thus we have (45) and (46 a), dropping the Σ and putting $T_i=0=U_i$, as two simple equations to determine the two unknowns $\frac{Y_i}{S_i}$, $\frac{Z_i}{S_i}$.

We thus get

$$D.Y_i=\frac{S_ik\beta a^{\frac{1}{2}}}{i(i+1)}\left[2J'_{i+\frac{1}{2}}(k\beta a)+\{k^2\beta^2a^2-2i(i+1)+1\}\frac{J_{i+\frac{1}{2}}(k\beta a)}{k\beta a}\right]\quad\ldots\ldots\ldots\ldots(48),$$

$$D.Z_i=\frac{S_ia^{\frac{1}{2}}}{k\alpha}\left[\frac{3J_{i+\frac{1}{2}}(k\alpha a)}{k\alpha a}-2J'_{i+\frac{1}{2}}(k\alpha a)\right]\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(49);$$

where D is the value of the determinant

$$\begin{vmatrix} J_{i+\frac{1}{2}}(k\alpha a)\left\{(m+n)-\frac{2n(i^2+i+1)}{k^2\alpha^2a^2}\right\}+\frac{4n}{k\alpha a}J'_{i+\frac{1}{2}}(k\alpha a), & \beta kna^{-1}\left[J_{i+\frac{1}{2}}(k\beta a)\frac{3}{k\beta a}-2J'_{i+\frac{1}{2}}(k\beta a)\right] \\ \frac{1}{k\alpha}\left\{\frac{3J_{i+\frac{1}{2}}(k\alpha a)}{k\alpha a}-2J'_{i+\frac{1}{2}}(k\alpha a)\right\}, & \frac{k\beta}{i(i+1)}\left[2J'_{i+\frac{1}{2}}(k\beta a)+\{k^2\beta^2a^2-2i(i+1)+1\}\frac{J_{i+\frac{1}{2}}(k\beta a)}{k\beta a}\right] \end{vmatrix}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(50).$$

* If instead of being themselves surface harmonics, $U_i=-\frac{dS''_i}{d\theta}$, and $T_i=\frac{1}{\sin\theta}\frac{dS''_i}{d\phi}$, where S''_i is a harmonic of the ith degree, the right-hand side becomes $i(i+1)S''_i$ and the Σ may be dropped from both sides.

Digitized by Google

The values of Y_i and Z_i shew that for a given frequency of vibration the ratios $Y_i : S_i$, and $Z_i : S_i$ are, for the same sphere, independent of the *form* of the spherical harmonic of the degree i. Thus we have the following important law.

The ratio of the amplitude of the forced vibrations to the amplitude of the constraining normal surface force of a given period depends only on the degree of the surface harmonic determining the law of distribution of the force, and not at all on the form of the surface harmonic.

Again the periods of the free vibrations, answering to $S_i = 0$, are given by $D = 0$, and so are independent of the form of the surface harmonic, depending only on a and i. Thus for a given sphere all free vibrations whose expression for u contains surface spherical harmonics of the same degree are of the same frequency, the frequency for degree i being given by $(50) = 0$. Thus we could obtain the law of frequency from a consideration of the simplest form of surface harmonic of given degree, say a Legendre's function.

We have thus a large variety of possible forms of vibration of the same frequency, and so the frequency alone is not sufficient to distinguish the vibration.

From the form of (50) it is obvious that the equation $D = 0$ may be regarded as an equation in ka, i.e. k and a occur always as a product. Thus for a given material and for a given form of vibration, the frequency varies inversely as the radius of the sphere. This remark applies to all the possible forms of free vibration, cf. (47 a).

§ 58. The case $i = 0$ is very easily treated, and is found in most books on elastic solids. Let us here however follow the Bessel's Function method. Proceeding as in the general case we get equation (10) with $i = 0$; whence, in place of (11),

$$\delta = \cos kt A' r^{-\frac{1}{2}} J_{\frac{1}{2}}(kar), \text{ where } A' \text{ is a constant,}$$

$$= \frac{A}{kar} \sin(kar) \cos kt \qquad (51),$$

if
$$A = A' \sqrt{\frac{2ka}{\pi}}.$$

δ is independent of θ or ϕ, and so

$$\delta = \frac{du}{dr} + \frac{2u}{r} = \frac{1}{r^2} \frac{d.ur^2}{dr} \qquad (52);$$

thus
$$\frac{d.ur^2}{dr} = \frac{A}{ka} r \sin(kar) \cos kt,$$

and so
$$ur^2 = \frac{A}{(ka)^2} \cos kt r \left[\frac{\sin(kar)}{kar} - \cos(kar)\right].$$

No constant is required as this gives $u = 0$ when $r = 0$,

$$\therefore\ u = \frac{A}{(ka)^2} \cos kt \frac{1}{r} \left\{\frac{\sin(kar)}{kar} - \cos(kar)\right\} \qquad (53).$$

Digitized by Google

The surface condition is $(m-n)\,\delta + 2n\dfrac{du}{dr}$ = normal traction, or, as is more convenient, $(m+n)\,\delta - 4n\dfrac{u}{r}$ = normal traction. If then this traction be $F\cos kt$ on the surface $r=a$,

$$F = A\left[(m+n)\frac{\sin(k\alpha a)}{k\alpha a} - \frac{4n}{k^2\alpha^2 a^2}\left\{\frac{\sin(k\alpha a)}{k\alpha a} - \cos(k\alpha a)\right\}\right] \quad\text{.....(54).}$$

For free vibrations $F=0$, and so

$$\frac{\tan(k\alpha a)}{k\alpha a} = \frac{1}{1 - \dfrac{m+n}{4n}k^2\alpha^2 a^2} \quad\text{.....(55).}$$

Possibly a simpler method than the preceding is to substitute $\dfrac{du}{dr} + \dfrac{2u}{r}$ at once for δ in (1). Then assuming $u \propto \cos kt$ we easily put the resulting equation into the form

$$\frac{d^2}{dr^2}(ur^{\frac{1}{2}}) + \frac{1}{r}\frac{d}{dr}(ur^{\frac{1}{2}}) + \left\{k^2\alpha^2 - \frac{\left(\frac{3}{2}\right)^2}{r^2}\right\}(ur^{\frac{1}{2}}) = 0 \quad\text{.....(56),}$$

whence

$$ur^{\frac{1}{2}} = A'' J_{\frac{3}{2}}(k\alpha r)\cos kt,$$

where A'' is a constant.

§ 59. The general solution is equally applicable to the vibrations of a spherical shell. There is required only the introduction of the second solution of the Bessel's equation. Since $i+\frac{1}{2}$ is neither an integer nor zero, the second solution is simply $J_{-(i+\frac{1}{2})}(k\alpha r)$.

Thus in place of (11), in the general case we should have

$$\delta = \cos kt\, r^{-\frac{1}{2}}\left[Y_i J_{i+\frac{1}{2}}(k\alpha r) + Y_i' J_{-(i+\frac{1}{2})}(k\alpha r)\right] \quad\text{.....(11'),}$$

where Y_i' is a surface harmonic of degree i, and (11′) is used to signify that the equation takes the place of (11).

It is much easier to introduce the additional terms in u, v and w than it was when dealing with the equilibrium of a shell, for in that case the coefficients of Y_i and Y_i', of Z_i and Z_i', were affected differently by the necessary differentiations with respect to r. This difficulty does not exist here. Thus for instance the coefficients of Y_i and Y_i' satisfy the same differential equation, and are operated on by exactly the same differentiations with respect to r. Further the resulting expressions are simplified only by a use of this same differential equation, and as $i+\frac{1}{2}$ occurs only as a square its sign has no influence. Thus all that is necessary is to introduce in (19), (34) and (35) wherever Y_i and $J_{i+\frac{1}{2}}(k\alpha r)$ occur a new term in Y_i' and $J_{-(i+\frac{1}{2})}(k\alpha r)$, and similarly in addition to the terms in Z_i, X_i, w_i and $J_{i+\frac{1}{2}}(k\beta r)$, new terms in Z_i', X_i', w_i' and $J_{-(i+\frac{1}{2})}(k\beta r)$, where the dashed letters are similar in signification to the undashed. Thus the complete solution for a shell is

$$u = \cos kt\left[\frac{1}{k^2\alpha^2}Y_i\left\{\frac{1}{2}r^{-\frac{3}{2}}J_{i+\frac{1}{2}}(k\alpha r) - r^{-\frac{1}{2}}\frac{d}{dr}J_{i+\frac{1}{2}}(k\alpha r)\right\}\right.$$
$$\left. + \frac{1}{k^2\alpha^2}Y_i'\left\{\frac{1}{2}r^{-\frac{3}{2}}J_{-(i+\frac{1}{2})}(k\alpha r) - r^{-\frac{1}{2}}\frac{d}{dr}J_{-(i+\frac{1}{2})}(k\alpha r)\right\} + Z_i r^{-\frac{3}{2}}J_{i+\frac{1}{2}}(k\beta r) + Z_i' r^{-\frac{3}{2}}J_{-(i+\frac{1}{2})}(k\beta r)\right] \quad\text{...(19'),}$$

Digitized by Google

$$v = \cos kt \frac{d}{d\theta}\left[-\frac{r^{-\frac{3}{2}}}{k^2\alpha^2}\left\{Y_i J_{i+\frac{1}{2}}(k\alpha r) + Y_i' J_{-(i+\frac{1}{2})}(k\alpha r)\right\}\right.$$

$$\left. + \frac{r^{-1}}{i(i+1)}\left\{Z_i \frac{d}{dr}.r^{\frac{1}{2}} J_{i+\frac{1}{2}}(k\beta r) + Z_i' \frac{d}{dr}.r^{\frac{1}{2}} J_{-(i+\frac{1}{2})}(k\beta r)\right\}\right] + \cos kt \frac{r^{-\frac{1}{2}}}{\sin\theta}\left\{X_i J_{i+\frac{1}{2}}(k\beta r) + X_i' J_{-(i+\frac{1}{2})}(k\beta r)\right\}$$

.....(34′),

$$w = \frac{\cos kt}{\sin\theta}\frac{d}{d\phi}\left[-\frac{r^{-\frac{3}{2}}}{k^2\alpha^2}\left\{Y_i J_{i+\frac{1}{2}}(k\alpha r) + Y_i' J_{-(i+\frac{1}{2})}(k\alpha r)\right\}\right.$$

$$\left. + \frac{r^{-1}}{i(i+1)}\left\{Z_i \frac{d}{dr}.r^{\frac{1}{2}} J_{i+\frac{1}{2}}(k\beta r) + Z_i' \frac{d}{dr}.r^{\frac{1}{2}} J_{-(i+\frac{1}{2})}(k\beta r)\right\}\right] + \cos kt r^{-\frac{1}{2}}\left\{w_i J_{i+\frac{1}{2}}(k\beta r) + w_i' J_{-(i+\frac{1}{2})}(k\beta r)\right\}$$

.....(35′),

where $$\frac{dw_i}{d\phi} = -\frac{dX_i}{d\theta} \text{ and } \frac{dw_i'}{d\phi} = -\frac{dX_i'}{d\theta},$$

and $$\delta = \cos kt r^{-\frac{1}{2}}\{Y_i J_{i+\frac{1}{2}}(k\alpha r) + Y_i' J_{-(i+\frac{1}{2})}(k\alpha r)\} \text{ .....(11′).}$$

Supposing surface forces to exist over the bounding surfaces $r = a$, $r = a'$, we get as usual 3 series of equations at each surface, and thence, exactly as in the case of equilibrium of a shell, determine the 6 series of unknowns.

If the surface forces on the shell be purely normal the equations get much simplified. There occur two expressions $\frac{d}{d\theta}f(\theta, \phi)$ and $\frac{1}{\sin\theta}\frac{d}{d\phi}f(\theta, \phi)$ vanishing all over the surface, whence there results the one equation $f(\theta, \phi) = 0$.

We thus get for a shell 4 simple equations to determine Y_i, Y_i', Z_i, Z_i' in terms of the normal forces $S_i \cos kt$ and $S_i' \cos kt$. The coefficients of these quantities are as before constants, so there is no theoretical difficulty in the process.

§ 60. We may by this means find for example the periods of the free "transversal" vibrations. We get exactly as we got (47 a), putting $U_i = T_i = 0$ over both the surfaces $r = a$, $r = a'$,

$$X_i\left[J'_{i+\frac{1}{2}}(k\beta a) - \frac{3}{2}\frac{J_{i+\frac{1}{2}}(k\beta a)}{k\beta a}\right] + X_i'\left[J'_{-(i+\frac{1}{2})}(k\beta a) - \frac{3}{2}\frac{J_{-(i+\frac{1}{2})}(k\beta a)}{k\beta a}\right] = 0,$$

$$X_i\left[J'_{i+\frac{1}{2}}(k\beta a') - \frac{3}{2}\frac{J_{i+\frac{1}{2}}(k\beta a')}{k\beta a'}\right] + X_i'\left[J'_{-(i+\frac{1}{2})}(k\beta a') - \frac{3}{2}\frac{J_{-(i+\frac{1}{2})}(k\beta a')}{k\beta a'}\right] = 0;$$

whence the periods are given by

$$\left\{J'_{i+\frac{1}{2}}(k\beta a) - \frac{3}{2}\frac{J_{i+\frac{1}{2}}(k\beta a)}{k\beta a}\right\}\left\{J'_{-(i+\frac{1}{2})}(k\beta a') - \frac{3}{2}\frac{J_{-(i+\frac{1}{2})}(k\beta a')}{k\beta a'}\right\}$$

$$-\left\{J'_{-(i+\frac{1}{2})}(k\beta a) - \frac{3}{2}\frac{J_{-(i+\frac{1}{2})}(k\beta a)}{k\beta a}\right\}\left\{J'_{i+\frac{1}{2}}(k\beta a') - \frac{3}{2}\frac{J_{i+\frac{1}{2}}(k\beta a')}{k\beta a'}\right\} = 0 \text{ ....(47′ a).}$$

If the shell be not very thin probably it is best to transform (47′ a) by means of the relations between consecutive Bessel's functions into

$$\{(i-1)J_{i-\frac{1}{2}}(k\beta a) - (i+2)J_{i+\frac{3}{2}}(k\beta a)\}\{(i-1)J_{-(i-\frac{1}{2})}(k\beta a') - (i+2)J_{-(i+\frac{3}{2})}(k\beta a')\}$$

$$-\{(i-1)J_{i-\frac{1}{2}}(k\beta a') - (i+2)J_{i+\frac{3}{2}}(k\beta a')\}\{(i-1)J_{-(i-\frac{1}{2})}(k\beta a) - (i+2)J_{-(i+\frac{3}{2})}(k\beta a)\} = 0 \text{ ...(47 b).}$$

Digitized by Google

For the special case $i=1$ this easily leads to

$$\frac{\tan(k\beta(a-a'))}{k\beta(a-a')} = \frac{3(k\beta)^2 aa' + 9}{(k\beta)^4 a^2 a'^2 - 3(k\beta)^2(a^2 - 3aa' + a'^2) + 9} \quad \text{.....(57).}$$

If the shell be very thin the use of the relations*

$$\left.\begin{aligned} J_{i+\frac{1}{2}}(k\beta a') &= J_{i+\frac{1}{2}}(k\beta a) - (a-a')\,k\beta\, J'_{i+\frac{1}{2}}(k\beta a) \\ k\beta a' J'_{i+\frac{1}{2}}(k\beta a') &= k\beta a J'_{i+\frac{1}{2}}(k\beta a) + (a-a')\,a\left\{k^2\beta^2 - \frac{\left(i+\frac{1}{2}\right)^2}{a^2}\right\} J_{i+\frac{1}{2}}(k\beta a) \end{aligned}\right\} \quad \text{.....(58)}$$

in (47′ a) gives at once the frequency $\frac{k}{2\pi}$ from the equation

$$k^2\beta^2 a^2 = (i-1)(i+2) \quad \text{.....(59)†.}$$

§ 61. The radial vibrations of a spherical shell may be treated similarly to those of a solid sphere.

Possibly it is best to use the equation in u got by our second method for the solid sphere, viz. (56).

The two solutions are $J_{\frac{1}{2}}(kar)$, and $J_{-\frac{1}{2}}(kar)$.

The value of $J_{-\frac{1}{2}}(kar)$ is $-\sqrt{\frac{2}{\pi kar}}\left\{\frac{\cos(kar)}{kar} + \sin(kar)\right\}$;

∴ the complete solution of (56) may be written in the form

$$u = \frac{\cos kt}{(ka)^2}\,\frac{1}{r}\left[A\left\{\frac{\sin(kar)}{kar} - \cos(kar)\right\} + B\left\{\frac{\cos(kar)}{kar} + \sin(kar)\right\}\right] \quad \text{.....(60),}$$

where A and B are constants.

Also
$$\delta = \frac{1}{r^2}\,\frac{d.ur^2}{dr} = \cos kt\,\frac{A\sin(kar) + B\cos(kar)}{kar} \quad \text{.....(61).}$$

The surface conditions are $(m+n)\,\delta - 4n\frac{u}{r} =$ normal force, on the surfaces $r=a$, $r=a'$. Supposing then $F\cos kt$ and $F'\cos kt$ to be the values of this expression at the two surfaces, we get to determine A and B in terms of F and F' the equations

$$(m+n)\,\delta - 4n\frac{u}{r} = F\cos kt \text{ when } r=a \quad \text{.....(62),}$$

$$(m+n)\,\delta - 4n\frac{u}{r} = F'\cos kt \text{ when } r=a' \quad \text{.....(63).}$$

So A and B can be expressed at once from (60), (61), (62) and (63) in terms of F and F'.

* See *The Messenger of Mathematics*, Vol. xv., 1885, p. 22, noticing specially equation (9).

† See Professor Lamb's paper in *The Proceedings of the London Mathematical Society*, Vol. xiv. p. 50.

Digitized by Google

Probably it is more interesting to find the period of the free vibrations. Putting then $F=0=F'$, we get

$$A\left[(m+n)\frac{\sin(k\alpha a)}{k\alpha a}-\frac{4n}{(k\alpha a)^2}\left\{\frac{\sin(k\alpha a)}{k\alpha a}-\cos(k\alpha a)\right\}\right]$$
$$+B\left[(m+n)\frac{\cos(k\alpha a)}{k\alpha a}-\frac{4n}{(k\alpha a)^2}\left\{\frac{\cos(k\alpha a)}{k\alpha a}+\sin(k\alpha a)\right\}\right]=0\text{.....}(64),$$

and a similar equation writing a' for a.

Thence, eliminating $A:B$, we get

$$\left[(m+n)\frac{\sin(k\alpha a)}{k\alpha a}-\frac{4n}{(k\alpha a)^2}\left\{\frac{\sin(k\alpha a)}{k\alpha a}-\cos(k\alpha a)\right\}\right]\left[(m+n)\frac{\cos(k\alpha a')}{k\alpha a'}-\frac{4n}{(k\alpha a')^2}\left\{\frac{\cos(k\alpha a')}{k\alpha a'}+\sin(k\alpha a')\right\}\right]$$
$=$ similar expression changing a into a' and a' into a.

Whence, after reduction, we get

$$\frac{\tan(k\alpha\overline{a-a'})}{k\alpha\overline{a-a'}}=\frac{4n\{(m+n)k^2\alpha^2aa'+4n\}}{(m+n)^2(k^2\alpha^2aa')^2-4n(m+n)k^2\alpha^2(a^2+a'^2)+16n^2(1+k^2\alpha^2aa')}\text{.....}(65).$$

This equation is obviously satisfied by $k=0$, a vibration of infinite period.

§ 62. If the shell be very thin, (65) gives us a simple result directly. In all cases α is very small, thus when $a-a'$ is very small we have, unless k be extremely large, $\frac{\tan k\alpha(a-a')}{k\alpha(a-a')}=1$. Using this relation in the left-hand side of (65), and on the right neglecting terms in $a-a'$, we find that the constant terms cut out, and are left with the equation

$$(k\alpha a)^4(m+n)^2-4n(3m-n)(k\alpha a)^2=0\text{.....}(66).$$

Thus $k\alpha a=0$, as was seen by inspection in the general case, answering to a vibration of infinite period, or else

$$(k\alpha a)^2=\frac{4n(3m-n)}{(m+n)^2}\text{.....}(67).$$

Since $\alpha^2=\frac{\rho}{m+n}$, this may also be written in the form

$$k^2=\frac{4n(3m-n)}{\rho a^2(m+n)}\text{.....}(67\text{ a})^*.$$

As usual the frequency varies inversely as the radius.

To get some idea of the actual frequency of this kind of vibration, suppose we consider a shell of brass. For this as already stated Thomson and Tait quote the result $\frac{m-n}{2m}=\cdot387$, which gives $\frac{n}{m}=\cdot226$. Kohlrausch's *Physical Measurements* give $\rho=8\cdot5$, and $\frac{n}{m}(3m-n)$, i.e. Young's *Modulus*, $=9\times10^6$ C. G. S.

* See footnote to Professor Lamb's paper *l. c.* p. 50.

Digitized by Google

These results give approximately $ka = 18600$, supposing a expressed in centimetres. Thus the number of vibrations per second, being $\frac{k}{2\pi}$, is roughly $\frac{2960}{a}$; or a thin shell of one centimetre radius would perform nearly 3000 radial vibrations per second. The middle note of a piano would be given by a thin shell of about 11 centimetres radius.

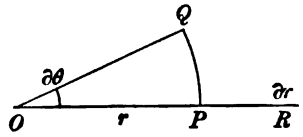
[November 16, 1888. If the surface forces on a solid sphere be derivable from a potential $\cos kt\, \Sigma \frac{1}{i} S_i r^i / a^{i-1}$, as in the foot-note Sect. VII., the right-hand side of (46 a) reduces to $\Sigma (i+1) S_i$. Thus the Σ may be dropped, and Y_i, Z_i are given explicitly by (45) and (46 a) in terms of S_i, while X_i vanishes by (47 a). In a shell the corresponding case may be similarly treated.]

SECTION IX.

Fundamental equations in cylindrical co-ordinates.

§ 63. In finding the equations of equilibrium or of vibration in cylindrical co-ordinates we have only to follow the method adopted for polars, i.e. transfer the Cartesian expression for the potential to cylindrical co-ordinates.

Let the cylindrical co-ordinates of any point be as usual r, θ, z, the latter being measured parallel to the axis of the cylinder, and let u, v, and w denote displacements radially, perpendicular to the radius in the transverse section, and parallel to the axis respectively. Then at every point the direction w is measured in is the same, so the only values to be transformed are those connected with α, β, x, y in Cartesians, following Thomson and Tait's notation. The figure shews how to transform at once.



P is point r, θ,
Q..... r, $\theta + \partial\theta$,
R..... $r + \partial r$, θ.

The axes at Q make angles $\partial\theta$ with those at P, while those at R are parallel to those at P.

Thus, regarding the axes at P as the fixed Cartesian axes, we get

$$e = \frac{d\alpha}{dx} = \frac{du}{dr} \quad \text{.....(1)},$$

$$f = \frac{d\beta}{dy} = \text{Lt.}\ \frac{(v+\partial v)\cos\partial\theta - v + (u+\partial u)\sin\partial\theta}{r\partial\theta} = \frac{1}{r}\frac{dv}{d\theta} + \frac{u}{r} \quad \text{.....(2)},$$

$$g = \frac{dw}{dz} \quad \text{.....(3)},$$

$$a = \frac{d\beta}{dz} + \frac{d\gamma}{dy} = \frac{dv}{dz} + \frac{1}{r}\frac{dw}{d\theta} \quad \text{.....(4)},$$

Digitized by Google

$$b=\frac{d\alpha}{dz}+\frac{d\gamma}{dx}=\frac{du}{dz}+\frac{dw}{dr}\text{.....}(5),$$

$$c=\frac{d\beta}{dx}+\frac{d\alpha}{dy}=\text{Lt.}\ \frac{v+\partial v-v}{\partial r}+\text{Lt.}\ \frac{(u+\partial u)\cos\partial\theta-u-(v+\partial v)\sin\partial\theta}{r\partial\theta}=\frac{dv}{dr}-\frac{v}{r}+\frac{1}{r}\frac{du}{d\theta}\text{...}(6).$$

Also
$$\delta=e+f+g=\frac{1}{r}\frac{d.ur}{dr}+\frac{1}{r}\frac{dv}{d\theta}+\frac{dw}{dz}\text{.....}(7).$$

If ϖ_0 denote the potential energy per unit volume of the strain u, v, w, we have as for polar co-ordinates,

$$2\varpi_0=(m+n)\,\delta^2+n\,\{a^2+b^2+c^2-4\,(ef+eg+fg)\}\text{.....}(8).$$

Thus if ϖ denote the potential energy for the entire solid, we have

$$2\varpi=\iiint r\,d\theta\,dr\,dz\left[(m+n)\left(\frac{1}{r}\frac{d.ur}{dr}+\frac{1}{r}\frac{dv}{d\theta}+\frac{dw}{dz}\right)^2+n\left\{-4\frac{du}{dr}\left(\frac{u}{r}+\frac{1}{r}\frac{dv}{d\theta}+\frac{dw}{dz}\right)\right.\right.$$
$$\left.\left.-4\frac{dw}{dz}\left(\frac{u}{r}+\frac{1}{r}\frac{dv}{d\theta}\right)+\left(\frac{dv}{dz}+\frac{1}{r}\frac{dw}{d\theta}\right)^2+\left(\frac{du}{dz}+\frac{dw}{dr}\right)^2+\left(r\frac{d.\frac{v}{r}}{dr}+\frac{1}{r}\frac{du}{d\theta}\right)^2\right\}\right]\text{.....}(9).$$

§ 64. Denote the variations in ϖ due to the displacements ∂u, ∂v, ∂w by $\partial\varpi_1$, $\partial\varpi_2$ and $\partial\varpi_3$ respectively.

Then we shall find $\partial\varpi_1$, $\partial\varpi_2$, $\partial\varpi_3$ each to consist of two series of terms, one for the interior, the other for the surface. To find these we have only to follow exactly the same method as in the case of polar co-ordinates in Section I. The algebra in the present case also is easier, so it will be sufficient to give the results. Suppose λ, μ, ν to be the direction cosines of the angles made by the normal at any point of the surface with the fundamental directions at the point; so that, dS denoting an element of surface,

$$r\,d\theta\,dz=\lambda dS,\qquad dr\,dz=\mu dS,\qquad r\,d\theta\,dr=\nu dS.$$

Then taking $\partial\varpi\equiv\partial\varpi_1+\partial\varpi_2+\partial\varpi_3$, we find

$$\partial\varpi=\iint dS\left[\partial u\left\{\lambda\left((m+n)\,\delta-2n\left(\frac{u}{r}+\frac{1}{r}\frac{dv}{d\theta}+\frac{dw}{dz}\right)\right)+\mu n\left(r\frac{d.\frac{v}{r}}{dr}+\frac{1}{r}\frac{du}{d\theta}\right)+\nu n\left(\frac{du}{dz}+\frac{dw}{dr}\right)\right\}\right.$$
$$+\partial v\left\{\lambda n\left(r\frac{d.\frac{v}{r}}{dr}+\frac{1}{r}\frac{du}{d\theta}\right)+\mu\left((m+n)\,\delta-2n\left(\frac{du}{dr}+\frac{dw}{dz}\right)\right)+\nu n\left(\frac{dv}{dz}+\frac{1}{r}\frac{dw}{d\theta}\right)\right\}$$
$$\left.+\partial w\left\{\lambda n\left(\frac{du}{dz}+\frac{dw}{dr}\right)+\mu n\left(\frac{dv}{dz}+\frac{1}{r}\frac{dw}{d\theta}\right)+\nu\left((m+n)\,\delta-2n\left(\frac{du}{dr}+\frac{u}{r}+\frac{1}{r}\frac{dv}{d\theta}\right)\right)\right\}\right]$$
$$-\iiint r\,d\theta\,dr\,dz\left[\partial u\left\{(m+n)\frac{d\delta}{dr}-\frac{n}{r}\frac{d}{d\theta}\left(\frac{1}{r}\frac{d.vr}{dr}-\frac{1}{r}\frac{du}{d\theta}\right)+n\frac{d}{dz}\left(\frac{du}{dz}-\frac{dw}{dr}\right)\right\}\right.$$

Digitized by Google

$$+\partial v\left\{\frac{m+n}{r}\frac{d\delta}{d\theta}-n\frac{d}{dz}\left(\frac{1}{r}\frac{dw}{d\theta}-\frac{dv}{dz}\right)+n\frac{d}{dr}\left(\frac{1}{r}\frac{d.vr}{dr}-\frac{1}{r}\frac{du}{d\theta}\right)\right\}$$

$$+\partial w\left\{(m+n)\frac{d\delta}{dz}-\frac{n}{r}\frac{d}{dr}\left(r\frac{du}{dz}-r\frac{dw}{dr}\right)+\frac{n}{r}\frac{d}{d\theta}\left(\frac{1}{r}\frac{dw}{d\theta}-\frac{dv}{dz}\right)\right\}\Bigg] \quad\text{.....(10).}$$

§ 65. Suppose now the forces acting on the solid are F, G, H along r, θ, z at any point on its surface per unit area, and R, Θ, Z per unit mass in the same standard directions in the interior. Then the work done by F, G, H, R, Θ, Z and the reversed effective forces in the arbitrary displacement ∂u, ∂v, ∂w, is

$$\iint dS\,(F\partial u+G\partial v+H\partial w)+\iiint r\,d\theta\,dr\,dz\,\rho\left[\partial u\left(R-\frac{d^2u}{dt^2}\right)+\partial v\left(\Theta-\frac{d^2v}{dt^2}\right)+\partial w\left(Z-\frac{d^2w}{dt^2}\right)\right]\text{...(11).}$$

This must be equal to the change in the potential energy; therefore since ∂u, ∂v, ∂w are arbitrary displacements their coefficients separately, both at the surface and in the interior, must be equal to the corresponding coefficients in the expression for $\partial\varpi$. Thus

$$F=\lambda\left\{(m+n)\,\delta-2n\left(\frac{u}{r}+\frac{1}{r}\frac{dv}{d\theta}+\frac{dw}{dz}\right)\right\}+\mu n\left(r\frac{d.\frac{v}{r}}{dr}+\frac{1}{r}\frac{du}{d\theta}\right)+\nu n\left(\frac{du}{dz}+\frac{dw}{dr}\right)\text{.....(12),}$$

$$G=\lambda n\left(r\frac{d.\frac{v}{r}}{dr}+\frac{1}{r}\frac{du}{d\theta}\right)+\mu\left\{(m+n)\,\delta-2n\left(\frac{du}{dr}+\frac{dw}{dz}\right)\right\}+\nu n\left(\frac{dv}{dz}+\frac{1}{r}\frac{dw}{d\theta}\right)\text{.....(13),}$$

$$H=\lambda n\left(\frac{du}{dz}+\frac{dw}{dr}\right)+\mu n\left(\frac{dv}{dz}+\frac{1}{r}\frac{dw}{d\theta}\right)+\nu\left\{(m+n)\,\delta-2n\left(\frac{du}{dr}+\frac{u}{r}+\frac{1}{r}\frac{dv}{d\theta}\right)\right\}\text{.....(14),}$$

$$\rho\left(\frac{d^2u}{dt^2}-R\right)=(m+n)\frac{d\delta}{dr}-\frac{n}{r}\frac{d}{d\theta}\left(\frac{1}{r}\frac{d.vr}{dr}-\frac{1}{r}\frac{du}{d\theta}\right)+n\frac{d}{dz}\left(\frac{du}{dz}-\frac{dw}{dr}\right)\text{.....(15),}$$

$$\rho\left(\frac{d^2v}{dt^2}-\Theta\right)=\frac{m+n}{r}\frac{d\delta}{d\theta}-n\frac{d}{dz}\left(\frac{1}{r}\frac{dw}{d\theta}-\frac{dv}{dz}\right)+n\frac{d}{dr}\left(\frac{1}{r}\frac{d.vr}{dr}-\frac{1}{r}\frac{du}{d\theta}\right)\text{.....(16),}$$

$$\rho\left(\frac{d^2w}{dt^2}-Z\right)=(m+n)\frac{d\delta}{dz}-\frac{n}{r}\frac{d}{dr}\left(r\frac{du}{dz}-r\frac{dw}{dr}\right)+\frac{n}{r}\frac{d}{d\theta}\left(\frac{1}{r}\frac{dw}{d\theta}-\frac{dv}{dz}\right)\text{.....(17).}$$

The first three of these are the surface conditions, the last the internal equations.

§ 66. The internal equations may be put in a very neat form.

Let

$$\mathfrak{A}=\frac{1}{r}\left(\frac{dw}{d\theta}-\frac{d.vr}{dz}\right)\text{.....(18),}$$

$$\mathfrak{B}=r\left(\frac{du}{dz}-\frac{dw}{dr}\right)\text{.....(19),}$$

$$\mathfrak{C}=\frac{1}{r}\left(\frac{d.vr}{dr}-\frac{du}{d\theta}\right)\text{.....(20);}$$

Digitized by Google

and then these equations are obviously identical with

$$(m+n)\,r\frac{d\delta}{dr}-n\frac{d\mathfrak{C}}{d\theta}+n\frac{d\mathfrak{B}}{dz}=\rho r\left(\frac{d^2u}{dt^2}-R\right)\quad\text{.....(21)},$$

$$(m+n)\frac{1}{r}\frac{d\delta}{d\theta}-n\frac{d\mathfrak{A}}{dz}+n\frac{d\mathfrak{C}}{dr}=\rho\left(\frac{d^2v}{dt^2}-\Theta\right)\quad\text{.....(22)},$$

$$(m+n)\,r\frac{d\delta}{dz}-n\frac{d\mathfrak{B}}{dr}+n\frac{d\mathfrak{A}}{d\theta}=\rho r\left(\frac{d^2w}{dt^2}-Z\right)\quad\text{.....(23)}.$$

These equations are of course really the same as those given by Lamé.

§ 67. The surface conditions may be much simplified if the cylinder be a right circular cylinder with flat ends.

On the curved surface $\lambda=1$, and so

$$F=(m+n)\,\delta-2n\left(\frac{u}{r}+\frac{1}{r}\frac{dv}{d\theta}+\frac{dw}{dz}\right)\quad\text{.....(24)},$$

$$G=n\left(r\frac{d.\frac{v}{r}}{dr}+\frac{1}{r}\frac{du}{d\theta}\right)\quad\text{.....(25)},$$

$$H=n\left(\frac{du}{dz}+\frac{dw}{dr}\right)\quad\text{.....(26)};$$

where F is the normal traction, G the tangential traction perpendicular to the axis, and H the tangential traction parallel to the axis.

For the flat ends $\qquad \nu^2=1,\ \lambda=0=\mu,$

$$F=n\left(\frac{du}{dz}+\frac{dw}{dr}\right)=\text{tangential traction along radius vector}\quad\text{.....(27)},$$

$$G=n\left(\frac{dv}{dz}+\frac{1}{r}\frac{dw}{d\theta}\right)=\text{tangential traction perpendicular to radius vector}\quad\text{.....(28)},$$

$$H=(m+n)\,\delta-2n\left(\frac{du}{dr}+\frac{u}{r}+\frac{1}{r}\frac{dv}{d\theta}\right)=\text{normal traction}\quad\text{.....(29)}.$$

In the case of an infinite right circular cylinder, i.e. the case where w is not considered and the forces on the flat ends are neglected, we have to consider only the curved surface, over which now

$$F=(m+n)\,\delta-2n\left(\frac{u}{r}+\frac{1}{r}\frac{dv}{d\theta}\right)\quad\text{.....(30)},$$

$$G=n\left(r\frac{d.\frac{v}{r}}{dr}+\frac{1}{r}\frac{du}{d\theta}\right)\quad\text{.....(31)}.$$

§ 68. By means of the surface equations we can also determine the elastic forces or "stresses" at any point in the interior of the solid. For suppose an imaginary

Digitized by Google

surface to pass through the point considered and to enclose a limited portion of the solid. Then our equations must apply to this limited solid, and our surface equations to its bounding surface. Thus by taking the normal to this imaginary surface at the point considered in any given direction we get the stresses exerted across a plane perpendicular to that direction.

To indicate the nature of the stresses at the point call them

$$R_r,\ R_\theta,\ R_z,\ \Theta_r,\ \Theta_\theta,\ \Theta_z,\ Z_r,\ Z_\theta,\ Z_z,$$

R_r being the normal stress on a plane perpendicular to the radius vector,

R_θ ... tangential stress parallel to the radius on a plane perpendicular to θ, &c.

Taking then $\lambda = 1$ we get the surface perpendicular to the radius, and so

$$R_r = (m+n)\,\delta - 2n\left(\frac{u}{r} + \frac{1}{r}\frac{dv}{d\theta} + \frac{dw}{dz}\right) \quad \text{(32)},$$

$$\Theta_r = n\left(r\frac{d\,.\,\frac{v}{r}}{dr} + \frac{1}{r}\frac{du}{d\theta}\right) \quad \text{(33)},$$

$$Z_r = n\left(\frac{du}{dz} + \frac{dw}{dr}\right) \quad \text{(34)}.$$

So taking $\mu = 1$,

$$R_\theta = n\left(r\frac{d\,.\,\frac{v}{r}}{dr} + \frac{1}{r}\frac{du}{d\theta}\right) \quad \text{(35)},$$

$$\Theta_\theta = (m+n)\,\delta - 2n\left(\frac{du}{dr} + \frac{dw}{dz}\right) \quad \text{(36)},$$

$$Z_\theta = n\left(\frac{dv}{dz} + \frac{1}{r}\frac{dw}{d\theta}\right) \quad \text{(37)}.$$

So taking $\nu = 1$,

$$R_z = n\left(\frac{du}{dz} + \frac{dw}{dr}\right) \quad \text{(38)},$$

$$\Theta_z = n\left(\frac{dv}{dz} + \frac{1}{r}\frac{dw}{d\theta}\right) \quad \text{(39)},$$

$$Z_z = (m+n)\,\delta - 2n\left(\frac{du}{dr} + \frac{u}{r} + \frac{1}{r}\frac{dv}{d\theta}\right) \quad \text{(40)}.$$

Whence $R_\theta = \Theta_r$, $R_z = Z_r$, $\Theta_z = Z_\theta$, which are the well-known relations between the tangential stresses.

Hereafter we shall denote R_r by R, Θ_θ by Θ, and Z_z by Z simply. No confusion with the bodily forces will ensue, because these will be expressed in terms of a potential.

Digitized by Google

§ 69. We shall now consider the tendency to rupture in a cylinder in which the displacement w is zero.

Regarding the radius at any point as axis of x, a parallel to the axis of the cylinder as axis of z, and the perpendicular to these two directions as axis of y, the equation to the stress-quadric is

$$Rx^2 + 2R_\theta xy + \Theta y^2 + Zz^2 = 1 \quad \text{.....(41).}$$

When transformed to its principal axes, the equation will become

$$R_1 x^2 + \Theta_1 y^2 + Zz^2 = 1 \quad \text{.....(41 a),}$$

where R_1, Θ_1, Z are the principal stresses.

By solid geometry $2R_1 = R + \Theta + S$, and $2\Theta_1 = R + \Theta - S$,

where
$$S = \sqrt{(R-\Theta)^2 + 4R_\theta^2} \quad \text{.....(42)}$$

taken positively, so that R_1 is the algebraically greatest stress in the cross section.

If e_1 and f_1 denote the principal strains in the cross section, we have

$$\left.\begin{aligned} R_1 &= (m-n)\,\delta + 2ne_1 \\ \Theta_1 &= (m-n)\,\delta + 2nf_1 \\ Z &= (m-n)\,\delta \end{aligned}\right\} \quad \text{.....(43),}$$

e_1 being algebraically not less than f_1, and $e_1 + f_1 = \delta$.

We see that the greatest strain is e_1, if positive, otherwise it is zero.

Also
$$e_1 = \frac{R_1 - \overline{m-n}\,\delta}{2n} = \frac{\delta}{2} + \frac{S}{4n} \quad \text{.....(44),}$$

since
$$R + \Theta = (R_1 + \Theta_1) = 2m\delta,$$

and
$$\therefore\ R_1 = m\delta + \frac{S}{2} \quad \text{.....(45).}$$

In accordance with the second theory of rupture stated in Section III., there are three cases under which may be treated the tendency to rupture in a cylinder for which $w = 0$.

Case (i) δ positive, and also $\delta^2 - \left(\frac{S}{2n}\right)^2$ positive.

Here Z is the least principal stress, and the maximum stress-difference is

$$R_1 - Z = \frac{S}{2} + n\delta \quad \text{.....(46).}$$

Case (ii) δ negative, and $\delta^2 - \left(\frac{S}{2n}\right)^2$ positive.

Here Z is the greatest principal stress, and the maximum stress-difference is

$$Z - \Theta_1 = \frac{S}{2} - n\delta \quad \text{.....(47).}$$

Digitized by Google

Case (iii) $$\delta^2 - \left(\frac{S}{2n}\right)^2 \text{ negative.}$$

Here Z is the mean principal stress, and the maximum stress-difference is

$$R_1 - \Theta_1 = S \qquad \text{...(48).}$$

The greatest strain is $\frac{1}{2}\delta + \frac{1}{4n}S$ if this be positive, otherwise it is zero. Thus on the third theory of rupture, that held by Saint-Venant, if $\frac{1}{2}\delta + \frac{1}{4n}S$ be at any point positive and greater than a certain limit, determined experimentally for the material, rupture will ensue; if this expression be negative there is no tendency to rupture.

It will be observed that in case (i) the two theories of rupture are practically in agreement; but in cases (ii) and (iii) they may differ widely.

SECTION X.

Equilibrium of cylinder, solution in ascending powers of r *and* z.

§ 70. Before considering the general solution in cylindrical co-ordinates, let us try whether it be possible to find a solution for δ, u, v, w in terms of powers of r and z. We may leave v out of consideration, or treat it separately; because v affects δ only through the term $\frac{dv}{d\theta}$. So at present we shall suppose v to be zero.

We have, see (7) Section IX.,

$$\delta = \frac{1}{r}\frac{d.ur}{dr} + \frac{dw}{dz} \qquad \text{...(1).}$$

The surface conditions are, see (24), (26), (27) and (29), Section IX.,

on the curved surface $$(m-n)\,\delta + 2n\frac{du}{dr} = \text{normal traction} \qquad \text{...(2),}$$

and $$n\left(\frac{du}{dz} + \frac{dw}{dr}\right) = \text{tangential traction parallel to axis} \qquad \text{...(3);}$$

on the flat ends $$n\left(\frac{du}{dz} + \frac{dw}{dr}\right) = \text{radial tangential traction} \qquad \text{...(4),}$$

and $$(m-n)\,\delta + 2n\frac{dw}{dz} = \text{normal traction} \qquad \text{...(5).}$$

If the bodily forces have a potential V, the internal equations are, see (21) and (23), Section IX., supposing V independent of θ,

$$(m+n)\,r\frac{d\delta}{dr} + n\frac{d\mathfrak{B}}{dz} = -\rho r\frac{dV}{dr} \qquad \text{...(6),}$$

$$(m+n)\,r\frac{d\delta}{dz} - n\frac{d\mathfrak{B}}{dr} = -\rho r\frac{dV}{dz} \qquad \text{...(7);}$$

where $$\mathfrak{B} = r\left(\frac{du}{dz} - \frac{dw}{dr}\right).$$

Digitized by Google

(6) and (7) give $$(m+n)\nabla^2\delta = -\rho\nabla^2 V \quad\text{.....(8)};$$

where as usual in cylindrical co-ordinates, when V is independent of θ,

$$\nabla^2 V \equiv \frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} + \frac{d^2V}{dz^2}.$$

Take as the particular solution of (8),

$$\delta = -\frac{\rho}{m+n}V \quad\text{.....(9)}.$$

For the general solution we have to find a series for δ in ascending powers of r and z, which will satisfy $\nabla^2\delta = 0$.....(10). We shall content ourselves with a few of the first terms of the series. For the present we shall treat V as being zero, for the values of u and w depending on V may be developed separately afterwards. Supposing also that the cylinder is solid, and so no inverse powers of r admissible, we get by trial from (10), denoting by A, B, etc. arbitrary constants,

$$\delta = A + Bz + C\left(z^2 - \frac{r^2}{2}\right) + D\left(z^3 - \tfrac{3}{2}r^2 z\right) + E\left(-8z^4 + 24z^2r^2 - 3r^4\right) + \ldots \quad\text{.....(11)}.$$

From (6), substituting its value for Θ, we get

$$(m+n)\,r\frac{d\delta}{dr} + nr\frac{d}{dz}\left(\frac{du}{dz} - \frac{dw}{dr}\right) = 0,$$

i.e. $$(m+n)\frac{d\delta}{dr} + n\left\{\frac{d^2u}{dz^2} - \frac{d}{dr}\left(\delta - \frac{du}{dr} - \frac{u}{r}\right)\right\} = 0;$$

$$\therefore \frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} + \frac{d^2u}{dz^2} = -\frac{m}{n}\frac{d\delta}{dr} \quad\text{.....(12)}$$

$$= \frac{m}{n}\left\{Cr + 3Drz + 12E\left(r^3 - 4z^2r\right) + \ldots\right\} \quad\text{.....(12 a)}.$$

For the present we shall consider terms in δ containing powers of r and z not higher than the second, and terms in u and w with powers not higher than the third.

A particular solution of (12 a) is then $u = \frac{m}{8n}Cr^3$.....(13),

and the general solution may be found by trial to be

$$u = \alpha r + \beta\left(2r^3 - 3z^2\right) + \gamma rz + \epsilon\left(z^3 - 2zr^2\right) + \eta\left(4z^2r - r^3\right) \quad\text{.....(14)};$$

$\therefore$ the complete solution is

$$u = \frac{m}{8n}Cr^3 + \alpha r + \beta\left(2r^3 - 3z^2\right) + \gamma rz + \epsilon\left(z^3 - 2zr^2\right) + \eta\left(4z^2r - r^3\right) \quad\text{.....(15)}.$$

From (7), eliminating u by means of (1), we get

$$\frac{d^2w}{dz^2} + \frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} = -\frac{m}{n}\frac{d\delta}{dz} = -\frac{m}{n}B - \frac{2m}{n}Cz \quad\text{.....(16)}.$$

Digitized by Google

A particular solution is $$w = -\frac{m}{2n} Bz^2 - \frac{m}{3n} Cz^3 \quad \text{.....(17)},$$

and the general solution may be found by trial to be

$$w = \alpha_1 z + \beta_1 (r^2 - 2z^2) + \epsilon_1 (2z^3 - 3zr^2) \quad \text{.....(18)};$$

$\therefore$ the complete solution is

$$w = -\frac{m}{2n} Bz^2 - \frac{m}{3n} Cz^3 + \alpha_1 z + \beta_1 (r^2 - 2z^2) + \epsilon_1 (2z^3 - 3zr^2) \quad \text{.....(19)}.$$

We have certain relations between the constants in δ, u and w in order that

$$\frac{dw}{dz} + \frac{1}{r}\frac{d.ur}{dr} = \delta.$$

This gives in fact

$$\alpha_1 - \frac{m}{n} Bz - \frac{m}{n} Cz^2 - 4\beta_1 z + 6\epsilon_1 \left(z^2 - \frac{r^2}{2}\right) + \frac{m}{2n} Cr^2 + 2\alpha + \beta\left(6r - \frac{3z^2}{r}\right) + 2\gamma z$$
$$+ \epsilon\left(\frac{z^3}{r} - 6zr\right) + \eta(8z^3 - 4r^2) = A + Bz + C\left(z^2 - \frac{r^2}{2}\right) \text{ identically.....(20)};$$

whence
$$\left.\begin{aligned} \alpha_1 + 2\alpha &= A, \\ 2\gamma - 4\beta_1 &= B\frac{m+n}{n}, \\ -\frac{m}{n} C + 6\epsilon_1 + 8\eta &= C, \\ -3\epsilon_1 + \frac{m}{2n} C - 4\eta &= -\frac{C}{2}. \\ \beta = 0 &= \epsilon. \end{aligned}\right\} \quad \text{.....(21)}.$$

These relations leave us six independent constants. Select A, B, C, α, β_1, and η. Then

$$\left.\begin{aligned} \alpha_1 &= A - 2\alpha, \\ \gamma &= \frac{B}{2}\left(\frac{m+n}{n}\right) + 2\beta_1, \\ \epsilon_1 &= \frac{C(m+n)}{6n} - \tfrac{4}{3}\eta, \end{aligned}\right\} \quad \text{.....(22)},$$

and the complete solution is

$$\delta = A + Bz + C\left(z^2 - \frac{r^2}{2}\right) \quad \text{.....(23)},$$

$$u = \alpha r + \left(\frac{B}{2}\frac{m+n}{n} + 2\beta_1\right) rz + \frac{m}{8n} Cr^2 + \eta(4z^2 r - r^3) \quad \text{.....(24)},$$

$$w = (A - 2\alpha)z - \frac{m}{2n} Bz^2 + \beta_1 (r^2 - 2z^2) - \frac{m}{3n} Cz^3 + (2z^3 - 3zr^2)\left(\frac{C\frac{m+n}{n} - 8\eta}{6}\right) \quad \text{.....(25)}.$$

Digitized by Google

Substituting these values in the surface conditions (2), (3), (4) and (5) we get, if a be the radius of the cylinder,

$$(m-n)\left\{A+Bz+C\left(z^2-\frac{a^2}{2}\right)\right\}+2n\left\{\alpha+z\left(\frac{B}{2}\frac{m+n}{n}+2\beta_1\right)+\frac{3m}{8n}Ca^2+\eta\left(4z^2-3a^2\right)\right\}$$

= normal traction on curved surface.....(26),

$$n\left\{a\left(\frac{B}{2}\frac{m+n}{n}+2\beta_1\right)+8\eta za+2\beta_1 a-za\left(C\frac{m+n}{n}-8\eta\right)\right\}$$

= tangential traction parallel to axis of cylinder on curved surface.....(27),

$$(m-n)\left\{A+Bz+C\left(z^2-\frac{r^2}{2}\right)\right\}+2n\left\{A-2\alpha-\frac{m}{n}Bz-4\beta_1 z-\frac{m}{n}Cz^2+\left(2z^2-r^2\right)\frac{C\frac{m+n}{n}-8\eta}{2}\right\}$$

= normal traction on flat end.....(28),

$$n\left\{r\left(\frac{B}{2}\frac{m+n}{n}+2\beta_1\right)+8\eta zr+2\beta_1 r-zr\left(C\frac{m+n}{n}-8\eta\right)\right\}$$

= tangential traction in a radial direction on flat end.....(29).

In the two last equations we substitute for z the values it has at the two ends of the cylinder.

§ 71. Let us first consider the terms of the lowest powers, viz.

$$\delta=A \quad\text{.....(30)},$$

$$u=\alpha r \quad\text{.....(31)},$$

$$w=(A-2\alpha)z \quad\text{.....(32)}.$$

These give

$(m-n)A+2n\alpha$ = normal traction on the curved surface $=-p_1$ say .....(33),

and $(m-n)A+2n(A-2\alpha)$ = normal traction on the flat ends $=-p_2$.....(34).

This gives the case of a cylinder under the action of different, or the same, uniform tractions or pressures on its curved and flat boundaries.

From (33) and (34) we get

$$\alpha=-\frac{p_1(m+n)-p_2(m-n)}{2n(3m-n)},$$

$$A-2\alpha=\frac{p_1(m-n)-p_2 m}{n(3m-n)};$$

whence

$$\left.\begin{aligned} u&=-r\frac{p_1(m+n)-p_2(m-n)}{2n(3m-n)},\\ w&=\ \ z\frac{p_1(m-n)-p_2 m}{n(3m-n)}.\end{aligned}\right\}\text{.....(35)}.$$

Several useful applications of this solution present themselves.

Digitized by Google

§ 72. Suppose $p_1 = p_2 = p$. This would apply to the case of a cylinder in a gaseous medium, and leads to

$$\left.\begin{aligned} u &= -\frac{pr}{3m-n}, \\ w &= -\frac{pz}{3m-n}, \\ \delta &= -\frac{3p}{3m-n}. \end{aligned}\right\} \quad \text{.....(36).}$$

In this case z is measured from the middle point of the axis.

§ 73. Suppose a uniform traction p on the flat ends, and no force on the curved surface. Then we get

$$\left.\begin{aligned} u &= -\frac{pr(m-n)}{2n(3m-n)}, \\ w &= \frac{pzm}{n(3m-n)}. \end{aligned}\right\} \quad \text{.....(37).}$$

This is the well known case of tension of a bar, and supposes z to be measured from one end. According to the usual definition

$$\text{Young's modulus} = p\Big/\frac{dw}{dz} = \frac{n}{m}(3m-n) \quad \text{.....(38);}$$

also

$$\frac{\text{lateral contraction}}{\text{longitudinal extension}} = \frac{-\dfrac{du}{dr}}{\dfrac{dw}{dz}} = \frac{m-n}{2m} \quad \text{.....(39).}$$

§ 74. Suppose $p_2 = 0$ and $p_1 = p$, answering to a uniform pressure on the curved surface of a cylinder whose ends are free. We get

$$\left.\begin{aligned} u &= -\frac{pr(m+n)}{2n(3m-n)}, \\ w &= \frac{pz(m-n)}{n(3m-n)}. \end{aligned}\right\} \quad \text{.....(40).}$$

Finally if we wish to produce no radial displacement, we get for the ratio of the pressures on the curved and flat boundaries

$$\frac{p_1}{p_2} = \frac{m-n}{m+n} \quad \text{.....(41);}$$

while if we wish no longitudinal extension we must have

$$\frac{p_1}{p_2} = \frac{m}{m-n} \quad \text{.....(42).}$$

These results are all, it will be observed, independent of the length or section of the cylinder.

Digitized by Google

Similar problems are considered by Prof. Clerk-Maxwell in the *Transactions of the Royal Society of Edinburgh for* 1850, but the method employed is entirely different from the preceding.

§ 75. Let us consider the solution given by (23), (24) and (25), more specially in the case when the surface forces are symmetrical with respect to the plane $z=0$, which we shall suppose to bisect the axis of the cylinder.

Then from symmetry u must be the same at corresponding points on opposite sides of the plane $z=0$, while at two such points the values of w must be opposite in sign while equal in magnitude.

It follows at once from (19) that B and β_1 must both vanish.

Suppose the length of the cylinder to be $2l$, then we get from the surface conditions (26), (27), (28), and (29), after a little reduction,

$$(m-n)A+2n\alpha+\frac{m+2n}{4}Ca^2-6n\eta a^2+z^2(C\overline{m-n}+8n\eta)$$

$$=\text{normal traction on curved surface}\ \ldots\ldots\ldots\ldots(26_a),$$

$$za\{16n\eta-C(m+n)\}=\text{tangential traction on curved surface}\ldots\ldots\ldots\ldots(27_a),$$

$$(m+n)A-4n\alpha+Cl^2(m+n)-16n\eta l^2+r^2\left\{8n\eta-\frac{3m+n}{2}C\right\}$$

$$=\text{normal traction on flat ends}\ldots\ldots\ldots\ldots\ldots\ldots(28_a),$$

$$lr\{16n\eta-C(m+n)\}=\text{tangential traction on flat ends}\ \ldots\ldots\ldots\ldots(29_a).$$

We can get rid of the tangential surface forces by taking

$$\eta=C\frac{m+n}{16n}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (43);$$

this leaves on the curved surface a normal traction

$$\left.\begin{array}{l}\overline{m-n}A+2n\alpha-\dfrac{m-n}{8}Ca^2+\dfrac{3m-n}{2}Cz^2,\\ \text{and on the flat ends a normal traction}\quad (m+n)A-4n\alpha-mCr^2\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(44).$$

We can get rid of the constant terms by suitably determining A and α in terms of C; but we can get rid of neither the traction $\propto z^2$ on the curved surface, nor the traction $\propto r^2$ on the flat ends unless we put $C=0$ when both vanish. $C=0$ returns to the case of uniform surface pressures already considered. This solution would apply to the case of pressures or tractions on the flat ends proportional to r^2, and gives that law of force which applied to the curved surface would maintain equilibrium. By means of the two constants A and α we can make the surface forces on the flat ends have a zero resultant, or vanish at any assigned distance from the centre, and the forces on the curved surface vanish over any transverse section at a given distance z from the centre of the axis.

Digitized by Google

§ 76. This solution may be applied to the case of a cylinder rotating with angular velocity ω about its axis. We may regard the cylinder as at rest under the action of a "centrifugal" force of potential $V=\frac{\omega^2 r^2}{2}$; whence, from (9),

$$\delta=-\frac{\rho}{m+n}\frac{\omega^2 r^2}{2} \quad\text{.....(45)}$$

is the additional term to be added to δ in (23), putting $B=0$. This is obviously a case of symmetry about the central transverse section of the cylinder, so equations (23), (24), (25), with $B=0=\beta_1$, apply.

A particular solution of (6) and (7), which also satisfies (1), is

$$u=-\frac{\omega^2\rho r^3}{8(m+n)} \quad\text{.....(46),}$$

$$w=0.$$

We may use the surface conditions (26_a)—(29_a) with the addition of

$$(m-n)\left(-\frac{\rho}{m+n}\frac{\omega^2 a^2}{2}\right)-\frac{3n}{4}\frac{\omega^2\rho a^2}{m+n},$$

i.e. $-\frac{\omega^2\rho a^2}{4}\frac{2m+n}{m+n}$, to the normal traction on the curved surface;

and
$$-\frac{m-n}{m+n}\frac{\omega^2\rho r^2}{2}$$

to the normal traction on the flat ends.

We can thus solve the problem of a rotating cylinder whose surface is exposed to no forces save normal pressures or tractions on the curved surface.

To do so, from (27_a) and (29_a), $\eta=C\frac{(m+n)}{16n}$; see (43). From (28_a), noticing the additional term $-\frac{m-n}{m+n}\frac{\omega^2\rho r^2}{2}$, we get

$$-\frac{m-n}{m+n}\frac{\omega^2\rho}{2}+8n\eta-\frac{3m+n}{2}C=0 \quad\text{.....(47);}$$

$$\therefore\ C=-\frac{m-n}{2m(m+n)}\omega^2\rho; \qquad \eta=-\frac{m-n}{32mn}\omega^2\rho \quad\text{.....(48).}$$

Also from the constant term in the same equation we have

$$(m+n)A-4na=0 \quad\text{.....(49).}$$

This leaves on the curved surface a normal *pressure*

$$-\frac{3m-n}{2}A+\frac{7m-n}{16m}\omega^2\rho a^2+\frac{(3m-n)(m-n)}{4m(m+n)}\omega^2\rho z^2 \quad\text{.....(50).}$$

Digitized by Google

We can get rid of the constant term by taking

$$A = \frac{7m-n}{8m(3m-n)}\,\omega^2\rho a^2 \quad\text{.....(51)};$$

or we can make the pressure vanish at the rim of the cylinder by taking

$$A = \left\{\frac{(7m-n)\,a^2}{4(3m-n)} + \frac{m-n}{m+n}\,l^2\right\}\frac{\omega^2\rho}{2m} \quad\text{.....(52)}.$$

This solution would suit the case of a cylinder rotating with its surface in contact with a smooth solid, in a cylindrical groove of which the motion is taking place, or the free rotation of a plate the square of whose thickness is negligible.

We can also get a solution for a rotating cylinder with only tangential surface forces, by taking

$$\eta = -C\,\frac{m-n}{8n} \quad\text{.....(53)},$$

from the coefficient of z^2 on the curved surface;

$$-8n\eta + \frac{3m+n}{2}\,C + \frac{m-n}{2(m+n)}\,\omega^2\rho = 0 \quad\text{.....(54)},$$

from the coefficient of r^2 on the flat ends; or

$$C = \frac{-(m-n)\,\omega^2\rho}{(m+n)(5m-n)},\quad \eta = \frac{(m-n)^2\,\omega^2\rho}{8n(m+n)(5m-n)} \quad\text{.....(55)};$$

and suitably determining A and α.

This leaves the following tangential stresses

$$\left.\begin{array}{ll} \omega^2\rho a z\,\dfrac{(3m-n)(m-n)}{(m+n)(5m-n)} & \text{on the curved surface parallel to the axis,} \\ \omega^2\rho l r\,\dfrac{(3m-n)(m-n)}{(m+n)(5m-n)} & \text{on the flat ends radially} \end{array}\right\} \quad\text{.....(56)}.$$

A third case occurs when no forces exist on the curved surface, and only normal forces on the flat ends, and comes at once.

Put $$C = 0 = \eta \quad\text{.....(57)},$$

$$(m-n)\,A + 2n\alpha - \omega^2\rho\,\frac{a^2(2m+n)}{4(m+n)} = 0 \quad\text{.....(58)}.$$

This leaves on the flat ends a normal pressure

$$\frac{m-n}{m+n}\,\frac{\omega^2\rho r^2}{2} - (3m-n)\,A + \frac{2m+n}{m+n}\,\frac{\omega^2\rho a^2}{2} \quad\text{.....(59)}.$$

This would suit a cylinder with its ends pressed against two yielding smooth surfaces.

If rim just free, $$A = \frac{3\omega^2\rho a^2 m}{2(m+n)(3m-n)} \quad\text{.....(60)};$$

$$\left.\begin{array}{l} \therefore\quad u = \dfrac{\omega^2\rho a^2 r(7m-n)}{8(m+n)(3m-n)} - \dfrac{\omega^2\rho r^2}{8(m+n)}, \\ \quad\;\; w = -\dfrac{\omega^2\rho a^2 z(m-n)}{4(m+n)(3m-n)} \end{array}\right\} \quad\text{.....(61)}.$$

Digitized by Google

The normal *traction* on the flat ends is, substituting for A in (59),

$$\frac{\omega^2\rho\,(a^2-r^2)\,(m-n)}{2\,(m+n)} \quad\text{.....(62).}$$

Or we might secure that no alteration would take place in the length of the cylinder by taking

$$A - 2\alpha = 0 \quad\text{.....(63);}$$

whence from (58),

$$A = \frac{\omega^2\rho a^2\,(2m+n)}{4m\,(m+n)},$$

and so

$$u = \frac{\omega^2\rho}{8\,(m+n)}\left\{a^2 r\,\frac{2m+n}{m} - r^3\right\} \quad\text{.....(64),}$$

$$w = 0;$$

and the normal traction on the flat ends

$$= \frac{\omega^2\rho\,(m-n)}{4\,(m+n)}\left\{a^2\,\frac{2m+n}{m} - 2r^2\right\} \quad\text{.....(65).}$$

This last hypothesis is really the same as that of an infinite cylinder and is considered fully presently.

§ 77. We pass to the case of an infinite right circular cylinder rotating with uniform angular velocity ω about its axis. By the term infinite we mean that the surface conditions at the flat ends need not be taken into account. We shall first consider the more general case of a shell the radii of whose bounding surfaces are a and a'. By putting $a'=0$, we get results applicable to a solid cylinder. We have

$$v = w = 0, \quad \delta = \frac{du}{dr} + \frac{u}{r} \quad\text{.....(66),}$$

and

$$(m+n)\,\frac{d\delta}{dr} + \omega^2\rho r = 0 \quad\text{.....(67).}$$

Substituting for δ in (67) from (66), we get

$$(m+n)\,\frac{d}{dr}\cdot\frac{1}{r}\frac{d\,.\,ur}{dr} = -\,\omega^2\rho r \quad\text{.....(68).}$$

By straightforward integration this leads to

$$u = -\,\frac{\omega^2\rho r^3}{8\,(m+n)} + Ar + \frac{B}{r} \quad\text{.....(69),}$$

where A and B are arbitrary constants, the latter however vanishing in the case of a solid cylinder. From (69) we find

$$\delta = -\,\frac{\omega^2\rho r^2}{2(m+n)} + 2A \quad\text{.....(70).}$$

Digitized by Google

To determine A and B, if we suppose the surface free of forces, we must apply at both the boundaries $r = a$ and $r = a'$ the surface equation

$$(m+n)\,\delta - 2n\,\frac{u}{r} = 0 \qquad \text{.....(71).}$$

This gives

$$2mA - 2nBa^{-2} = \frac{\omega^2 \rho a^2 (2m+n)}{4(m+n)},$$

$$2mA - 2nBa'^{-2} = \frac{\omega^2 \rho a'^2 (2m+n)}{4(m+n)};$$

whence

$$A = \frac{\omega^2 \rho (a^2 + a'^2)(2m+n)}{8m(m+n)} \qquad \text{.....(72),}$$

$$B = \frac{\omega^2 \rho a^2 a'^2 (2m+n)}{8n(m+n)} \qquad \text{.....(73).}$$

Thus

$$\delta = \frac{\omega^2 \rho}{4(m+n)} \left\{ \frac{2m+n}{m} (a^2 + a'^2) - 2r^2 \right\} \qquad \text{.....(74),}$$

$$u = \frac{\omega^2 \rho}{8(m+n)} \left\{ \frac{2m+n}{m}\, r\,(a^2 + a'^2) + \frac{2m+n}{n}\,\frac{a^2 a'^2}{r} - r^3 \right\} \qquad \text{.....(75),}$$

$$w = 0.$$

The principal stresses act along the directions of u, v and w at every point, and so are R, Θ, and Z, and their values are respectively

$$(m-n)\,\delta + 2n\frac{du}{dr}, \quad (m-n)\,\delta + 2n\,\frac{u}{r}, \quad \text{and} \quad (m-n)\,\delta.$$

Since r is intermediate in value between a and a' it is obvious from (74) and (75) that δ and $\frac{u}{r}$ are always positive.

Thus, supposing m greater than n, Θ and Z are certainly positive, or are tractions; and Θ is greater than Z. The sign of $\frac{du}{dr}$ depends on the value of r and on the ratio $m : n$.

It is easily seen however that $\frac{du}{dr}$ is always algebraically less than $\frac{u}{r}$, thus Θ is always the greatest stress, and $\frac{u}{r}$ the greatest strain.

The absolutely greatest value of $\frac{u}{r}$, which in accordance with Saint-Venant's theory measures the tendency to rupture, occurs at the inner surface $r = a'$ and is

$$\frac{\omega^2 \rho}{8mn} \{a^2 (2m+n) + na'^2\} \qquad \text{.....(76).}$$

Digitized by Google

It is unnecessary to determine the regions within which Z or R is the greater if we merely wish to find the greatest value of the maximum stress-difference, which in accordance with the stress-difference theory measures the tendency to rupture. It is in fact easily seen that the greatest value of $\Theta - Z$, which equals $2n\frac{u}{r}$, occurs when $r = a'$ and is

$$\frac{\omega^2 \rho n}{4m}\left\{a^2 \frac{2m+n}{n} + a'^2\right\} \quad \text{.....(77)};$$

while the greatest value of $\Theta - R$, which equals $2n\left(\frac{u}{r} - \frac{du}{dr}\right)$, also occurs when $r = a'$ and is

$$\frac{\omega^2 \rho}{2(m+n)}\{a^2(2m+n) + na'^2\} \quad \text{.....(78)}.$$

Now (78) bears to (77) the ratio $2m : m+n$, and thus is the greater, and so in accordance with the second theory measures the tendency to rupture.

On either hypothesis the tendency to rupture is greatest at the inner surface of the cylinder.

For a very thin shell (76) and (78) are approximately equal to

$$\frac{\omega^2 \rho a^2 (m+n)}{4mn} \quad \text{.....(79)},$$

and

$$\omega^2 \rho a^2 \quad \text{.....(80)},$$

respectively. If on the other hand a' be very small, or the cylinder solid nearly to its centre, (76) and (78) become approximately

$$\frac{\omega^2 \rho a^2 (2m+n)}{8mn} \quad \text{.....(81)},$$

and

$$\frac{\omega^2 \rho a^2 (2m+n)}{2(m+n)} \quad \text{.....(82)},$$

respectively.

For a solid cylinder we get from (74) and (75)

$$\delta = \frac{\omega^2 \rho}{4(m+n)}\left\{\frac{2m+n}{m} a^2 - 2r^2\right\} \quad \text{.....(83)},$$

$$u = \frac{\omega^2 \rho r}{8(m+n)}\left\{\frac{2m+n}{m} a^2 - r^2\right\} \quad \text{.....(84)},$$

and so

$$\frac{du}{dr} = \frac{\omega^2 \rho}{8(m+n)}\left\{\frac{2m+n}{m} a^2 - 3r^2\right\} \quad \text{.....(85)}.$$

Here, as for a hollow cylinder, δ and u are always positive; while Θ and Z are always tractions, of which the former is the greater. Inside the coaxial cylinder

$$r = a\sqrt{\frac{2m+n}{3m}} \quad \text{.....(86)},$$

Digitized by Google

$\frac{du}{dr}$ is positive, and so R greater than Z; outside this surface the reverse is the case. In both cases it is obvious that $\frac{u}{r}$ is algebraically greater than $\frac{du}{dr}$, thus throughout the whole cylinder Θ is the greatest stress. Inside the surface (86) the maximum stress-difference is

$$\Theta - Z \equiv 2n\frac{u}{r} = \frac{\omega^2\rho n}{4(m+n)}\left\{\frac{2m+n}{m}a^2 - r^2\right\} \quad \text{.....(87)},$$

whose greatest value occurs at the axis, and is

$$\frac{\omega^2\rho a^2 n(2m+n)}{4m(m+n)} \quad \text{.....(88)}.$$

Outside the surface (86) the maximum stress-difference is

$$\Theta - R \equiv 2n\left(\frac{u}{r} - \frac{du}{dr}\right) = \frac{\omega^2\rho r^2 n}{2(m+n)} \quad \text{.....(89)},$$

which has its greatest value, viz.

$$\frac{\omega^2\rho a^2 n}{2(m+n)} \quad \text{.....(90)},$$

at the surface. This is, however, less than (88), which in accordance with the stress-difference theory will thus measure the tendency to rupture in a solid cylinder.

The greatest strain is obviously $\frac{u}{r}$, which is always greater than $\frac{du}{dr}$ except at the axis, where they are equal. It is here, however, that it has its greatest value, viz.

$$\frac{\omega^2\rho a^2(2m+n)}{8m(m+n)} \quad \text{.....(91)}.$$

In accordance with the greatest strain theory (91) is thus the measure of the tendency to rupture.

It should be noticed that on either theory the tendency to rupture is greatest in the axis of the cylinder.

Comparing (78), (80), (82) and (88), or again (76), (79), (81) and (91), we see that the tendency to rupture in a solid cylinder is very much less than in a hollow cylinder of the same external diameter however small be the diameter of the core removed.

It is also noticeable that the absolute amount bored out of the solid cylinder is of comparatively small importance. The absolute ratios of the tendencies to rupture in a solid cylinder, in a cylinder from which an extremely small core has been removed, and in a cylinder of extreme thinness, all of the same external diameter, are from the theory of maximum stress-difference

$$n : 2m : 2m\frac{2(m+n)}{2m+n};$$

and from the theory of maximum strain

$$n : m+n : (m+n)\frac{2(m+n)}{2m+n}.$$

Digitized by Google

SECTION XI.

Equilibrium of infinite cylinder or cylindrical shell.

§ 78. Let us now consider the general case in two dimensions of an infinite cylinder at rest acted on by a series of bodily forces which have a potential V,—satisfying $\nabla^2 V = 0$, and which may be expanded in a series of sines and cosines of θ—, and by a series of surface forces expressible in a series of sines and cosines of θ.

We then have, see (7) Section IX.,

$$\delta = \frac{du}{dr} + \frac{u}{r} + \frac{1}{r}\frac{dv}{d\theta} = \frac{1}{r}\left(\frac{d.ur}{dr} + \frac{dv}{d\theta}\right) \quad \text{.....(1).}$$

The surface conditions are, see (24) and (25) Section IX.,

$$(m-n)\,\delta + 2n\frac{du}{dr} = \text{normal force} \quad \text{.....(2),}$$

$$n\left(r\frac{d.\frac{v}{r}}{dr} + \frac{1}{r}\frac{du}{d\theta}\right) = \text{tangential force} \quad \text{.....(3).}$$

The internal equations are, see (21) and (22) Section IX.,

$$(m+n)\,r\frac{d\delta}{dr} - n\frac{d\mathfrak{C}}{d\theta} = -\rho r\frac{dV}{dr} \quad \text{.....(4),}$$

$$\frac{m+n}{r}\frac{d\delta}{d\theta} + n\frac{d\mathfrak{C}}{dr} = -\frac{\rho}{r}\frac{dV}{d\theta} \quad \text{.....(5);}$$

where

$$\mathfrak{C} = \frac{1}{r}\left(\frac{d.vr}{dr} - \frac{du}{d\theta}\right) \quad \text{.....(6).}$$

Differentiate (4) with respect to r and (5) with respect to θ; adding we eliminate $\mathfrak{C}$ and get

$$(m+n)\,\nabla^2\delta = -\rho\nabla^2 V \quad \text{.....(7),}$$

where

$$\nabla^2 \equiv \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \frac{1}{r^2}\frac{d^2}{d\theta^2}.$$

Take as a particular solution

$$\delta = -\frac{\rho V}{m+n} \quad \text{.....(8).}$$

Let V be expanded in a series satisfying $\nabla^2 V = 0$, viz.

$$-V = \Sigma\cos i\theta\,(\alpha_i r^i + \alpha_i' r^{-i}) + \Sigma\sin i\theta\,(\beta_i r^i + \beta_i' r^{-i}) \quad \text{.....(9),}$$

where i is a positive integer and α_i, α_i', β_i, β_i' constants. The case $i = 0$ is considered specially afterwards.

Digitized by Google

The general solution of (7) is

$$\delta = \Sigma \cos i\theta \,(A_i r^i + A_i' r^{-i}) + \Sigma \sin i\theta \,(B_i r^i + B_i' r^{-i}) \quad\text{.....(10)},$$

where A_i, B_i, A_i', B_i' are arbitrary constants to be determined by the surface conditions.

If the cylinder be solid $\qquad A_i' = 0 = B_i'$.

It is unnecessary to retain the series for V or δ in $\sin i\theta$ in the work, as the corresponding series for u and v can be at once deduced from the parts of their values depending on the series in $\cos i\theta$ in V and δ. So leave them out at present and take

$$-V = \Sigma \cos i\theta \,(a_i r^i + a_i' r^{-i}) \quad\text{.....(11)},$$

$$\delta = \Sigma \cos i\theta \left\{ r^i \left(A_i + \frac{\rho a_i}{m+n} \right) + r^{-i} \left(A_i' + \frac{\rho a_i'}{m+n} \right) \right\} \quad\text{.....(12)}.$$

Substituting in (4) the value of $\mathfrak{C}$ given by (6), we get

$$\frac{1}{r}\frac{d^2u}{d\theta^2} - \frac{1}{r}\frac{d}{dr}\cdot r\frac{dv}{d\theta} = -\frac{m+n}{n}\Sigma \cos i\theta \,.\, i\,\{A_i r^i - A_i' r^{-i}\} \quad\text{.....(13)}.$$

Substitute $\qquad \dfrac{dv}{d\theta} = r\left(\delta - \dfrac{1}{r}\dfrac{d\,.\,ur}{dr}\right),$

and for δ the value (12) and we get, multiplying by r,

$$r^2\frac{d^2u}{dr^2} + 3r\frac{du}{dr} + u + \frac{d^2u}{d\theta^2} = \Sigma \cos i\theta \left[r^{i+1}\left\{ \frac{\rho a_i (i+2)}{m+n} - A_i \frac{mi-2n}{n} \right\} \right.$$
$$\left. - r^{-i+1}\left\{ \frac{\rho a_i' (i-2)}{m+n} - A_i' \frac{mi+2n}{n} \right\} \right] \quad\text{.....(14)}.$$

Let u_i denote the coefficient of r^{i+1} in u; then, equating coefficients of r^{i+1} on the two sides of (14), we get

$$\frac{d^2u_i}{d\theta^2} + (i+2)^2 u_i = \cos i\theta \left\{ \frac{\rho a_i (i+2)}{m+n} - A_i \frac{mi-2n}{n} \right\} \quad\text{.....(15)}.$$

The complete solution is

$$u_i = \frac{\cos i\theta}{4(i+1)}\left\{ \frac{\rho a_i (i+2)}{m+n} - A_i \frac{mi-2n}{n} \right\} + C_{i+2}\cos (i+2)\,\theta,$$

where C_{i+2} is an arbitrary constant.

Again, let u_i' denote the coefficient of r^{-i+1} in u. Then similarly the complete solution is

$$u_i' = C'_{i-2}\cos (i-2)\,\theta + \frac{\cos i\theta}{4(i-1)}\left\{ \frac{\rho a_i' (i-2)}{m+n} - A_i' \frac{mi+2n}{n} \right\} \quad\text{.....(16)},$$

where C'_{i-2} is an arbitrary constant.

Digitized by Google

Thus that part of u which contains $\cos i\theta$ is

$$u = \cos i\theta \left[\frac{r^{i+1}}{4(i+1)} \left\{ \frac{\rho\alpha_i(i+2)}{m+n} - A_i \frac{mi-2n}{n} \right\} + C_i r^{i-1} \right.$$
$$\left. + \frac{r^{-i+1}}{4(i-1)} \left\{ \frac{\rho\alpha_i'(i-2)}{m+n} - A_i' \frac{mi+2n}{n} \right\} + C_i' r^{-i-1} \right] \text{.....(17)},$$

where C_i, C_i' are arbitrary constants.

The corresponding terms in v come from (1), which gives, putting together the coefficients and integrating,

$$v = \sin i\theta \left[\frac{r^{i+1}}{4(i+1)} \left\{ A_i \frac{m(i+2)+2n}{n} - \frac{i\rho\alpha_i}{m+n} \right\} - C_i r^{i-1} \right.$$
$$\left. - \frac{r^{-i+1}}{4(i-1)} \left\{ A_i' \frac{m(i-2)-2n}{n} - \frac{i\rho\alpha_i'}{m+n} \right\} + C_i' r^{-i-1} \right] + f(r) \text{...(18)},$$

where $f(r)$ is a function of r only, which we shall here neglect.

The expressions for u and v depending on those terms in δ and V which contain $\sin i\theta$ can be at once deduced from (17) and (18). Thus in (17) write $\sin i\theta$ for $\cos i\theta$, B_i for A_i, B_i' for A_i', β_i for α_i, β_i' for α_i', D_i for C_i and D_i' for C_i', where D_i and D_i' are two new arbitrary constants, and we get the new part of u. While in (18) write $-\cos i\theta$ for $\sin i\theta$, B_i for A_i etc., and we get the new part of v. Calling (17) and (18) with these additions (17_a) and (18_a) respectively, then (17_a) is the complete value of u, and (18_a) of v depending on $\cos i\theta$ and $\sin i\theta$.

§ 79. To determine the constants we must have recourse to the surface conditions (2) and (3). So far as depends on the cosine series in δ and V, and clearly the cosine and sine series must separately satisfy the surface conditions, we find for the value of

$$(m-n)\delta + 2n\frac{du}{dr}$$

a series in cosines, which after reduction gives

$$(m-n)\delta + 2n\frac{du}{dr} = \Sigma \cos i\theta \left[r^i \left\{ -\frac{m(i-2)}{2} A_i + \frac{2m+ni}{2} \frac{\rho\alpha_i}{m+n} \right\} \right.$$
$$\left. + 2n(i-1) C_i r^{i-2} + r^{-i} \left\{ \frac{m(i+2)}{2} A_i' + \frac{2m-ni}{2} \frac{\rho\alpha_i'}{m+n} \right\} - 2n(i+1) C_i' r^{-i-2} \right] \text{.....(19)}.$$

Similarly,

$$n\left(\frac{r\frac{d.v}{r}}{dr} + \frac{1}{r}\frac{du}{d\theta} \right) = \Sigma \sin i\theta \left[r^i \left\{ \frac{mi}{2} A_i - \frac{ni}{2} \frac{\rho\alpha_i}{m+n} \right\} - 2n(i-1) C_i r^{i-2} \right.$$
$$\left. + r^{-i} \left\{ \frac{mi}{2} A_i' - \frac{ni}{2} \frac{\rho\alpha_i'}{m+n} \right\} - 2n(i+1) C_i' r^{-i-2} \right] \text{.....(20)}.$$

The complete value of (19), depending on both cosine and sine series for δ and V, is got by adding to (19) another series in sines, the term in $\sin i\theta$ being got from that in $\cos i\theta$ in (19) by writing B_i for A_i, B_i' for A_i' etc.

Digitized by Google

Similarly for the additional term in (20) write $-\cos i\theta$ for $\sin i\theta$, B_i for A_i, B_i' for A_i' etc. Call these complete values (19_a) and (20_a).

Then we have

$(19_a)_{r=a}$ = Normal force on outer surface,

$(19_a)_{r=a'}$ = Normal force on inner surface,

$(20_a)_{r=a}$ = Tangential force on outer surface,

$(20_a)_{r=a'}$ = Tangential force on inner surface.

Since at the surface the cosine and sine terms must be identically satisfied separately, we can consider separately the terms depending on the cosine and sine series in δ and V. Thus suppose those parts of the surface forces depending on $\cos i\theta$ and $\sin i\theta$ to be

$S_i \cos i\theta + V_i \sin i\theta$ normally on outer surface,

$S_i' \cos i\theta + V_i' \sin i\theta$ normally on inner surface,

$T_i \sin i\theta - W_i \cos i\theta$ tangentially on outer surface,

$T_i' \sin i\theta - W_i' \cos i\theta$ tangentially on inner surface.

Then A_i, A_i', C_i and C_i' are determined in terms of α_i, α_i', S_i, S_i', T_i, and T_i';
while B_i, B_i', D_i and D_i' are determined in terms of β_i, β_i', V_i, V_i', W_i and W_i'.

Also the value of B_i can be deduced from that of A_i by writing β_i for α_i, β_i' for α_i', V_i for S_i, V_i' for S_i', W_i for T_i, and W_i' for T_i'; and the same substitution deduces B_i' from A_i', D_i from C_i, and D_i' from C_i'. We need thus consider only one set of surface conditions, say that depending on S_i, T_i, S_i' and T_i'. Doing so we get

$$-\frac{m(i-2)}{2}A_i a^i + \frac{2m+ni}{2}\frac{\rho\alpha_i}{m+n}a^i + 2n(i-1)C_i a^{i-2} + \frac{m(i+2)}{2}A_i' a^{-i}$$
$$+\frac{2m-ni}{2}\frac{\rho\alpha_i'}{m+n}a^{-i} - 2n(i+1)C_i' a^{-i-2} = S_i \quad\text{.....(21)},$$

$$-\frac{m(i-2)}{2}A_i a'^i + \frac{2m+ni}{2}\frac{\rho\alpha_i}{m+n}a'^i + 2n(i-1)C_i a'^{i-2} + \frac{m(i+2)}{2}A_i' a'^{-i}$$
$$+\frac{2m-ni}{2}\frac{\rho\alpha_i'}{m+n}a'^{-i} - 2n(i+1)C_i' a'^{-i-2} = S_i' \quad\text{.....(22)},$$

$$\frac{mi}{2}A_i a^i - \frac{ni}{2}\frac{\rho\alpha_i}{m+n}a^i - 2n(i-1)C_i a^{i-2} + \frac{mi}{2}A_i' a^{-i}$$
$$-\frac{ni}{2}\frac{\rho\alpha_i'}{m+n}a^{-i} - 2n(i+1)C_i' a^{-i-2} = T_i \quad\text{.....(23)},$$

$$\frac{mi}{2}A_i a'^i - \frac{ni}{2}\frac{\rho\alpha_i}{m+n}a'^i - 2n(i-1)C_i a'^{i-2} + \frac{mi}{2}A_i' a'^{-i}$$
$$-\frac{ni}{2}\frac{\rho\alpha_i'}{m+n}a'^{-i} - 2n(i+1)C_i' a'^{-i-2} = T_i' \quad\text{.....(24)}.$$

Digitized by Google

From (21), (22), (23), and (24) we determine the four unknowns A_i, A_i', C_i, C_i' in terms of the knowns S_i, S_i', T_i, T_i', α_i, α_i'.

The values are determinate in every case except when $i=1$, which makes the coefficients of C_i throughout zero.

Thus if the values were expressed in determinants one whole column of the denominator would vanish when $i=1$.

To avoid a bodily displacement we must put $C_1=0$, and to ensure statical equilibrium we find two relations (one in a solid cylinder) between S_1, S_1', T_1, T_1', α_1 and α_1'.

In a solid cylinder $A_i'=C_i'=\alpha_i'=0$, and we get two equations (21) and (23) to determine A_i and C_i in terms of S_i, T_i and α_i. We get at once in that case

$$A_i=\frac{S_i+T_i}{m}a^{-i}-\frac{\rho\alpha_i}{m+n}\quad\text{.....(25)},$$

$$C_i=\frac{iS_i+(i-2)T_i}{4n(i-1)}a^{-i+2}-\frac{\rho i\alpha_i a^2}{4n(i-1)}\quad\text{.....(26)};$$

whence typical terms in the solution for a solid cylinder are

$$u=\cos i\theta\left[\frac{r^{i+1}}{4(i+1)}\left\{\frac{i\rho\alpha_i}{n}-\frac{mi-2n}{mn}\frac{S_i+T_i}{a^i}\right\}-\frac{r^{i-1}a^2}{4n(i-1)}\left\{i\rho\alpha_i-\frac{iS_i+(i-2)T_i}{a^i}\right\}\right]\quad\text{...(27)},$$

$$v=\sin i\theta\left[\frac{r^{i+1}}{4n(i+1)}\left\{-(i+2)\rho\alpha_i+\frac{m(i+2)+2n}{m}\frac{S_i+T_i}{a^i}\right\}\right.$$
$$\left.+\frac{r^{i-1}a^2}{4n(i-1)}\left\{i\rho\alpha_i-\frac{iS_i+(i-2)T_i}{a^i}\right\}\right]\quad\text{...(28)}.$$

Supposing $\frac{n}{m}$ negligible, then the parts of u, and also of v, which depend on $\rho\alpha_i$ and on $-\frac{S_i}{a^i}$ are absolutely identical. This exactly corresponds to the result we obtained for the case of a sphere in § 13, and seems rather curious. In words, this signifies that the displacements at any point of an infinite solid cylinder due to a surface *pressure* $-S_i\cos i\theta$ and to a bodily force whose potential *per unit volume* is *$S_i\left(\frac{r}{a}\right)^i\cos i\theta$ are, if $\frac{n}{m}$ be negligible, absolutely identical.

§ 80. The general equations (21), (22), (23) and (24), though slightly long, are very convenient for the purpose of eliminating the various constants. Proceed as follows.

Multiply the sum of (21) and (23) by a^{i+2}, and from the result subtract the sum of (22) and (24) multiplied by a'^{i+2}; we thus eliminate C_i and C_i', the result being

$$mA_i(a^{2i+2}-a'^{2i+2})+m(i+1)A_i'(a^2-a'^2)=(S_i+T_i)a^{i+2}-(S_i'+T_i')a'^{i+2}$$
$$-\frac{\rho m\alpha_i}{m+n}(a^{2i+2}-a'^{2i+2})-\frac{m-ni}{m+n}\rho\alpha_i'(a^2-a'^2)\quad\text{...(29)}.$$

* Notice that potential $V=-\Sigma\cos i\theta\,(\alpha_i r^i)$, &c.

Digitized by Google

Again, subtract (21) from (23) and multiply the result by a^{-i+2}, and from this subtract the result obtained by subtracting (22) from (24) each multiplied by a'^{-i+2}. We thus get

$$m(i-1)A_i(a^2-a'^2)-mA_i'(a^{-2i+2}-a'^{-2i+2})=(T_i-S_i)a^{-i+2}-(T_i'-S_i')a'^{-i+2}$$
$$+\frac{m+ni}{m+n}\rho\alpha_i(a^2-a'^2)+\frac{m\rho\alpha_i'}{m+n}(a^{-2i+2}-a'^{-2i+2})\text{.....}(30).$$

(29) and (30) give A_i and A_i' at once. Then (21) added to (23) gives C_i' in terms of A_i, A_i', and known quantities. Similarly (21) subtracted from (23) gives C_i.

Thus

$$4nC_i'(i+1)a^{-i-2}=mA_ia^i+m(i+1)A_i'a^{-i}-(S_i+T_i)+\frac{\rho m\alpha_i}{m+n}a^i$$
$$+\frac{m-ni}{m+n}\rho\alpha_i'a^{-i}\text{ .....}(31),$$

$$4nC_i(i-1)a^{i-2}=m(i-1)A_ia^i-mA_i'a^{-i}+(S_i-T_i)-\frac{m+ni}{m+n}\rho\alpha_ia^i$$
$$-\frac{m\rho\alpha_i'}{m+n}a^{-i}\text{.....}(32).$$

After eliminating A_i' from (29) and (30) and simplifying, we get

$$mA_i\left[i^2(a^2-a'^2)^2-a^2a'^2\left\{\left(\frac{a}{a'}\right)^i-\left(\frac{a'}{a}\right)^i\right\}^2\right]$$
$$=a^{i+2}a'^{-2i+2}\left[S_i\left\{(i+1)\left(\frac{a'^2}{a^2}\right)^i-i\left(\frac{a'^2}{a^2}\right)^{i-1}-1\right\}+T_i\left\{(i+2)\left(\frac{a'^2}{a^2}\right)^{i-1}-(i+1)\left(\frac{a'^2}{a^2}\right)^i-1\right\}\right]$$
$$+a'^{i+2}a^{-2i+2}\left[S_i'\left\{(i+1)\left(\frac{a^2}{a'^2}\right)^i-i\left(\frac{a^2}{a'^2}\right)^{i-1}-1\right\}+T_i'\left\{(i+2)\left(\frac{a^2}{a'^2}\right)^{i-1}-(i+1)\left(\frac{a^2}{a'^2}\right)^i-1\right\}\right]$$
$$+\frac{\rho\alpha_i}{m+n}\left[m\left\{i(a^2-a'^2)^2+a^2a'^2\left(\left|\frac{a}{a'}\right|^i-\left|\frac{a'}{a}\right|^i\right)^2\right\}+ni(i+1)(a^2-a'^2)^2\right]$$
$$+i\rho\alpha_i'(a^2-a'^2)(a^{-2i+2}-a'^{-2i+2})\text{ .....}(33).$$

Similarly

$$mA_i'\left[i^2(a^2-a'^2)^2-a^2a'^2\left\{\left(\frac{a}{a'}\right)^i-\left(\frac{a'}{a}\right)^i\right\}^2\right]$$
$$=a^{-i+2}a'^{2i+2}\left[S_i\left\{i\left(\frac{a^2}{a'^2}\right)^{i+1}-(i-1)\left(\frac{a^2}{a'^2}\right)^i-1\right\}-T_i\left\{(i-1)\left(\frac{a^2}{a'^2}\right)^i-(i-2)\left(\frac{a^2}{a'^2}\right)^{i+1}-1\right\}\right]$$
$$+a'^{-i+2}a^{2i+2}\left[S_i'\left\{i\left(\frac{a'^2}{a^2}\right)^{i+1}-(i-1)\left(\frac{a'^2}{a^2}\right)^i-1\right\}-T_i'\left\{(i-1)\left(\frac{a'^2}{a^2}\right)^i-(i-2)\left(\frac{a'^2}{a^2}\right)^{i+1}-1\right\}\right]$$
$$-i\rho\alpha_i(a^2-a'^2)(a^{2i+2}-a'^{2i+2})$$
$$-\frac{\rho\alpha_i}{m+n}\left[m\left\{i(a^2-a'^2)^2-a^2a'^2\left(\left|\frac{a}{a'}\right|^i-\left|\frac{a'}{a}\right|^i\right)^2\right\}-ni(i-1)(a^2-a'^2)^2\right]\text{.....}(34).$$

Digitized by Google

Since $x=1$ is a root of $(i+1)\begin{cases}x^n\\x^{n-1}\end{cases}-i\begin{cases}x^{n-1}\\x^n\end{cases}-1=0$ for all values of i and of n, it is obvious that $a^2-a'^2$ divides both sides of both (33) and (34). This gives $A_i=\frac{0}{0}$, and $A_i'=\frac{0}{0}$, when $a=a'$.

If $\frac{a'}{a}$ be very nearly unity, i.e. if the shell be very thin, putting $a'=a(1-\epsilon)$ and neglecting ϵ^2 when terms in ϵ exist, the resulting values of A_i and A_i' are much simplified. We thus get, after some simple reduction,

$$A_i=3\,\frac{i\,(S_i-S_i')-(T_i-T_i')}{ma^i i^2\,(i^2-1)\,\epsilon^2}-\frac{3\rho}{mi\epsilon^2}\left(\frac{\alpha_i}{i-1}+\frac{a^{-2i}\alpha_i'}{i+1}\right)\quad\text{.....(35).}$$

Similarly

$$A_i'=3\,\frac{(T_i-T_i')-i\,(S_i-S_i')}{ma^{-i}i^2\,(i^2-1)\,\epsilon^2}+\frac{3\rho}{mi\epsilon^2}\left(\frac{\alpha_i'}{i+1}-\frac{a^{2i}\alpha_i}{i-1}\right)\quad\text{.....(36).}$$

It is quite simple to get C_i and C_i' now from (31) and (32).

If ϵ be very small the parts of A_i, A_i', C_i and C_i' which depend on the surface-forces are much more important than those depending on the bodily forces. The same remark of course applies to the displacements and to the stresses.

Substituting the values obtained for A_i, A_i', C_i, C_i' in (17) and (18) we obtain the general solution for a thin shell.

§ 81. Let us now consider specially the case of a solid uniform cylinder exposed only to surface forces, viz. a normal traction

$$S_i\cos i\theta+U_i\sin i\theta,$$

and a tangential traction $\qquad T_i\sin i\theta-W_i\cos i\theta.$

Then from (27) and (28), paying attention to the remarks already made as to how to get the terms depending on U_i and W_i from those depending on S_i and T_i, we get

$$u=\frac{a\cos i\theta}{4n}\left\{\frac{iS_i+(i-2)\,T_i}{i-1}\left(\frac{r}{a}\right)^{i-1}-\frac{mi-2n}{m(i+1)}(S_i+T_i)\left(\frac{r}{a}\right)^{i+1}\right\}$$
$$+\frac{a\sin i\theta}{4n}\left\{\frac{iU_i+(i-2)\,W_i}{i-1}\left(\frac{r}{a}\right)^{i-1}-\frac{mi-2n}{m\,(i+1)}(U_i+W_i)\left(\frac{r}{a}\right)^{i+1}\right\}\quad\text{.....(37),}$$

$$v=\frac{a\sin i\theta}{4n}\left\{\frac{m\,(i+2)+2n}{m\,(i+1)}(S_i+T_i)\left(\frac{r}{a}\right)^{i+1}-\frac{iS_i+(i-2)\,T_i}{i-1}\left(\frac{r}{a}\right)^{i-1}\right\}$$
$$+\frac{a\cos i\theta}{4n}\left\{\frac{iU_i+(i-2)\,W_i}{i-1}\left(\frac{r}{a}\right)^{i-1}-\frac{m\,(i+2)+2n}{m\,(i+1)}(U_i+W_i)\left(\frac{r}{a}\right)^{i+1}\right\}\text{.....(38),}$$

$$\delta=\cos i\theta\,\frac{S_i+T_i}{m}\left(\frac{r}{a}\right)^i+\sin i\theta\,\frac{U_i+W_i}{m}\left(\frac{r}{a}\right)^i\quad\text{.....(39).}$$

Digitized by Google

The elastic forces are, see (32), (36), (35), and (40) Section IX.,

$$R = (m - n)\,\delta + 2n\frac{du}{dr},$$

$$\Theta = (m + n)\,\delta - 2n\frac{du}{dr},$$

$$R_\theta = n\left(r\frac{d.\frac{v}{r}}{dr} + \frac{1}{r}\frac{du}{d\theta}\right),$$

$$Z = (m - n)\,\delta.$$

We thus get

$$R = \cos i\theta\left\{\frac{iS_i + (i-2)\,T_i}{2}\left(\frac{r}{a}\right)^{i-2} - \frac{i-2}{2}(S_i + T_i)\left(\frac{r}{a}\right)^i\right\}$$
$$+ \sin i\theta\left\{\frac{iU_i + (i-2)\,W_i}{2}\left(\frac{r}{a}\right)^{i-2} - \frac{i-2}{2}(U_i + W_i)\left(\frac{r}{a}\right)^i\right\} \quad \text{.....(40),}$$

$$\Theta = \cos i\theta\left\{\frac{i+2}{2}(S_i + T_i)\left(\frac{r}{a}\right)^i - \frac{iS_i + (i-2)\,T_i}{2}\left(\frac{r}{a}\right)^{i-2}\right\}$$
$$+ \sin i\theta\left\{\frac{i+2}{2}(U_i + W_i)\left(\frac{r}{a}\right)^i - \frac{iU_i + (i-2)\,W_i}{2}\left(\frac{r}{a}\right)^{i-2}\right\} \quad \text{.....(41)}$$

$$R_\theta = \cos i\theta\left\{\frac{iU_i + (i-2)\,W_i}{2}\left(\frac{r}{a}\right)^{i-2} - \frac{i}{2}(U_i + W_i)\left(\frac{r}{a}\right)^i\right\}$$
$$+ \sin i\theta\left\{\frac{i}{2}(S_i + T_i)\left(\frac{r}{a}\right)^i - \frac{iS_i + (i-2)\,T_i}{2}\left(\frac{r}{a}\right)^{i-2}\right\} \quad \text{.....(42),}$$

$$Z = \frac{m-n}{m}\left(\frac{r}{a}\right)^i\{\cos i\theta\,(S_i + T_i) + \sin i\theta\,(U_i + W_i)\} \quad \text{.....(43).}$$

It is easy to find the principal stresses from these values.

The value of S or $\sqrt{(R - \Theta)^2 + 4R_\theta^2}$ is very simple.

From above

$$R - \Theta = \cos i\theta\left[\{iS_i + (i-2)\,T_i\}\left(\frac{r}{a}\right)^{i-2} - i\,(S_i + T_i)\left(\frac{r}{a}\right)^i\right]$$
$$+ \sin i\theta\left[\{iU_i + (i-2)\,W_i\}\left(\frac{r}{a}\right)^{i-2} - i\,(U_i + W_i)\left(\frac{r}{a}\right)^i\right] \quad \text{.....(44).}$$

Noticing the similarity of this to the value of $2R_\theta$, we get

$$S^2 = \left[\{iS_i + (i-2)\,T_i\}\left(\frac{r}{a}\right)^{i-2} - i\,(S_i + T_i)\left(\frac{r}{a}\right)^i\right]^2$$
$$+ \left[\{iU_i + (i-2)\,W_i\}\left(\frac{r}{a}\right)^{i-2} - i\,(U_i + W_i)\left(\frac{r}{a}\right)^i\right]^2 \quad \text{.....(45).}$$

Digitized by Google

This is independent of θ which is very curious.

S can vanish in the following cases:

$$\left.\begin{array}{l} i=2,\ r=0, \text{ when } S_2=0=U_2, \\ i>2,\ r=0 \end{array}\right\} \quad \text{.....(46)};$$

$$\left.\begin{array}{l} U_i=0=W_i, \text{ and } \left.\dfrac{r}{a}\right|^2=\dfrac{iS_i+(i-2)T_i}{i(S_i+T_i)}; \\ S_i=0=T_i, \text{ and } \left.\dfrac{r}{a}\right|^2=\dfrac{iU_i+(i-2)W_i}{i(U_i+W_i)}; \\ \text{or} \quad \dfrac{iS_i+(i-2)T_i}{S_i+T_i}=\dfrac{iU_i+(i-2)W_i}{U_i+W_i}=i\left(\dfrac{r}{a}\right)^2 \end{array}\right\} \quad \text{.....(47)}.$$

This last requires $\qquad S_iW_i=T_iU_i$ .....(47 a).

This condition includes the preceding cases

$$S_2=0=U_2, \quad U_i=0=W_i, \quad \text{and} \quad S_i=0=T_i.$$

$S=0$ implies that the stress quadric is one of revolution about a parallel to the axis of the cylinder, and also that there is no stress-difference in the transverse section.

We must of course have $\dfrac{r}{a}<1$ in order that the solution may apply to our problem. Clearly this is the case in (47) if S_i and T_i have the same sign, and if U_i and W_i have the same sign, i being $\not< 2$.

§ 82. We shall consider more fully the two most interesting cases,

(i) no tangential, (ii) no normal surface-forces.

Case (i) $\qquad T_i=0=W_i,$

$$S^2=\left\{i\left(\frac{r}{a}\right)^{i-2}\left(1-\frac{r^2}{a^2}\right)\right\}^2(S_i^2+U_i^2) \quad \text{.....(48)}.$$

Thus $S=0$ only when $r=a$, or when $r=0$ and $i>2$.

The surface $S^2-(2n\delta)^2=0$ which separates the regions in which S is and is not the maximum stress-difference may be at once determined from (39) and (48). For our present purpose however it will be sufficient to determine the maximum values of S and of $\dfrac{S}{2}\pm n\delta$ and the points where these exist.

If $i=2$, S has its maximum value

$$2\sqrt{S_2^2+U_2^2} \quad \text{.....(49)},$$

at the axis.

The greatest value of $\dfrac{S}{2}\pm n\delta$ also occurs at the axis, but is only half the above

Digitized by Google

as $\delta = 0$ when $r = 0$. By § 69, taking δ positive we have $\frac{S}{4n} + \frac{\delta}{2}$ for the greatest strain. Thus if $i = 2$ the tendency to rupture is certainly greatest at the axis on either theory.

If $i > 2$, S has its maximum value

$$2\left(\frac{i-2}{i}\right)^{\frac{i-2}{2}} \sqrt{S_i^2 + U_i^2} \quad \text{.....(50)*},$$

over the surface of the cylinder

$$r = a\sqrt{\frac{i-2}{i}} \quad \text{.....(51)}.$$

The + or − being so taken as to make the term in δ positive, $\frac{S}{2} \pm n\delta$ has its maximum value

$$\left(\frac{i-2}{i - \frac{2n}{m}}\right)^{\frac{i-2}{2}} \sqrt{S_i^2 + U_i^2} \quad \text{.....(52)*},$$

when

$$\tan i\theta = \pm \frac{U_i}{S_i} \quad \text{.....(53)},$$

and

$$r = a\sqrt{\frac{i-2}{i - \frac{2n}{m}}} \quad \text{.....(54)}.$$

Which of the two expressions (50) or (52) is the greater must be determined specially for each value of i, greater than 2, and of $\frac{n}{m}$. In any given case the determination would be easy, and the correct measure of the tendency to rupture and the locus of its occurrence on the stress-difference theory would follow at once.

On the theory of greatest strain, dividing (52) by $2n$ we have a measure of the tendency to rupture, cf. § 69; and the locus where the tendency is greatest is given by (53) and (54), taking in the former the + or − sign as is required to make δ positive.

§ 83. Case (ii) $S_i = 0 = U_i$,

$$S^2 = \left(\frac{r}{a}\right)^{2i-4}\left(i - 2 - i\frac{r^2}{a^2}\right)^2 (T_i^2 + W_i^2) \quad \text{.....(55)}.$$

Thus S vanishes when

$$r = a\sqrt{\frac{i-2}{i}} \quad \text{.....(56)}.$$

When $r = 0$, R_1, Θ_1, Z all vanish if $i \nless 2$; the same is true of the strains, thus there is never any tendency to rupture at the axis.

When $i =$ or > 2, S has its greatest maximum value

$$2\sqrt{T_i^2 + W_i^2} \quad \text{.....(57)},$$

when $r = a$; here also occurs, for

$$\tan i\theta = \pm \frac{W_i}{T_i} \quad \text{.....(58)},$$

* Treating these as indeterminate forms we can deduce the results for the case $i = 2$.

Digitized by Google

the greatest value of $\frac{S}{2} \pm n\delta$, which is

$$\left(1 + \frac{n}{m}\right) \sqrt{T_i^2 + W_i^2} \quad \text{.....(59).}$$

The former is always the greater supposing $m > n$. Referring to § 69, we see that on either theory the tendency to rupture is greatest at the surface.

If $i > 2$, S has a second maximum value

$$2\left(\frac{i-2}{i}\right)^{i-1} \sqrt{T_i^2 + W_i^2} \quad \text{.....(60),}$$

over the cylinder

$$r = a\frac{i-2}{i} \quad \text{.....(61).}$$

The sign being taken so as to make δ positive, $\frac{S}{2} \pm n\delta$ has also a maximum value

$$\frac{(i-2)^{i-1}}{i\left\{i\left(i - \frac{2n}{m}\right)\right\}^{\frac{i-2}{2}}} \sqrt{T_i^2 + W_i^2} \quad \text{.....(62),}$$

when

$$\tan i\theta = \pm \frac{W_i}{T_i} \quad \text{.....(63),}$$

and

$$r = a\frac{i-2}{\sqrt{i\left(i - \frac{2n}{m}\right)}} \quad \text{.....(64).}$$

On the theory of greatest strain (62) divided by $2n$ may be looked on as a second maximum of the measure of the tendency to rupture, the locus where this maximum occurs being given by (63) and (64), taking in the former the + or − sign so as to make δ positive.

It is obvious however that (57) is always greater than (60) or (62), and likewise (59) greater than (62), so these second maxima need not be further considered.

To compare the effects of normal and of tangential forces in producing rupture, suppose $S_i^2 + U_i^2 = T_i^2 + W_i^2$. Then from (49) and (57) for $i = 2$, on the stress-difference theory, the tendencies to rupture are the same; while from the remarks after (49) and (59), on the greatest strain theory, it follows that the tangential forces are most apt to cause rupture. If $i > 2$, it is easily seen that (57) is greater than (50) or (52), and (59) greater than (52).

Thus on either theory if $i > 2$, tangential forces are more apt to rupture the cylinder than normal forces of the same magnitude.

Digitized by Google

SECTION XII.

Vibrations of infinite solid cylinder or cylindrical shell,—including the case of a cylinder of finite length whose length is maintained constant.

§ 84. We shall first consider the case of a solid cylinder.

If
$$\mathfrak{C} = \frac{1}{r}\left(\frac{d\,.\,vr}{dr} - \frac{du}{d\theta}\right) \quad \text{.....(1)},$$

and
$$\delta = \frac{1}{r}\left(\frac{d\,.\,ur}{dr} + \frac{dv}{d\theta}\right) \quad \text{.....(2)};$$

then, supposing there to be no bodily forces, the equations of motion are, in accordance with the results of Section IX.,

$$(m+n)\, r\,\frac{d\delta}{dr} - n\,\frac{d\mathfrak{C}}{d\theta} = \rho r\,\frac{d^2u}{dt^2} \quad \text{.....(3)},$$

$$(m+n)\,\frac{1}{r}\,\frac{d\delta}{d\theta} + n\,\frac{d\mathfrak{C}}{dr} = \rho\,\frac{d^2v}{dt^2} \quad \text{.....(4)}.$$

Differentiate (3) with respect to r, and (4) with respect to θ; then adding, we get

$$(m+n)\left(\frac{d^2\delta}{dr^2} + \frac{1}{r}\,\frac{d\delta}{dr} + \frac{1}{r^2}\,\frac{d^2\delta}{d\theta^2}\right) = \rho\,\frac{d^2\delta}{dt^2} \quad \text{.....(5)}.$$

We may assume by Fourier's theorem that δ can be expanded in a series of sines and cosines of multiple angles. Suppose then that the term in $\cos i\theta$ in δ, when a vibration of frequency $\frac{k}{2\pi}$ is taking place, is given by

$$\delta = A_i \cos kt \cos i\theta\, \phi(r),$$

where A_i is an arbitrary constant, and i is an integer.

Substituting this expression in (5) we find that $\phi(r)$ must satisfy the equation

$$\frac{d^2\phi(r)}{dr^2} + \frac{1}{r}\,\frac{d\phi(r)}{dr} + \left(\frac{k^2\rho}{m+n} - \frac{i^2}{r^2}\right)\phi(r) = 0 \quad \text{.....(6)}.$$

If we denote
$$\left.\begin{aligned} \frac{\rho}{m+n} &\text{ by } \alpha^2 \\ \frac{\rho}{n} &\text{ by } \beta^2 \end{aligned}\right\} \quad \text{.....(7)},$$
and

then the solution of (6), rejecting the second Bessel's function as applicable only to a hollow cylinder, is

$$\phi(r) = J_i(k\alpha r) \quad \text{.....(8)}.$$

Thus the typical solution is

$$\delta = A_i \cos kt \cos i\theta\, J_i(k\alpha r) \quad \text{.....(9)}.$$

The corresponding term for the series in sines of multiple angles is similarly

$$\delta = B_i \cos kt \sin i\theta\, J_i(k\alpha r).$$

Digitized by Google

This at present we shall neglect, as it can easily be introduced subsequently when required.

To determine u and v, we proceed exactly as in the case of equilibrium.

Substitute in (3) the value of $\mathfrak{C}$ from (1), and substitute from (2) the result

$$\frac{dv}{d\theta} = r\delta - \frac{d\,.\,ur}{dr};$$

and finally use the value (9) for δ.

Carrying out the necessary differentiations in due order, we find

$$\frac{d^2u}{dr^2} + \frac{3}{r}\frac{du}{dr} + \frac{u}{r^2} + \frac{1}{r^2}\frac{d^2u}{d\theta^2} - \frac{\rho}{n}\frac{d^2u}{dt^2} = A_i \cos kt \cos i\theta \left\{\frac{2}{r} J_i(kar) - \frac{m}{n}\frac{d}{dr} J_i(kar)\right\} \quad \text{.....(10)},$$

as the differential equation giving that part of u depending on the typical term in

$$\cos kt \cos i\theta.$$

Assume $$u \propto \cos kt \cos i\theta \quad \text{.....(11)},$$

and we get

$$\frac{d^2u}{dr^2} + \frac{3}{r}\frac{du}{dr} + u\left\{k^2\beta^2 - \frac{i^2-1}{r^2}\right\} = A_i \cos kt \cos i\theta \left\{\frac{2}{r} J_i(kar) - \frac{m}{n}\frac{d}{dr} J_i(kar)\right\} \quad \text{.....(12)},$$

as the equation giving u as a function of r.

(12) may be written in the form

$$\frac{d^2\,.\,ur}{dr^2} + \frac{1}{r}\frac{d\,.\,ur}{dr} + \left(k^2\beta^2 - \frac{i^2}{r^2}\right) ur = A_i \cos kt \cos i\theta \left\{2J_i(kar) - \frac{m}{n} r \frac{d}{dr} J_i(kar)\right\} \quad \text{.....(13)}.$$

It is comparatively easy to get a particular solution of (13).

For, using Bessel's equation, it is easy to prove that

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \left(k^2\beta^2 - \frac{i^2}{r^2}\right)\right] r \frac{d}{dr} J_i(kar) = k^2(\beta^2 - \alpha^2)\, r \frac{d}{dr} J_i(kar) - 2k^2\alpha^2 J_i(kar) \quad \text{...(14)}.$$

But $$\beta^2 - \alpha^2 = \rho\left(\frac{1}{n} - \frac{1}{m+n}\right) = \frac{m}{n}\,\frac{\rho}{m+n} \quad \text{.....(15)};$$

therefore the right-hand side of (14) is

$$\frac{k^2\rho}{m+n}\left\{-2J_i(kar) + \frac{m}{n} r \frac{d}{dr} J_i(kar)\right\}.$$

Thus the particular solution of (13) is simply

$$ur = -A_i \cos kt \cos i\theta \frac{m+n}{k^2\rho} r \frac{d}{dr} J_i(kar) \quad \text{.....(16)}.$$

The general solution is obviously

$$ur = C_i \cos kt \cos i\theta\, J_i(k\beta r) \quad \text{.....(17)},$$

where C_i is an arbitrary constant.

Digitized by Google

Thus, using the notation (7), the complete solution of (10), and so the typical term of u, is

$$u = \cos kt \cos i\theta \left\{ \frac{C_i}{r} J_i(k\beta r) - A_i \frac{1}{k^2\alpha^2} \frac{d}{dr} J_i(k\alpha r) \right\} \quad \text{.....(18).}$$

To find the corresponding typical term in v, we have

$$\frac{dv}{d\theta} = r\delta - \frac{d \,.\, ur}{dr}$$

$$= \cos kt \cos i\theta \left[A_i \left\{ rJ_i(k\alpha r) + \frac{1}{k^2\alpha^2} \frac{d}{dr} r \frac{d}{dr} J_i(k\alpha r) \right\} - C_i \frac{d}{dr} J_i(k\beta r) \right] \text{.....(19);}$$

whence, omitting the solution $v = f(r, t)$, and using Bessel's equation, we get

$$v = \cos kt \frac{\sin i\theta}{i} \left[\frac{A_i}{k^2\alpha^2} \frac{i^2}{r} J_i(k\alpha r) - C_i \frac{d}{dr} J_i(k\beta r) \right] \quad \text{.....(20).}$$

In (9), (18) and (20) we have a complete representation of the solution. To determine C_i and A_i we have over the surface $r = a$ the conditions

$$(m - n)\,\delta + 2n \frac{du}{dr} = \text{normal traction} \quad \text{.....(21).}$$

$$n \left(r \frac{d \,.\, \frac{v}{r}}{dr} + \frac{1}{r} \frac{du}{d\theta} \right) = \text{tangential traction} \quad \text{.....(22).}$$

If we suppose

$$\text{a normal surface-traction} = S_i \cos i\theta \cos kt \quad \text{.....(23),}$$

and

$$\text{a tangential surface-traction} = T_i \sin i\theta \cos kt \quad \text{.....(24);}$$

then A_i and C_i are by means of (21) and (22) expressible in terms of S_i and T_i. We may suppose (23) and (24) to be merely typical terms in the expansion by Fourier's theorem of the expressions for normal and tangential forces, so that the solution is general.

Just as in the case of equilibrium if we take the still more general forms, viz.

$$\left.\begin{aligned} &\text{normal surface-traction} = \cos kt\,(S_i \cos i\theta + U_i \sin i\theta), \\ &\text{tangential surface-traction} = \cos kt\,(T_i \sin i\theta - W_i \cos i\theta) \end{aligned}\right\} \text{.....(25),}$$

we require in addition to (9), (18) and (20) the typical terms

$$\left.\begin{aligned} \delta &= \cos kt \sin i\theta\, B_i J_i(k\alpha r), \\ u &= \cos kt \sin i\theta \left\{ \frac{1}{r} D_i J_i(k\beta r) - \frac{B_i}{k^2\alpha^2} \frac{d}{dr} J_i(k\alpha r) \right\}, \\ v &= -\cos kt \frac{\cos i\theta}{i} \left\{ \frac{B_i}{k^2\alpha^2} \frac{i^2}{r} J_i(k\alpha r) - D_i \frac{d}{dr} J_i(k\beta r) \right\} \end{aligned}\right\} \text{.....(26),}$$

where B_i and D_i are determined in terms of U_i and W_i by means of (21) and (22).

Also B_i may be at once deduced from A_i by writing U_i for S_i and W_i for T_i, and D_i is got from C_i by the same substitution. Thus we need consider only the case given by (23) and (24).

Digitized by Google

§ 85. Noticing that $\frac{\rho}{m+n} = \alpha^2$, and $\frac{\rho}{n} = \beta^2$, using Bessel's equation and the notation

$$\frac{dJ_i(k\beta r)}{d.k\beta r} = J_i'(k\beta r), \text{ etc.},$$

we may reduce (21) and (22) to the forms

$$A_i \frac{k\rho a}{\alpha}\left\{\left(1 - \frac{2i^2}{k^2\beta^2a^2}\right)\frac{J_i(k\alpha a)}{k\alpha a} + \frac{2}{k^2\beta^2a^2}J_i'(k\alpha a)\right\}$$
$$+ 2\frac{k\rho C_i}{\beta a}\left\{J_i'(k\beta a) - \frac{J_i(k\beta a)}{k\beta a}\right\} = S_i \text{.....(27)},$$

and

$$\frac{2A_i\rho i}{k\alpha a\beta^2}\left\{J_i'(k\alpha a) - \frac{J_i(k\alpha a)}{k\alpha a}\right\}$$
$$+ C_i\frac{\rho k^2\beta a}{i}\left\{\frac{2}{k^2\beta^2a^2}J_i'(k\beta a) + \left(1 - \frac{2i^2}{k^2\beta^2a^2}\right)\frac{J_i(k\beta a)}{k\beta a}\right\} = T_i \text{ .....(28)}.$$

For free vibrations $\qquad S_i = 0 = T_i$.

Eliminating $A_i : C_i$ from (27) and (28), we get after reduction

$$\frac{J_i(k\alpha a)J_i(k\beta a)}{k\alpha a\,.\,k\beta a}\{k^4\beta^4a^4 - 4i^2k^2\beta^2a^2 + 4i^2(i^2-1)\}$$
$$+ 2k^2\beta^2a^2\left\{\frac{J_i(k\alpha a)}{k\alpha a}J_i'(k\beta a) + J_i'(k\alpha a)\frac{J_i(k\beta a)}{k\beta a}\right\} - 4(i^2-1)J_i'(k\alpha a)J_i'(k\beta a) = 0 \text{.....(29)},$$

which gives the frequency of the free vibrations.

The above shows that for cylinders of different radii performing the same form of free vibration k, and so the frequency of vibration, varies inversely as the diameter.

(27) and (28) determine A_i and C_i without ambiguity in terms of S_i and T_i. It is obvious of course that if the period of the surface-force coincide with one of the periods of free vibration the values for A_i and C_i become infinite, for the denominator of these values is the left-hand side of (29).

The practical method of dealing with (29) at least for cylinders of small cross section, is to substitute the ordinary convergent series giving the Bessel's in ascending odd or even powers of $k\alpha a$ and $k\beta a$, and retain as many terms as supply the required degree of approximation. It will be found that the terms of the lowest order cut out.

§ 86. There are two important forms of vibration not included in the general case. These correspond to $i = 0$, u being the only variable; and to $v = f(r, t)$, $u = 0$.

Case $i = 0$.

δ is a function only of u, therefore

$$\delta = \frac{du}{dr} + \frac{u}{r} \text{ ....(30)}.$$

Digitized by Google

Assuming $u \propto \cos kt$, the equation (3) of the general solution then gives, since $\mathfrak{C} = 0$,

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + u\left\{\frac{k^2\rho}{m+n} - \frac{1}{r^2}\right\} = 0 \quad \text{.....(31)},$$

$$\therefore \quad u = A \cos kt\, J_1(kar) \quad \text{.....(32)},$$

where A is an arbitrary constant and $a^2 = \dfrac{\rho}{m+n}$.

The surface-forces must be purely normal, suppose them of type $F \cos kt$.

Then
$$(m-n)\,\delta + 2n\frac{du}{dr} = F \cos kt,$$

or
$$(m+n)\frac{du}{dr} + (m-n)\frac{u}{r} = F \cos kt\,;$$

therefore from (32)

$$A = \frac{F}{ka\left\{(m+n)J_1'(kaa) + (m-n)\dfrac{J_1(kaa)}{kaa}\right\}} \quad \text{.....(33)}.$$

The periods of the free vibrations are given by $F = 0$, and therefore by

$$(m+n)J_1'(kaa) + (m-n)\frac{J_1(kaa)}{kaa} = 0,$$

which by well-known properties of Bessel's functions leads to

$$\frac{J_0\left(ka\sqrt{\dfrac{\rho}{m+n}}\right)}{n} = \frac{J_2\left(ka\sqrt{\dfrac{\rho}{m+n}}\right)}{m} \quad \text{.....(34)}.$$

§ 87. Case $u = 0$, $v = f(r, t)$.

This gives $\delta = 0$, and
$$\mathfrak{C} = \frac{dv}{dr} + \frac{v}{r} \quad \text{.....(35)};$$

therefore from (4) of the general solution

$$\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2} = \frac{\rho}{n}\frac{d^2v}{dt^2} \quad \text{.....(36)}.$$

Assume $v \propto \cos kt$, and write $\dfrac{\rho}{n} = \beta^2$.

Then
$$\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} + v\left(k^2\beta^2 - \frac{1}{r^2}\right) = 0 \quad \text{.....(37)},$$

determines v as a function of r;

$$\therefore \quad v = B \cos kt\, J_1(k\beta r) \quad \text{.....(38)},$$

where B is an arbitrary constant, gives the part of v depending on $\cos kt$.

The surface-conditions (21) and (22) show that this case must be due to purely tangential surface-forces, of which $G \cos kt$ will be a type.

To determine B in terms of G, we get from (22)

$$B = \frac{\beta G}{k\rho\left\{J_1'(k\beta a) - \dfrac{J_1(k\beta a)}{k\beta a}\right\}} \quad \text{.....(39)}.$$

Digitized by Google

For the free vibrations $G = 0$, and so

$$J_1'(k\beta a) - \frac{J_1(k\beta a)}{k\beta a} = 0\,;$$

$$\text{i.e. } J_2(k\beta a) = 0 \quad \text{.....(40)},$$

is the equation giving the frequency $\frac{k}{2\pi}$ of the free vibrations of this form. This result depends only on n, not at all on m.

This is a very simple result, and the arithmetical value of k would be easily calculated from a table of Bessel's functions.

§ 88. For the radial vibrations of a hollow cylinder we have from (31)

$$u = \cos kt\,\{AJ_1(k\alpha r) + BY_1(k\alpha r)\} \quad \text{.....(41)},$$

where Y_1 denotes the second solution of the Bessel's equation, and A, B are arbitrary constants.

If $F\cos kt$, $F'\cos kt$ denote the normal forces on the boundaries, the surface-conditions may be put in the form

$$(m+n)\frac{du}{dr} + (m-n)\frac{u}{r} = F\cos kt \text{ when } r = a,$$
$$= F'\cos kt \text{ when } r = a'.$$

These lead to

$$\left.\begin{array}{l} k\alpha\left[A\left\{(m+n)J_1'(k\alpha a) + (m-n)\dfrac{J_1(k\alpha a)}{k\alpha a}\right\} + B\left\{(m+n)Y_1'(k\alpha a) + (m-n)\dfrac{Y_1(k\alpha a)}{k\alpha a}\right\}\right] = F \\ \qquad\qquad \text{and a similar expression, writing } a' \text{ for } a, = F' \end{array}\right\} (42);$$

whence we can determine A and B in terms of F and F'.

Putting $F = 0 = F'$, and eliminating between these two equations the ratio $A : B$, we get an equation for the frequency of the free vibrations. If the shell be very thin, then by employing results analogous to (58), Sect. VIII., and equating to zero the coefficient of $k\alpha a\{J_1(k\alpha a)\,Y_1'(k\alpha a) - Y_1(k\alpha a)\,J_1'(k\alpha a)\}$, an expression* which is independent of k, we immediately find

$$k^2\alpha^2 a^2 = \frac{4mn}{(m+n)^2} \quad \text{.....(43)},$$

or

$$k^2 = \frac{4mn}{\rho a^2(m+n)} \quad \text{.....(43 a)}.$$

§ 89. The case $v = f(r, t)$, $u = 0$, is also easily extended to a shell bounded by the cylinders $r = a$, $r = a'$.

From (37) the solution is, A and B denoting arbitrary constants,

$$v = \cos kt\,\{AJ_1(k\beta r) + BY_1(k\beta r)\} \quad \text{.....(44)}.$$

The surface-conditions lead to equations of the form

$$n\left(\frac{dv}{dr} - \frac{v}{r}\right) = G\cos kt \text{ when } r = a,$$
$$= G'\cos kt \text{ when } r = a'.$$

* See *Messenger of Mathematics*, Vol. xv., p. 22; or Forsyth's *Differential Equations*, p. 166.

Digitized by Google

From this, using the relations between successive Bessel's, we find

$$\left.\begin{array}{l} -k\beta\{AJ_2(k\beta a)+BY_2(k\beta a)\}=\dfrac{G}{n} \;\ldots\ldots \\ \text{and} \quad \text{a similar expression, writing } a' \text{ for } a, =\dfrac{G'}{n} \;\ldots\ldots \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(45);$$

whence we know A and B in terms of the knowns G and G'.

For the free vibrations, putting $G=0=G'$, and eliminating $A:B$, we find, as the equation giving the frequencies,

$$J_2(k\beta a)\,Y_2(k\beta a')-J_2(k\beta a')\,Y_2(k\beta a)=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(46).$$

For a very thin shell this leads to

$$\{Y_2(k\beta a)\}^2\frac{d}{da}\left[\frac{J_2(k\beta a)}{Y_2(k\beta a)}\right]=0,$$

an equation having no finite root.

§ 90. The following seems an interesting case of vibrations due to bodily forces.

A solid infinite cylinder, or one whose length is maintained constant, is rotating about its axis with an angular velocity ω, which is not a constant but is capable of being analysed into a series of harmonic expressions of the time, to find the consequent vibrations and stresses in the cylinder.

We may regard the cylinder as not rotating but as acted on by bodily forces which are radial, and at distance r from the axis are $=\omega^2 r$; under the action of these forces it vibrates radially.

Assume $$\omega=\omega_0+\omega_1\cos k_1t+\omega_2\cos k_2t+\ldots\ldots+\omega_i\cos k_it+\ldots\ldots\ldots\ldots\ldots\ldots(47).$$

The equation of motion is, see § 66,

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}+\frac{\omega^2\rho r}{m+n}=\frac{\rho}{m+n}\frac{d^2u}{dt^2} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(48),$$

putting $\delta=\dfrac{du}{dr}+\dfrac{u}{r}$ at once in the first equation of motion and also $R=\omega^2 r$.

Now $$\omega^2=\omega_0^2+\Sigma\omega_i^2\frac{1+\cos 2k_it}{2}+\Sigma\omega_i\omega_j\{\cos(k_i+k_j)t+\cos(k_i-k_j)t\},$$

$$=\Omega^2+\Sigma\frac{\omega_i^2}{2}\cos 2k_it+\Sigma\omega_i\omega_j\cos(k_i+k_j)t+\Sigma\omega_i\omega_j\cos(k_i-k_j)t \;\ldots\ldots\ldots\ldots(49),$$

where $$\Omega^2=\omega_0^2+\tfrac{1}{2}\Sigma\omega_i^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(50),$$

and j is any integer other than i, including zero when k_j is supposed to vanish.

The solution of (48) will be of the form

$$u=u_0+\Sigma u_i\cos 2k_it+\Sigma u_{i+j}\cos(k_i+k_j)t+\Sigma u_{i-j}\cos(k_i-k_j)t+U\cos\kappa t\ldots\ldots(51);$$

where u_0 is the solution of

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}+\frac{\Omega^2\rho r}{m+n}=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(52),$$

Digitized by Google

$u_i \cos 2k_i t$ the particular solution of

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}+\frac{\omega_i^2\rho r\cos(2k_it)}{2(m+n)}=\frac{\rho}{m+n}\frac{d^2u}{dt^2}, \text{ etc.} \quad\text{.....(53),}$$

and $U\cos\kappa t$ the solution of

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}=\frac{\rho}{m+n}\frac{d^2u}{dt^2},$$

which is suitable for a solid cylinder. In fact U is a type of the free vibrations.

We have
$$u_0=-\frac{\Omega^2\rho r^3}{8(m+n)}+A_0r \quad\text{.....(54);}$$

$$U=AJ_1(\kappa\alpha r) \quad\text{.....(55),}$$

where κ is given by
$$\frac{J_0(\kappa\alpha a)}{n}=\frac{J_2(\kappa\alpha a)}{m} \quad\text{.....(56),}$$

where as usual
$$\alpha^2=\frac{\rho}{m+n}.$$

To determine u_i, we get

$$\frac{d^2u_i}{dr^2}+\frac{1}{r}\frac{du_i}{dr}-\frac{u_i}{r^2}+\frac{4k_i^2\rho u_i}{m+n}+\frac{\omega_i^2\rho r}{2(m+n)}=0 \quad\text{.....(57).}$$

The solution is
$$u_i=-\frac{\omega_i^2r}{8k_i^2}+A_iJ_1(2k_i\alpha r) \quad\text{.....(58).}$$

We get similar results for the u_{i+j} and u_{i-j} terms.

Thus the complete solution is

$$u=A_0r-\frac{\Omega^2\rho r^3}{8(m+n)}+\Sigma A\cos(\kappa t)J_1(\kappa\alpha r)$$
$$+\Sigma\cos(2k_it)\left\{A_iJ_1(2k_i\alpha r)-\frac{\omega_i^2r}{8k_i^2}\right\}+\Sigma\cos(k_i+k_j)t\left\{A_{i+j}J_1(\overline{k_i+k_j}\alpha r)-\frac{\omega_i\omega_jr}{(k_i+k_j)^2}\right\}$$
$$+\Sigma\cos(k_i-k_j)t\left\{A_{i-j}J_1(\overline{k_i-k_j}\alpha r)-\frac{\omega_i\omega_jr}{(k_i-k_j)^2}\right\} \quad\text{.....(59);}$$

where A_0, A_i, A_{i+j}, A_{i-j} are constants to be determined by the surface-conditions.

The only surface-condition is, see (24), Sect. IX.,

$$(m+n)\delta-2n\frac{u}{r}=0, \quad\text{when } r=a \quad\text{.....(60).}$$

This must be satisfied independently by the expressions involving the different cosines, and by the term independent of the time.

From the last we get
$$A_0=\frac{\Omega^2\rho a^2(2m+n)}{8m(m+n)} \quad\text{.....(61).}$$

The type $A\cos\kappa tJ_1(\kappa\alpha r)$ of the terms derived from the free vibrations gives of course no term towards the surface force.

As an example of the others consider fully

$$\cos 2k_it\left\{A_iJ_1(2k_i\alpha r)-\frac{\omega_i^2r}{8k_i^2}\right\}.$$

Digitized by Google

This gives
$$(m+n)\left[\left\{2k_i\alpha J_1'(2k_i\alpha a)+2k_i\alpha\frac{J_1(2k_i\alpha a)}{2k_i\alpha a}\right\}A_i-\frac{\omega_i^2}{4k_i^2}\right]$$
$$-2n\left\{2k_i\alpha A_i\frac{J_1(2k_i\alpha a)}{2k_i\alpha a}-\frac{\omega_i^2}{8k_i^2}\right\}=0;$$

therefore by an elementary property of consecutive Bessel's functions,

$$A_i=\frac{\omega_i^2 m}{8k_i^2\alpha\{mJ_0(2k_i\alpha a)-nJ_2(2k_i\alpha a)\}}\quad\text{.....(62).}$$

Similarly we should find

$$A_{i+j}=\frac{2\omega_i\omega_j m}{(k_i+k_j)^2\alpha\{mJ_0(\overline{k_i+k_j}\alpha a)-nJ_2(\overline{k_i+k_j}\alpha a)\}}\quad\text{.....(63),}$$

$$A_{i-j}=\frac{2\omega_i\omega_j m}{(k_i-k_j)^2\alpha\{mJ_0(\overline{k_i-k_j}\alpha a)-nJ_2(\overline{k_i-k_j}\alpha a)\}}\quad\text{.....(64).}$$

Thus the complete solution is

$$u=\frac{\Omega^2\rho}{8(m+n)}\left\{\frac{a^2r(2m+n)}{m}-r^3\right\}+\Sigma AJ_1(\kappa\alpha r)\cos(\kappa t)$$
$$+\Sigma\frac{\cos(2k_it)}{(2k_i)^2}\frac{\omega_i^2}{2}\left[\frac{mJ_1(2k_i\alpha r)}{k_i\alpha\{mJ_0(2k_i\alpha a)-nJ_2(2k_i\alpha a)\}}-r\right]$$
$$+\Sigma\frac{\cos(k_i+k_j)t}{(k_i+k_j)^2}\omega_i\omega_j\left[\frac{2mJ_1(\overline{k_i+k_j}\alpha r)}{(k_i+k_j)\alpha\{mJ_0(\overline{k_i+k_j}\alpha a)-nJ_2(\overline{k_i+k_j}\alpha a)\}}-r\right]$$
$$+\Sigma\frac{\cos(k_i-k_j)t}{(k_i-k_j)^2}\omega_i\omega_j\left[\frac{2mJ_1(\overline{k_i-k_j}\alpha r)}{(k_i-k_j)\alpha\{mJ_0(\overline{k_i-k_j}\alpha a)-nJ_2(\overline{k_i-k_j}\alpha a)\}}-r\right]\text{...(65).}$$

If any of the denominators vanish the corresponding vibration becomes infinite. This happens if any one of the following series of equations is satisfied, viz.:

$$mJ_0(2k_i\alpha a)-nJ_2(2k_i\alpha a)=0,$$
$$mJ_0(\overline{k_i+k_j}\alpha a)-nJ_2(\overline{k_i+k_j}\alpha a)=0,$$
$$mJ_0(\overline{k_i-k_j}\alpha a)-nJ_2(\overline{k_i-k_j}\alpha a)=0.$$

But the equation determining the frequency of the free normal vibrations is

$$mJ_0(\kappa\alpha a)-nJ_2(\kappa\alpha a)=0.$$

If then $\kappa=2k_i$, or $=k_i+k_j$, or $=k_i-k_j$, the corresponding vibration becomes infinite; and the consequent stresses and stress-difference also become infinite.

Now we included 0 as one of the values of j, answering to the terms

$$2\omega_0\omega_i\cos k_it,\text{ etc. in }\omega^2,$$

and k_j then stands for zero. Thus our last result signifies finally that if $\kappa=k_i$ the vibration becomes infinite; and the conclusions we have come to lead to the following law.

In a rotating cylinder if the angular velocity be analysed into its component harmonics, and if the frequency of one of those harmonics or double its frequency, or if the sum or the difference of the frequencies of two of the harmonics coincide with the frequency of one of the radial vibrations natural to the cylinder, then the corresponding

Digitized by Google

radial vibration will become infinite, and the tendency to rupture will also become infinite. A near approach to coincidence will similarly cause a large radial vibration and a great tendency to rupture.

The forced vibration of frequency $\frac{1}{2\pi} k_i$ is proportional to ω_0, while those of the frequencies $\frac{1}{2\pi} \cdot 2k_i$, $\frac{1}{2\pi}(k_i + k_j)$ and $\frac{1}{2\pi}(k_i - k_j)$, where j is now other than zero, are quite independent of ω_0.

Thus when the cylinder is being very rapidly rotated the first is in general much the most important.

These results seem not unlikely to throw light on the dangers attending the working of some kinds of machinery.

SECTION XIII.

Equilibrium of a finite cylinder under purely surface forces.

§ 91. In now proceeding to consider the equilibrium of a right circular cylinder finite in all dimensions, let us as a preliminary briefly regard the differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y\left(-k^2 - \frac{i^2}{x^2}\right) = 0.$$

This differs from the ordinary Bessel's equation only in the sign of k^2, and consequently the two solutions may be conveniently written under the forms

$$J_i(kx\sqrt{-1}) \quad \text{and} \quad Y_i(kx\sqrt{-1}).$$

The second solution is infinite when $x = 0$ and would be required only for a hollow cylinder. We shall here consider only solid cylinders and so may confine our attention to the former solution. This may be expanded in a series differing from $J_i(kx)$ only in the fact that all the terms are of the same sign. It contains only consecutive odd or even powers of $kx\sqrt{-1}$, and thus by the selection of a proper constant multiplier may be regarded as altogether real. In our subsequent work we may suppose this to be the case. Let us denote $\frac{1}{k\sqrt{-1}}\frac{d}{dr}J_i(kr\sqrt{-1})$ by $J_i'(kr\sqrt{-1})$, so that $\sqrt{-1}\ J_i'(kr\sqrt{-1})$ may be regarded as wholly real if $J_i(kr\sqrt{-1})$ is so. It will also be useful to notice that if $J_i(kr\sqrt{-1})$ be defined as the form taken by $J_i(kr)$ when $k\sqrt{-1}$ is written for k, so that it is now no longer necessarily real, we get relations among these consecutive functions exactly similar to the well-known relations for the Bessel's, viz. to

$$\left.\begin{aligned} J_i(kr) &= \frac{kr}{2i}\{J_{i-1}(kr) + J_{i+1}(kr)\} \\ J_i'(kr) &= \tfrac{1}{2}\{J_{i-1}(kr) - J_{i+1}(kr)\} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Digitized by Google

§ 92. Supposing no bodily forces to act, the internal equations of equilibrium are, see Sect. IX.,

$$(m+n)\, r\, \frac{d\delta}{dr} - n\, \frac{d\mathfrak{C}}{d\theta} + n\, \frac{d\mathfrak{B}}{dz} = 0 \quad \text{.....(2)},$$

$$(m+n)\, \frac{1}{r}\, \frac{d\delta}{d\theta} - n\, \frac{d\mathfrak{A}}{dz} + n\, \frac{d\mathfrak{C}}{dr} = 0 \quad \text{.....(3)},$$

$$(m+n)\, r\, \frac{d\delta}{dz} - n\, \frac{d\mathfrak{B}}{dr} + n\, \frac{d\mathfrak{A}}{d\theta} = 0 \quad \text{.....(4)};$$

where

$$\delta = \frac{1}{r}\, \frac{d\,.\,ur}{dr} + \frac{1}{r}\, \frac{dv}{d\theta} + \frac{dw}{dz} \quad \text{.....(5)},$$

$$\mathfrak{A} = \frac{1}{r}\left(\frac{dw}{d\theta} - \frac{d\,.\,vr}{dz}\right) \quad \text{.....(6)},$$

$$\mathfrak{B} = r\left(\frac{du}{dz} - \frac{dw}{dr}\right) \quad \text{.....(7)},$$

$$\mathfrak{C} = \frac{1}{r}\left(\frac{d\,.\,vr}{dr} - \frac{du}{d\theta}\right) \quad \text{.....(8)}.$$

Differentiate (2) with respect to r, (3) with respect to θ, and (4) with respect to z; then adding we get

$$\frac{d^2\delta}{dr^2} + \frac{1}{r}\, \frac{d\delta}{dr} + \frac{1}{r^2}\, \frac{d^2\delta}{d\theta^2} + \frac{d^2\delta}{dz^2} = 0 \quad \text{.....(9)}.$$

§ 93. Assuming $\delta = \Sigma \cos i\theta\, f_1(r) f_2(z)$,

where $f_1(r)$, $f_2(z)$ are respectively functions of r only and of z only, we easily find as a type of the solution

$$\delta = \cos i\theta\, [J_i(kr)\,(A \cosh kz + B \sinh kz) + J_i(kr\sqrt{-1})\,(C \cos kz + D \sin kz)] \text{.....(10)},$$

where A, B, C, D are arbitrary constants.

An exactly similar series in sines of multiples of θ of course exists.

From (4), (5), (6) and (7) we get

$$\frac{d^2w}{dr^2} + \frac{1}{r}\, \frac{dw}{dr} + \frac{1}{r^2}\, \frac{d^2w}{d\theta^2} + \frac{d^2w}{dz^2} = -\frac{m}{n}\, \frac{d\delta}{dz} = -\frac{km}{n} \cos i\theta\, [J_i(kr)\,(A \sinh kz + B \cosh kz)$$
$$+ J_i(kr\sqrt{-1})\,(D \cos kz - C \sin kz)] \text{.....(11)}.$$

By treating (11) in different ways we get two different particular solutions which seem adapted for application to different problems and will be considered separately.

In either case we must suppose w to depend on $\cos i\theta$ and so may replace $\frac{d^2w}{d\theta^2}$ by $-i^2w$. Thus let us assume

$$\frac{w}{\cos i\theta} = w'\,(A \sinh kz + B \cosh kz) + w''\,(D \cos kz - C \sin kz)\,;$$

Digitized by Google

then w', w'' are determined as functions of r by

$$\frac{d^2w'}{dr^2}+\frac{1}{r}\frac{dw'}{dr}+\left(k^2-\frac{i^2}{r^2}\right)w'=-\frac{mk}{n}J_i(kr),$$

$$\frac{d^2w''}{dr^2}+\frac{1}{r}\frac{dw''}{dr}-\left(k^2+\frac{i^2}{r^2}\right)w''=-\frac{mk}{n}J_i(kr\sqrt{-1}).$$

Using the Bessel's equation, it is easily seen that

$$w'=\frac{m}{2nk}\,r\frac{d}{dr}J_i(kr);$$

similarly

$$w''=-\frac{m}{2nk}\,r\frac{d}{dr}J_i(kr\sqrt{-1}).$$

Thus form (i) of the particular solution of (11) is

$$w=\frac{m}{2nk}\cos i\theta\left[r\frac{d}{dr}J_i(kr)(A\sinh kz+B\cosh kz)\right.$$
$$\left.+\,r\frac{d}{dr}J_i(kr\sqrt{-1})(C\sin kz-D\cos kz)\right]\ldots\ldots\ldots\ldots(12).$$

For that part of the general solution of (11) depending on $\cos i\theta$ we have obviously, referring to (9), terms similar to (10).

Thus as a type of the complete value of w we get

$$w=\cos i\theta\left[\frac{m}{2nk}\left\{r\frac{d}{dr}J_i(kr)(A\sinh kz+B\cosh kz)\right.\right.$$
$$\left.+\,r\frac{d}{dr}J_i(kr\sqrt{-1})(C\sin kz-D\cos kz)\right\}$$
$$\left.+\,J_i(kr)(A'\sinh kz+B'\cosh kz)+J_i(kr\sqrt{-1})(C'\sin kz+D'\cos kz)\right]\ldots\ldots(13),$$

where A', B', C', D' are constants.

Substituting from (6), (7) and (8) for $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ in (2) and (3), and then for $\frac{dw}{dz}$ from (5), and lastly introducing the typical value (10) for δ, we obtain two differential equations of the second order in u and v. In the first, v appears only as $\frac{dv}{d\theta}$, and in the second u appears only as $\frac{du}{d\theta}$. For a typical solution answering to (10) and (12) assuming $u\propto\cos i\theta$ and $v\propto\sin i\theta$, and eliminating v between the two equations, we get a differential equation of the fourth order to determine u as a function of r and z.

So far as z is concerned the solution will consist, as in the case of w, partly of exponential and partly of circular functions. In obtaining the corresponding functions of r we may replace $\frac{d^2}{dz^2}$ by $+k^2$ in the first case, and by $-k^2$ in the second. The consequent differential equations of the fourth order in r may be regarded as resulting from the successive application of two operators*, in the one case such as occur in Bessel's equations of orders $i-1$ and $i+1$, in the other case such as occur in the corresponding

* For $\pm k^2$, $\left[r^2\frac{d^2}{dr^2}+r\frac{d}{dr}\pm r^2k^2-(i+1)^2\right]\left[r^2\frac{d^2}{dr^2}+r\frac{d}{dr}\pm r^2k^2-(i-1)^2\right]$.

Digitized by Google

equations where k is imaginary. It may thus be deduced that the solution giving the typical term is

$$u = \cos i\theta \left[-\frac{m}{2n} \left\{ rJ_i(kr)(A \cosh kz + B \sinh kz) + rJ_i(kr\sqrt{-1})(C \cos kz + D \sin kz) \right\} \right.$$
$$+ \frac{1}{r} J_i(kr)(A_1 \cosh kz + B_1 \sinh kz) + \frac{d}{dr} J_i(kr)(A_1' \cosh kz + B_1' \sinh kz)$$
$$\left. + \frac{1}{r} J_i(kr\sqrt{-1})(C_1 \cos kz + D_1 \sin kz) + \frac{d}{dr} J_i(kr\sqrt{-1})(C_1' \cos kz + D_1' \sin kz) \right] \ldots (14),$$

where A_1, B_1, A_1', B_1', C_1, D_1, C_1', D_1' are constants.

A similar treatment leads to a corresponding differential equation of the fourth order in v, which is however much simpler than that in u as the right-hand side vanishes.

The typical solution is

$$v = \sin i\theta \left[\frac{1}{r} J_i(kr)(A_2' \cosh kz + B_2' \sinh kz) + \frac{d}{dr} J_i(kr)(A_2 \cosh kz + B_2 \sinh kz) \right.$$
$$\left. + \frac{1}{r} J_i(kr\sqrt{-1})(C_2' \cos kz + D_2' \sin kz) + \frac{d}{dr} J_i(kr\sqrt{-1})(C_2 \cos kz + D_2 \sin kz) \right] \ldots\ldots (15),$$

where A_2', B_2', A_2, B_2, C_2', D_2', C_2, D_2 are constants.

The constants we have introduced in (10), (13), (14) and (15) are not all arbitrary, as the existence of the identity (5) necessitates the following relations:

$$\left.\begin{aligned}
\frac{A_2'}{i} &= -A_1' = \frac{1}{k^2 n}\{A(m+n) - nkA'\} \\
\frac{B_2'}{i} &= -B_1' = \frac{1}{k^2 n}\{B(m+n) - nkB'\} \\
iA_2 &= -A_1 \\
iB_2 &= -B_1 \\
-\frac{C_2'}{i} &= C_1' = \frac{1}{k^2 n}\{C(m+n) - nkC'\} \\
-\frac{D_2'}{i} &= D_1' = \frac{1}{k^2 n}\{D(m+n) + nkD'\} \\
iC_2 &= -C_1 \\
iD_2 &= -D_1
\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16).$$

Thus the solutions in u and v are

$$u = \cos i\theta \left[J_i(kr) \left\{ -\frac{m}{2n} r(A \cosh kz + B \sinh kz) + \frac{1}{r}(A_1 \cosh kz + B_1 \sinh kz) \right\} \right.$$
$$- \frac{1}{k^2 n} \frac{d}{dr} J_i(kr) \left\{ (A\overline{m+n} - nkA') \cosh kz + (B\overline{m+n} - nkB') \sinh kz \right\}$$
$$+ J_i(kr\sqrt{-1}) \left\{ -\frac{m}{2n} r(C \cos kz + D \sin kz) + \frac{1}{r}(C_1 \cos kz + D_1 \sin kz) \right\}$$
$$\left. + \frac{1}{k^2 n} \frac{d}{dr} J_i(kr\sqrt{-1}) \left\{ (C\overline{m+n} - nkC') \cos kz + (D\overline{m+n} + nkD') \sin kz \right\} \right] \ldots (17),$$

Digitized by Google

$$v = \sin i\theta \left[\frac{iJ_i(kr)}{k^2 nr} \left\{ (A\overline{m+n} - nkA')\cosh kz + (B\overline{m+n} - nkB')\sinh kz \right\} \right.$$

$$- \frac{1}{i}\frac{d}{dr} J_i(kr)(A_1 \cosh kz + B_1 \sinh kz)$$

$$- \frac{iJ_i(kr\sqrt{-1})}{k^2 nr} \left\{ (C\overline{m+n} - nkC')\cos kz + (D\overline{m+n} + nkD')\sin kz \right\}$$

$$\left. - \frac{1}{i}\frac{d}{dr} J_i(kr\sqrt{-1})(C_1 \cos kz + D_1 \sin kz) \right] \text{.....(18).}$$

A type of the complete solution of the internal equations is thus supplied by (17), (18) and (13).

§ 94. For the special case $i = 0$ the equations are much simplified. The expression for v is quite independent of those for u and w and is to be considered separately.

The typical solution may be easily seen to be

$$u = -J_0(kr)\frac{m}{2n} r (A \cosh kz + B \sinh kz)$$

$$- \frac{1}{k^2 n}\frac{d}{dr} J_0(kr) \left[\{A(m+n) - nkA'\}\cosh kz + \{B(m+n) - nkB'\}\sinh kz \right]$$

$$- J_0(kr\sqrt{-1})\frac{m}{2n} r (C \cos kz + D \sin kz)$$

$$+ \frac{1}{k^2 n}\frac{d}{dr}(J_0 kr\sqrt{-1}) \left[\{C(m+n) - nkC'\}\cos kz + \{D(m+n) + nkD'\}\sin kz \right] \text{.....(19),}$$

$$w = \frac{m}{2nk} r \frac{d}{dr} J_0(kr)(A \sinh kz + B \cosh kz) + J_0(kr)(B' \cosh kz + A' \sinh kz)$$

$$+ \frac{m}{2nk} r \frac{d}{dr} J_0(kr\sqrt{-1})(C \sin kz - D \cos kz) + J_0(kr\sqrt{-1})(D' \cos kz + C' \sin kz) \text{ ....(20),}$$

$$\delta = J_0(kr)(A \cosh kz + B \sinh kz) + J_0(kr\sqrt{-1})(C \cos kz + D \sin kz) \text{.....(21).}$$

For v there is the independent typical solution, deducible from (3) directly by supposing u and w to vanish and v to be independent of θ,

$$v = J_1(kr)(E' \cosh kz + E \sinh kz) + J_1(kr\sqrt{-1})(N' \cos kz + N \sin kz) \text{ .....(22),}$$

where E, E', N, N' are arbitrary constants.

If there be symmetry about the central section $z = 0$ we must take in these equations B, D, B', D', E, N all zero; for w must change sign with z, and u and v must not. In this case the expressions are thus much simplified.

The terms in the solution involving exponential and circular functions occur separately, and seem suitable for application to distinct problems.

Digitized by Google

§ 95. The surface-conditions are given in equations (24)—(29), Sect. IX.

The solution obtained above is not in general capable of satisfying these when the surface-forces are given arbitrarily.

For instance, supposing $z=0$ a plane of symmetry which includes the case when it is held so that w only vanishes with z, it is possible from the solution to satisfy the condition that the curved surface of the cylinder shall be exposed to any system of purely normal forces.

But the arbitraries left limit us to a choice of two alternatives, either an accompanying perfectly determinate system of tangential forces on the curved surface, or a determinate system of normal tractions on the flat ends.

§ 96. The most manageable part of the solution seems to be that given in equation (22). Confining ourselves to the case when v vanishes over the plane $z=0$, and taking first the circular function, we have as a type

$$v = N \sin kz\, J_1(kr\sqrt{-1}) \quad \text{.....(22 a).}$$

The surface-stresses are, on the curved surface $r=a$,

$$G = R_\theta = nr\frac{d.\frac{v}{r}}{dr} = -nkN\sqrt{-1}\, J_2(ka\sqrt{-1}) \sin kz \quad \text{.....(23);}$$

on the flat end $z=l$,

$$G = Z_\theta = n\frac{dv}{dz} = nkNJ_1(kr\sqrt{-1}) \cos kl \quad \text{.....(24);}$$

on the flat end $z=0$, $$G = Z_\theta = nkNJ_1(kr\sqrt{-1}) \quad \text{.....(24 a).}$$

By taking $k=(2i+1)\frac{\pi}{2l}$, where i is any integer, we get rid of all surface-forces except those on the end $z=0$ and R_θ on the curved surface. Thus from (23) we can determine N so that (22 a) may represent the solution for the case of a cylinder, one end of which is held, over whose curved surface there is applied any distribution of tangential forces which act perpendicularly to the axis and are symmetrical round it.

N is real or imaginary according as $J_1(kr\sqrt{-1})$ is real or imaginary, i.e. according as $J_2(kr\sqrt{-1})$ is imaginary or real.

Since all the terms of $J_2(ka\sqrt{-1})$ have the same sign it cannot vanish for a finite real value of k, and so we cannot from (23) and (24) make R_θ vanish over the curved surface, and Z_θ arbitrary over the flat ends. This we can accomplish however by using the exponential part of (22), viz. the type

$$v = E \sinh kz J_1(kr) \quad \text{.....(22 b).}$$

The surface-forces are, on the curved surface

$$G = R_\theta = -nkE \sinh kz\, J_2(ka) \quad \text{.....(25);}$$

on the flat end $z=l$, $$G = Z_\theta = nkE \cosh kl\, J_1(kr) \quad \text{.....(26);}$$

on the flat end $z=0$, $$G = Z_\theta = nkEJ_1(kr) \quad \text{.....(26 b).}$$

Digitized by Google

By taking for k the roots of $J_2(ka)=0$ we can thus get rid of the tangential force on the curved surface.

It is also easily proved that if k and k' be any two of those roots, then

$$\int_0^a rJ_1(kr)\,J_1(k'r)\,dr=0.$$

We can thus expand functions of r in a series of terms $J_1(kr)$ by the usual method and thus solve the case where, over a flat end of a cylinder, whose other end is held, there are applied arbitrary tangential forces, symmetrical round the axis, perpendicular to the radius vector at every point. This solution would thus have an extensive application to the more complicated forms of torsion in circular cylinders.

As an example suppose a torsional force Tr^3 over the end $z=l$.

Suppose Tr^3 expanded in a series $\Sigma nkE\cosh kl\,J_1(kr)$, where k is a root of

$$J_2(ka)=0.$$

Then to determine E, we have

$$Enk\cosh kl\int_0^a r\,\{J_1(kr)\}^2\,dr=T\int_0^a r^4J_1(kr)\,dr \quad\text{.....(27).}$$

In the right-hand side of (27) using the Bessel's equation and the relation

$$J_0'(kr)=-J_1(kr),$$

and integrating by parts, we find

$$\int_0^a r^4J_1(kr)\,dr=\frac{a^3}{k^3}\{3J_1(ka)-kaJ_1'(ka)\}+\frac{8a}{k^4}\{kaJ_1'(ka)-J_1(ka)\}\quad\text{.....(28).}$$

But since $J_2(ka)=0$, we have

$$\frac{J_1(ka)}{ka}=\tfrac{1}{2}J_0(ka)=J_1'(ka),$$

and the right-hand side of (28) reduces to $\dfrac{a^4}{k}J_0(ka)$.

Thus using the result*

$$2\int_0^a r\,\{J_1(kr)\}^2\,dr=a^2\left[\{J_1'(ka)\}^2+\left(1-\frac{1}{k^2a^2}\right)\{J_1(ka)\}^2\right],$$

and the above values of $J_1(ka)$ and $J_1'(ka)$, we finally obtain

$$E=\frac{8T}{nk^4J_0(ka)\cosh kl};$$

whence the solution is

$$v=\frac{8T}{n}\Sigma\frac{J_1(kr)\sinh kz}{k^4J_0(ka)\cosh kl}\quad\text{.....(29),}$$

where the summation includes all values of k satisfying

$$J_2(ka)=0\quad\text{.....(30).}$$

* Lord Rayleigh's *Sound*, Vol. I. Equation (16), p. 270.

Digitized by Google

The stresses at any point (r, z) in the cylinder are

$$R_\theta = -8T\,\Sigma\,\frac{J_2(kr)\sinh kz}{k^2 J_0(ka)\cosh kl} \quad\text{.....(31)},$$

$$Z_\theta = 8T\,\Sigma\,\frac{J_1(kr)\cosh kz}{k^2 J_0(ka)\cosh kl} \quad\text{.....(32)}.$$

§ 97. We now proceed to the second form of solution.

Returning to (11), the general solution is the same as before, while the particular solution consists of two parts, the one depending on $J_i(kr)$, the other on $J_i(kr\sqrt{-1})$. For the first part assume $w \propto \cos i\theta J_i(kr)$; then to determine w as a function of z, we have

$$\frac{d^2w}{dz^2} - k^2 w = -k\frac{m}{n}(A\sinh kz + B\cosh kz)\cos i\theta J_i(kr);$$

of which a solution is

$$w = -\frac{mz}{2n}(A\cosh kz + B\sinh kz)\cos i\theta J_i(kr).$$

For the second part we get

$$w = -\frac{mz}{2n}(D\sin kz + C\cos kz)\cos i\theta J_i(kr\sqrt{-1});$$

therefore the type of the complete solution is

$$w = \cos i\theta\left[J_i(kr)\left\{B'\cosh kz + A'\sinh kz - \frac{mz}{2n}(A\cosh kz + B\sinh kz)\right\}\right.$$
$$\left. + J_i(kr\sqrt{-1})\left\{D'\cos kz + C'\sin kz - \frac{mz}{2n}(C\cos kz + D\sin kz)\right\}\right] \quad\text{.....(33)}.$$

In obtaining u and v the method to be adopted closely resembles that sketched out in treating the first form of solution. It leads to the typical solution

$$u = \cos i\theta\left[-\frac{d}{dr}J_i(kr)\left\{\frac{A(m+2n) - 2nkA'}{2k^2n}\cosh kz + \frac{B(m+2n) - 2nkB'}{2k^2n}\sinh kz\right.\right.$$
$$\left. + \frac{mz}{2nk}(A\sinh kz + B\cosh kz)\right\} + \frac{1}{r}J_i(kr)(A_1\cosh kz + B_1\sinh kz)$$
$$+ \frac{d}{dr}J_i(kr\sqrt{-1})\left\{\frac{C(m+2n) - 2nkC'}{2k^2n}\cos kz + \frac{D(m+2n) + 2nkD'}{2k^2n}\sin kz\right.$$
$$\left.\left. - \frac{mz}{2nk}(C\sin kz - D\cos kz)\right\} + \frac{1}{r}J_i(kr\sqrt{-1})(C_1\cos kz + D_1\sin kz)\right] \quad\text{.....(34)},$$

$$v = \sin i\theta\left[\frac{i}{r}J_i(kr)\left\{\frac{A(m+2n) - 2nkA'}{2k^2n}\cosh kz + \frac{B(m+2n) - 2nkB'}{2k^2n}\sinh kz\right.\right.$$
$$\left. + \frac{mz}{2nk}(A\sinh kz + B\cosh kz)\right\} - \frac{1}{i}\frac{d}{dr}J_i(kr)(A_1\cosh kz + B_1\sinh kz)$$
$$-\frac{i}{r}J_i(kr\sqrt{-1})\left\{\frac{C(m+2n) - 2nkC'}{2k^2n}\cos kz + \frac{D(m+2n) + 2nkD'}{2k^2n}\sin kz - \frac{mz}{2nk}(C\sin kz - D\cos kz)\right\}$$
$$\left. - \frac{1}{i}\frac{d}{dr}J_i(kr\sqrt{-1})(C_1\cos kz + D_1\sin kz)\right] \quad\text{.....(35)}.$$

Digitized by Google

For the case of symmetry about the plane $z=0$ we have

$$B = D = B' = D' = B_1 = D_1 = 0,$$

and the equations are much shortened.

It will be noticed how simple is the relation between u and v in this solution.

§ 98. For the special case $i=0$, we have the type

$$u = -\frac{d}{dr} J_0(kr) \left\{ \frac{A(m+2n)-2nkA'}{2k^2n} \cosh kz + \frac{B(m+2n)-2nkB'}{2k^2n} \sinh kz + \frac{mz}{2nk}(A \sinh kz + B \cosh kz) \right\}$$

$$+ \frac{d}{dr} J_0(kr\sqrt{-1}) \left\{ \frac{C(m+2n)-2nkC'}{2k^2n} \cos kz + \frac{D(m+2n)+2nkD'}{2k^2n} \sin kz - \frac{mz}{2nk}(C \sin kz - D \cos kz) \right\}$$

$$\text{.....(36).}$$

$$w = J_0(kr) \left\{ B' \cosh kz + A' \sinh kz - \frac{mz}{2n}(A \cosh kz + B \sinh kz) \right\}$$

$$+ J_0(kr\sqrt{-1}) \left\{ D' \cos kz + C' \sin kz - \frac{mz}{2n}(C \cos kz + D \sin kz) \right\} \text{.....(37),}$$

$$\delta = J_0(kr)(A \cosh kz + B \sinh kz) + J_0(kr\sqrt{-1})(C \cos kz + D \sin kz) \text{.....(38).}$$

The independent solution in v is the same as by the former method.

In the case of symmetry about the plane $z=0$, we have B, D, B', D', B_1, D_1 all zero as before. Of course there is absolutely no connection between the constants in the two different forms of the solution, the same letters being used merely to shew that the two are not distinct solutions.

The difficulties in dealing with the surface-conditions are exactly analogous to those occurring in the first form of solution.

§ 99. A special case of (36) and (37) which seems of interest occurs when the cylinder extends to infinity and is subjected to normal forces on the plane end $z=0$. In the equations referred to suppose $B=-A$ and $B'=-A'$, then the solution in exponentials is typefied by

$$u = -\frac{d}{dr} J_0(kr)\, e^{-kz} \left\{ \frac{A(m+2n)-2nkA'}{2k^2n} - \frac{Amz}{2nk} \right\} \text{.....(39),}$$

$$w = -J_0(kr)\, e^{-kz} \left(A' + \frac{Amz}{2n} \right) \text{.....(40),}$$

$$\delta = Ae^{-kz} J_0(kr) \text{.....(41).}$$

The stresses are

$$(m-n)\,\delta + 2n\frac{dw}{dz} = e^{-kz} J_0(kr) \{2nkA' + A(mkz - n)\} \text{.....(42),}$$

$$n\left(\frac{du}{dz} + \frac{dw}{dr}\right) = e^{-kz} \frac{d}{dr} J_0(kr) \left\{ A\left(\frac{m+n}{k} - mz\right) - 2nA' \right\} \text{.....(43),}$$

Digitized by Google

$$(m-n)\,\delta + 2n\frac{du}{dr} = e^{-kz}\left[J_0(kr)\,\{A\,(2m+n-mkz) - 2nkA'\}\right.$$

$$\left. + \frac{1}{r}\frac{d}{dr}J_0(kr)\left\{\frac{A\,(m+2n)-2nkA'}{k^2} - \frac{Amz}{k}\right\}\right] \quad \text{.....(44).}$$

By taking k such that $J_0'(ka) = 0$, we get rid of the tangential traction over the curved surface $r = a$; and by putting $A' = A\,\dfrac{m+n}{2nk}$ we get rid of the tangential traction on the end $z = 0$. The normal traction on the flat end $z = 0$ is then simply $AmJ_0(kr)$; and, as $J_0'(ka) = 0$, the normal *pressure* on the curved surface reduces to

$$-AmJ_0(ka)\,e^{-kz}\,(1-kz) \quad \text{.....(45).}$$

If k be large this last expression diminishes rapidly as z increases.

Since $J_0'(ka) = 0$ we can expand any function of r in a series of Bessels of which $J_0(kr)$ is a type, and thus this solution would apply to the most general symmetrical distribution of normal tractions or pressures over the flat end of the cylinder. Writing in the value of A', the solution is

$$u = \frac{A}{2k}\,e^{-kz}\left(1 - \frac{mkz}{n}\right)J_1(kr) \quad \text{.....(46),}$$

$$w = -\frac{A}{2k}\,e^{-kz}\left(1 + \frac{m}{n} + \frac{mkz}{n}\right)J_0(kr) \quad \text{.....(47).}$$

It is obvious that the strains and stresses become very small, and may be neglected at considerable distances from the bounding plane $z = 0$, especially for the larger values of k. Since the equation determining k gives certain constant values for ka, the solution would be particularly suitable to the case of a cylinder of small radius.

ERRATA.

p. 260, equation (12), insert $\frac{1}{r^2}$ in the two last terms.

p. 297, equation (30), read $\frac{1}{2}$ for 2.

p. 298, equation (33), read $\frac{1}{2}$ for 2 in the last term.

Digitized by Google

XIII. *On Solution and Crystallization.* By G. D. Liveing, M.A., Professor of Chemistry in the University of Cambridge.

[*Read* May 21, 1888.]

The nature of solution has often occupied the attention of chemists, because the homogeneous character of a solution and the definite proportions of saturated solutions at given temperatures point plainly to an analogy between solution and chemical combination. Moreover many substances certainly form chemical compounds with water, alcohol, &c., and there is no reason why there should not be, at ordinary temperatures, liquid as well as solid compounds of this kind. The heat evolved in the dilution of many solutions has also been supposed to indicate a concurrent chemical action; and the fact that such an evolution is most marked on adding water to solutions of those salts which form solid crystallized hydrates lends support to that supposition. The gradual variation in the amount of the substance which will saturate a given menstruum when the temperature is gradually varied is the great stumblingblock in the way of the theory of chemical combination in solution, though this may be pretty well matched by the gradual dissociation of many chemical compounds by a rise of temperature. Nevertheless though chemical action doubtless produces its own effects in some solutions, and it is probable that some substances will not dissolve without combining with the solvent, the regular course of solution seems capable of explanation as the result of the molecular energies which we see displayed in surface tension. Between two liquids which mix with each other in all proportions there is no surface tension, and it is well known that the surface tension of liquids varies when substances are dissolved in the liquids. If we consider the case of water and ether; as the water dissolves some of the ether and the ether dissolves some of the water the surface tension at the junction increases, and when it has acquired a certain value solution ceases. That value must be such that the entropy is a maximum, and this is probably the case, supposing the temperature constant, when the increase of entropy by interdiffusion of the liquids is balanced by the diminution of entropy in the increased surface energy. That an increase of entropy attends solution is certain because work has to be done on a solution in order to separate the substance dissolved from the solvent. Either the latter has to be evaporated away, or cooled below the temperature of the surroundings (supposing it in thermal equilibrium with its surroundings to begin with), or something has to be done which implies work. The more completely two liquids are miscible the smaller will be the dissipation of energy which could be effected if we supposed that their interdiffusion were carried beyond the point

Digitized by Google

of saturation, and the smaller therefore the surface tension which will balance the tendency to such further interdiffusion. Thus when hydrochloric acid, or hydriodic acid, is dissolved in water, the stronger the solution the less is the amount of heat evolved when more hydrochloric or hydriodic acid is dissolved in it; that is to say the dissipation due to solution diminishes as the strength of the solution increases. This accords with Quincke's observation that the more freely two liquids are miscible the smaller is the surface tension between them.

If the substance dissolved be solid instead of liquid the change of entropy due to its liquefaction will have to be taken into account, but the point of saturation at a given temperature will still be determined by the consideration that the increase of entropy due to further solution is balanced by the diminution arising from increased surface tension. A change of temperature will affect both the surface tension and the amount of change of entropy due to liquefaction, and will consequently affect the point of saturation. The greater the surface tension at the junction of the substance with the solvent the greater must be the availability of the energy necessary to increase that surface tension and the more difficult will it be to increase the amount of the substance dissolved. If the substance is more soluble at higher temperatures, and if, after the solution has been saturated at a given temperature, the temperature be subsequently lowered, there will be a tendency to a reverse action, to a deposition of part of the substance in the solid state. But this can only occur *when this deposition will be attended with an immediate increase of entropy.* If there is in the solution a piece of the undissolved solid deposition can occur upon this without the development of any new surface tension, since the new surface will be of the same kind as the old and only a very little larger, and there will be no surface tension between the new deposit and the mass on which it is deposited; and this will therefore afford the most favourable circumstance for the depletion of the solution. The entropy will continuously increase until the point of saturation corresponding to the temperature is reached. If there be no undissolved portion of the substance in the liquid deposition may occur at any place where the solution comes in contact with anything else, provided the replacement of the surface tension at that place by the two new surface tensions produced by the deposition is attended with an increase of entropy. When once a deposit has been formed there must be a tendency for this deposit to grow rather than for new deposits to be formed, for the reason above given. The formation of a new surface in the interior of a uniform liquid means a storage of energy which may exceed the availability of the energies in the neighbourhood, that is to say the immediate result of the formation of such a new surface may be a diminution of entropy. Nevertheless the diminution of entropy by the formation of the new surface may be so small in some cases that a trifling mechanical disturbance in the liquid may suffice to render the necessary amount of energy available to produce the deposition. All this is in agreement with what is observed in super-saturated solutions. These remarks about the formation of a new surface in the interior of a solution will apply also to the deposition at any part of the solution where it is in contact with another substance, if it happened that there would be at first a loss of entropy by a deposition at that place. The immediate loss may be very small and the mechanical action of rubbing the

Digitized by Google

surface, as by rubbing a glass rod against the side of the vessel containing the solution, may suffice to supply energy enough in a form available to carry the change past the point of minimum entropy.

Whenever a substance separates out of a solution the new arrangement of the mass must, if there be equilibrium, be such that the entropy is a maximum, and so far as the external surface is concerned it must tend to take such a shape as shall make the surface energy a minimum. If it separate in the fluid state it will generally appear in spherical drops, which have the minimum surface for their mass. If it separate in the solid state and be a crystallizable substance the arrangement of the particles throughout the mass must be crystalline, because the tendency of crystalloids to take the crystalline structure is a proof that such an arrangement of the particles corresponds to a maximum entropy. A crystalline arrangement of the particles is however consistent with any external form, yet it is a rare thing that we find a crystalloid separating from a solution in globular masses. We must therefore conclude that when there is a crystalline arrangement of the particles the surface tension is a minimum in planes bearing certain relations to the arrangement of the particles. In liquids we suppose the molecules to have a certain freedom of motion amongst themselves whereby diffusion occurs, but in solids the freedom is restricted so that we suppose each molecule, though capable of considerable motion, to maintain on the average the same relative position to the other molecules. This makes a definite arrangement of the molecules possible, and the optical and other properties of crystals render such an arrangement almost a certainty.

If then a crystalloid separates from solution in the solid form its molecules arrange themselves in a definite way depending on their mutual actions, and the external surface of the solid assumes *such a shape that its surface energy is a minimum.*

Also, if the surface tension tend to contract the surface, the *surface energy per unit of area will, ceteris paribus, be a minimum when the approximation of the molecules of the surface to one another is a maximum.*

The application of these principles gives us, as I hope to shew, a solution to a first, and near, approximation of the problems of the external shape and of the cleavages of crystals.

Cubic System.

If the substance be one which crystallizes in the regular system we may suppose the law of arrangement of the molecules to be as follows. Let space be supposed divided into cubes of a uniform size by three sets of parallel planes, each set at right angles to the other two, and suppose each corner of a cube, or each point of intersection of three planes, to be the mean place of a molecule of the crystal. This arrangement satisfies the condition imposed by the isotropic character of the crystal.

Now of all surfaces which can be drawn through the system of molecules arranged in the way described the planes which form the cubes above mentioned contain the greatest number of molecules per unit of area.

For let *oa* in fig. 1 be one of the cubes and consider the set of planes parallel to a face of it, *bac*. The distances between contiguous molecules in that plane is the least possible with the arrangement of molecules supposed. Moreover if we take in OX, points x_1, x_2, x_3, &c. successively at distances all equal to Ox_1, and in OY points y_1, y_2, y_3,

Digitized by Google

&c., and in OZ points z_1, z_2, z_3, &c., such that their distances from each other are all equal to Oy_1 or Ox_1, every plane which intersects the axis OX in one of the points x_1, x_2, &c.,

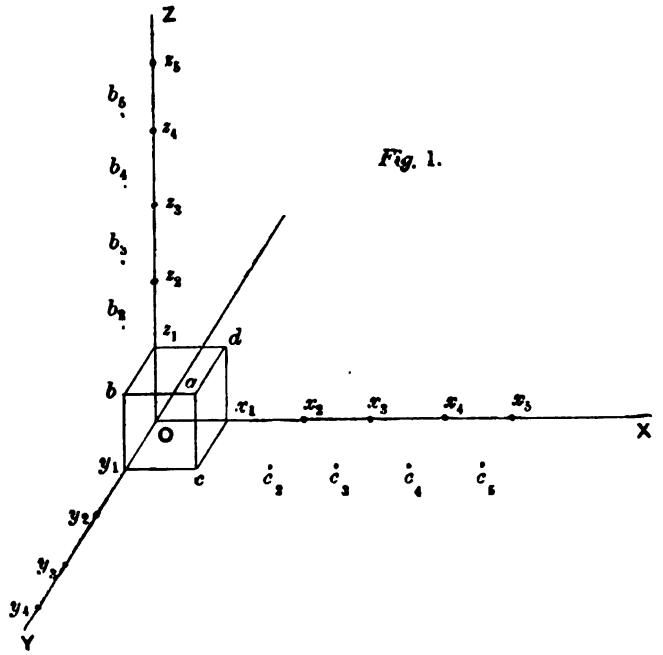


Fig. 1.

the axis OY in one of the points y_1, y_2, &c., and the axis OZ in one of the points z_1, z_2, &c., will be a surface of maximum concentration of the molecules.

For returning to the plane $bacy_1$ it meets a succession of molecules c, c_2, c_3, &c. in the plane YOX, and a similar succession in every one of the planes parallel to YOX through z_1, z_2, &c. But if we turn the plane $bacy_1$ through a very small angle about by_1 it will pass through none of these molecules except those which lie in the plane ZOY, namely y_1, b, b_2, b_3, &c., and it will pass through no other molecules nearer than the line in which it intersects the plane ZOX. If we suppose it to meet OX at a point x_p, the place of one of the molecules, it will then be a plane of maximum concentration of molecules, for if it be turned either way about y_1b through a small angle the molecule nearest to y_1b which it will meet will be farther away than x_p. Also the nearer x_p is to O the less the distance y_1x_p and the greater the concentration in the plane by_1x_p. The plane by_1x_p will intersect every one of the planes parallel to ZOX in a row of molecules, and the successive distances of these rows measured in the plane YOX will be all equal to y_1x_p.

Moreover the whole set of planes drawn through O, x_1, $x_2 \ldots x_p \ldots$, all parallel to by_1x_p, will together contain all the molecules; and the arrangement of the molecules in each plane of the set will be the same. And if we compare two sets of such planes, parallel to by_1x_p and by_1x_q respectively, which pass through unit volume of the crystal, each set

Digitized by Google

will pass through all the molecules, but the number of planes out of each set which intersect the unit volume will be inversely as the distance between the successive planes of the set. Hence the concentration of molecules in each plane of a set will, when p varies, be inversely as the perpendicular distance between successive planes of that set.

Similar reasoning will shew that if we make the plane by_1x_p revolve about y_1x_p it will come to successive positions of maximum concentration as it intersects OZ in the successive points which are the positions of molecules. Each plane $z_ny_1x_p$ will be a plane of maximum concentration, the whole set of planes parallel to it will contain all the molecules of the mass, and the concentration in each plane, when n is varied, will vary inversely as the perpendicular distance between the successive planes of the set. The same reasoning may now be extended by supposing the plane $z_ny_1x_p$ to be turned about z_nx_p, every plane $z_ny_mx_p$ will be a plane of maximum concentration, and the concentration when n, m and p are varied will vary inversely as the perpendicular distance between the successive planes of the set, which is the same as the length of the perpendicular from O upon the nearest plane of the set, because every set has one plane through O.

Every plane $x_py_mz_n$ is by the crystallographic law a possible face of the crystal and if M be the least common multiple of p, m and n, omitting any one of them which is infinite, $\frac{M}{p}, \frac{M}{m}, \frac{M}{n}$ will be the indices of the face in Miller's notation.

This shews that for such an arrangement of molecules as is here supposed, plane surfaces following the law of indices will be surfaces of minimum energy. The substance when separating from solution must therefore tend to take a form bounded by such plane surfaces.

From the symmetry of the arrangement of molecules it is evident that any interchange of the three symbols will not alter the concentration in the set of planes represented. That is, every face of the same "form" of crystallographers will have the same concentration and therefore the same surface energy when in the same surroundings.

If the distance Ox_1 be taken as the unit of distance the perpendicular from O upon the plane $x_py_mz_n$ will be the reciprocal of $\sqrt{\frac{1}{p^2}+\frac{1}{m^2}+\frac{1}{n^2}}$. Also since the set of planes parallel to $x_py_mz_n$ passes through every molecule, one plane of the set must pass through each of the molecules $x_1, x_2 \ldots x_p$ and one through each of the molecules $y_1, y_2 \ldots y_m$, and one through each of the molecules $z_1, z_2 \ldots z_n$, and if none of the numbers p, m, n have a common measure the whole number of the set of planes between O and the plane $x_py_mz_n$ will be pmn, and in general it will be M, where M is the least common multiple of p, m and n, so that the perpendicular distance between them will be the reciprocal of $M\sqrt{\frac{1}{p^2}+\frac{1}{m^2}+\frac{1}{n^2}}$, or the reciprocal of $\sqrt{\left(\frac{M}{p}\right)^2+\left(\frac{M}{m}\right)^2+\left(\frac{M}{n}\right)^2}$, or if h, k, l be the symbols of the corresponding face, the reciprocal of $\sqrt{h^2+k^2+l^2}$. We may designate this distance as P.

Digitized by Google

For the cube, 100, $P=1$,

for the dodecahedron, 110, $P=\frac{1}{\sqrt{2}}$,

for the octahedron, 111, $P=\frac{1}{\sqrt{3}}$,

for the tetrakishexahedron, 210, $P=\frac{1}{\sqrt{5}}$,

for the eikositessarahedron, 211, $P=\frac{1}{\sqrt{6}}$,

for the triakisoctahedron, 221, $P=\frac{1}{3}$,

and so on, P diminishing as the indices grow larger, and the concentration also diminishing and the surface energy consequently increasing.

It should be observed that although the surface energy must increase as the concentration diminishes it does not follow that these quantities are reciprocally proportional: that will depend on the law of force between the molecules. Hence the values of P above given indicate the *order* only and not the exact amounts of surface energy of the different planes. It will be observed too that the molecules are assumed to be symmetrical, or to act on each other merely as centres of force, so that their mutual influence will be the same in all directions, otherwise we should have to take into account the orientation of the planes of each set as well as their relative distances. It is probable that to a first approximation we may treat the molecules as so symmetrical, though for the complete solution of the problem we must regard them as unsymmetrical.

Hence supposing that the substance is one like common salt which crystallizes in the cubic system, there must be a tendency, when it separates from solution in the solid state, to assume that form which has a minimum surface energy, and as the cube has the minimum surface energy *per unit of surface* there is a tendency on the part of salt, as well as of all other substances that crystallize in the same system, to take the form of a cube. Salt does form cubic crystals, but other substances such as alum which belong to the same system crystallize more frequently in octahedra. How is this?

The law of energy requires the surface energy of the whole solid mass to be a minimum, and it does not follow necessarily that for a given mass of the solid the integral of the surface energy over a whole cube will be less than the integral of the surface energy over a whole octahedron when the two figures have equal volumes. In fact for a given volume the surface of the cube is greater than that of the octahedron in the ratio of $\sqrt[3]{2} : \sqrt[6]{3}$ or 1·049 to 1. Bearing in mind what has been indicated above, that although the surface energy must, *ceteris paribus*, decrease as the concentration of the surface increases we do not know as yet the law of force between the molecules and therefore cannot tell the exact ratio in which the decrease occurs for any given increase of concentration, we can see that it may very well happen that with salt the total surface

Digitized by Google

energy of a cube may be less than that of an octahedron of equal mass, while with alum the reverse may be true.

Again, by a combination of forms we may easily obtain for a given mass a total surface much less than that of the cube. Thus by replacing the edges of the cube by facets of a dodecahedron, and the corners by facets of an octahedron, while retaining the same total volume, the surface will be sensibly diminished. It will be noted that any form of which the surface has minimum energy will be a surface of equilibrium so that any of the forms developed by the crystallographic law are possible. When a crystal has once begun to form there will be a tendency in general to continue to develop the same form, because the development of a face of any other form will imply the development of a surface with a different surface energy. Still for the reasons just given we may have edges and angles replaced by surfaces of greater energy per unit of area than the surfaces of the primary form. When one edge or angle is replaced however other similar edges and angles will be similarly modified, for the following reason. The tensions of a plane surface will have no resultant in any direction out of that plane, but at an edge, or angle, the surface tensions will have a resultant directed to the interior of the mass and, since the arrangement of the molecules is one of equilibrium, that resultant must be equilibrated, either by a corresponding resultant on the opposite side of the crystal, or by an equivalent resultant produced by internal displacement of the molecules. In general such an internal displacement implies that the solid is not formed with the maximum entropy, and hence when one edge or angle of a form is developed all the others will as a rule be developed, and all similar edges and angles will be similarly modified. Occasionally the equilibrium may be attained when only half the faces of a form are developed. Thus in the tetrahedron the resultants due to opposite edges balance one another. Still when hemihedral forms occur in the cubic system we often have evidence from the optical or other properties of the crystal that there is an internal strain or deformation of the arrangement of molecules. Thus cubes of sodium chlorate which have half their angles replaced by faces of a tetrahedron rotate the plane of polarized light, cubes of boracite with similar replacements exhibit pyroelectricity, and so on.

Whenever a solid is broken two new surfaces are developed each with its own surface energy. Those surfaces which are surfaces of minimum energy will be produced with the greater ease. Hence cleavage surfaces of crystals must follow the law of the external forms and be planes of maximum concentration of molecules. The only cleavages of crystals of the regular system are parallel either to the faces of the cube, dodecahedron, or octahedron, which correspond to the three greatest degrees of concentration. It is plain however, that mere division is not quite the same problem as the development of the complete bounding surface of a mass. In the former case the all important consideration is the surface energy per unit of area of the two new surfaces produced, while in the latter it is the integral of surface energy over the whole solid.

Now for many crystals, such as rock-salt, galena, cobaltine, pyrite, we find the cubical cleavage readily obtained, while sometimes a dodecahedral and more often an octahedral cleavage is also obtainable, though less easily. For these crystals the arrangement of the

Digitized by Google

molecules above supposed seems adequate to account for the facts observed. But there are some crystals of the cubic system such as fluor, fahlerz, cuprite and alum which do not cleave parallel to the faces of a cube but give, more or less easily, an octahedral cleavage. For them the planes of the octahedron must for some reason be planes of less surface energy than those of the cube. It may be either that the mutual action of the molecules is not the same in all directions but less in the plane of the octahedron than in that of the cube, or that the arrangement of the molecules is different from that above supposed.

There is another arrangement of the molecules in the crystal which will equally well satisfy the optical requirements of the problem: this is to suppose the molecules to occupy the centres of the several faces of the cubes instead of the angular points of the cubes. The same sets of planes as before will be planes of maximum condensation, but the relative degrees of condensation in the several planes will in some cases be different. In fact the arrangement of molecules will be the same as if three sets of molecules, each set arranged at the corners of equal cubes, were combined. Thus if O, O', O'' (Fig. 2),

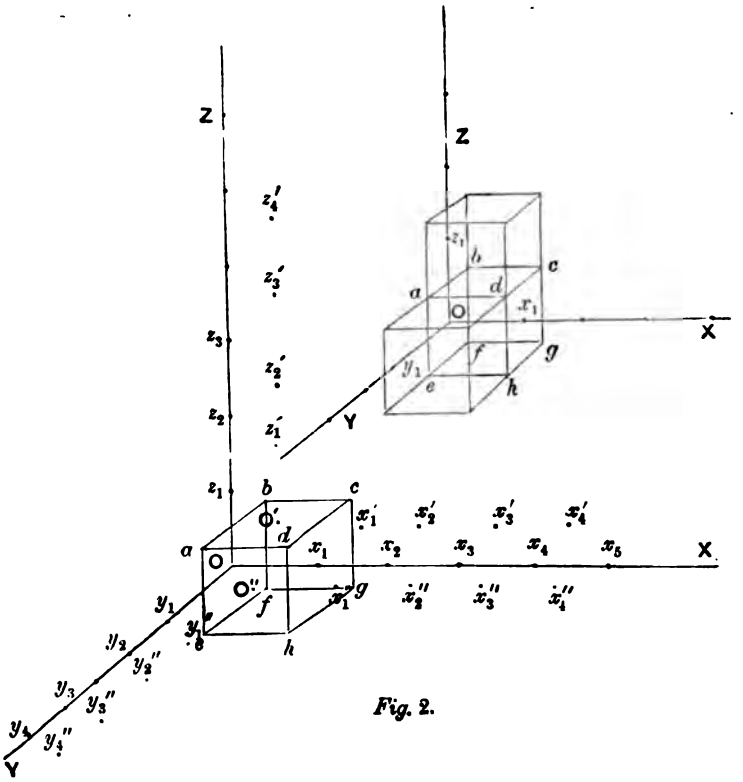


Fig. 2.

be the centres of three adjacent faces of one of the cubes *ag*, namely O the centre of the face *abfe*, O' that of *abcd*, and O'' that of *adhe*, we shall have one system of molecules which lie in the centres of all those faces of the cubes which are in the plane of *abfe* or are parallel to it; a second system of molecules, similarly arranged and at the same distances apart, which lie in the centres of all the faces in the plane *abcd* or

Digitized by Google

parallel to it; and a third system, similar in all respects to the former two, lying in the centres of all the faces in the plane *adhe* or parallel to it. Hence for any set of planes belonging to the first system and parallel to a plane with the symbol *hkl* there will in general be a similar set of planes parallel to them belonging to each of the other two systems. If there were three such sets for all values of *hkl* the relative condensations in planes having different directions would be the same as in the simpler case first taken. But in some cases two of the three sets of planes coincide, and in others all three sets coincide, and thus in these cases the condensation in the planes which coincide is much increased.

Two sets of the planes will coincide when

1. Two of the indices *hkl* are equal to one another. For in that case that plane of the first set which passes through x_1, z_1 will also pass through O', that which passes through x_2, z_2 will pass through x_1', z_1', and so on. In the particular case in which two of the indices are zero the plane parallel to YOZ which passes through O' also passes through O'', and we have a similar coincidence of corresponding planes throughout the systems.

2. If two of the indices are odd numbers. Thus a plane passing through z_1, x_3 also passes through x_1', and a plane through z_1, x_5 passes through x_2'; and generally a plane through z_1, x_{2n+1}, where n is any integer, passes through x_n'. The same will be true if two indices be equimultiples of some odd numbers.

When two sets of planes coincide the condensation in every alternate plane will be doubled, while in the remainder the condensation will be that which belongs to only one of the three systems; so that the average condensation in the planes considered will be one and a half times that due to a single one of the three systems.

The three sets of planes will coincide whenever they are all three odd numbers. Thus the plane through x_1, y_5, z_3 passes through both x_1' and y_2'', and so on.

Comparing now the relative condensations in the several planes symbolized below, under the arrangement of molecules above described, we find them to stand thus:

octahedron	*o*	111	condensation		·5774	or	1·1548
cube	*a*	100	„	„	·5000		1·0000
dodekahedron	*d*	110	„	„	·3536		·7071
	m	113	„	„	·3015		·6030
	q	133	„	„	·2294		·4588
	n	112	„	„	·2041		·4083
	p	122	„	„	·1666		·3333
	f	013	„	„	·1581		·3162
	e	012	„	„	·1491		·2982
	s	123	„	„	·1336		·2673
	w	137	„	„	·1302		·2604
		114	„	„	·1179		·2357
		233	„	„	·1066		·2132
		014	„	„	·0808		·1616
		124	„	„	·0727		·1455

Digitized by Google

If then this were the molecular arrangement we should expect the octahedron to be the predominant form, though the cube has a condensation but little less.

Perhaps such substances as cuprite, fahlerz, and alum may have their molecules arranged in this way, while others such as most metals, galena, garnet, analcime, and rock salt have the other and simpler arrangement.

Pyramidal System.

Turning now to the pyramidal system we may as before suppose space divided into similar and equal rectangular parallelopipeds by three sets of parallel planes; the first set parallel to the plane of the two equal crystallographic axes OX, OY, and equidistant from one another at some distance which we may call c; the second set parallel to a plane passing through the third axis OZ and through one of the equal axes OX, and equidistant from one another at some distance a, where the ratio $a : c$ is that of one of the equal axes to third axis of the crystal; and the third set parallel to ZOY and also at the distance a from one another. If now we suppose the molecules arranged so that one of them is in each point of intersection of three of the planes, the arrangement will satisfy the optical requirements.

Also as before every plane which intersects the three axes OX, OY, OZ in points x_p, y_q, z_r respectively, where x_p, y_q and z_r are the places of molecules, will be a plane of maximum condensation, and therefore a possible face of the crystal. And as before the relative condensations in the several sets of planes of maximum condensation will be proportional to the perpendicular distances between the individual planes of the sets.

If c be greater than a, the planes parallel to XOY, that is to the face 001, will have the greatest maximum of condensation, and planes parallel to the faces 100 will have the next less maximum of condensation.

If a be greater than c, 100 will be the plane of greatest maximum, and 001 or 110 will come next after it according as c or $\frac{a}{\sqrt{2}}$ is the greater.

If P be the length of the perpendicular from O upon the plane through x_p, y_q, z_r we shall have P equal to the reciprocal of

$$\sqrt{\left(\frac{1}{pa}\right)^2+\left(\frac{1}{qa}\right)^2+\left(\frac{1}{rc}\right)^2},$$

and if M be the least common multiple of p, q, r, omitting any of them which may be infinite, P/M will be the perpendicular distance between the successive planes of the set, and will be the reciprocal of

$$\sqrt{\left(\frac{M}{pa}\right)^2+\left(\frac{M}{qa}\right)^2+\left(\frac{M}{rc}\right)^2},$$

which is the reciprocal of $$\sqrt{\left(\frac{h}{a}\right)^2+\left(\frac{k}{a}\right)^2+\left(\frac{l}{c}\right)^2},$$

where h, k, l are the symbols of the face in Miller's notation.

Digitized by Google

Example 1. Idocrase. $c/a = \cdot 5351$.

Distance between	planes	parallel to	100 = 1·0000
"	"	"	110 = ·7071
"	"	"	001 = ·5351
"	"	"	101 = ·4718
"	"	"	210 = ·4472
"	"	"	111 = ·4267
"	"	"	201 = ·3653
"	"	"	211 = ·3431
"	"	"	310 = ·3162
"	"	"	221 = ·2950
"	"	"	112 = ·2502

Now the forms most commonly met with in idocrase are 100, 110, 001, 101 and 310, of which the first four are the four planes of greatest condensation if the molecules have the arrangement supposed. Moreover the cleavages are parallel to 100 and 110 and less distinct parallel to 001.

Example 2. Rutile. $c/a = \cdot 6441$.

(*a*)	*P* for 100 = 1·0000		*P* for 310 = ·3162
(*m*)	" 110 = ·7071		" 221 = ·3100
(*c*)	" 001 = ·6441		" 301 = ·2964
(*e*)	" 101 = ·5416		" 320 = ·2774
(*s*)	" 111 = ·4762		" 321 = ·2548
(*h*)	" 210 = ·4472		" 410 = ·2425
	" 201 = ·3950		" 313 = ·1790
	" 211 = ·3674		

The most common forms of rutile are the first five of the above, and the cleavages are parallel to 100 and 110.

Example 3. Anatase.

If we take the axes as given by Miller, we have $c/a = 1 \cdot 7771$, and

(*a*)	for 100,	*P* = 1·0000
(*m*)	" 110,	*P* = ·7071
(*c*)	" 001,	*P* = 1·7771
(*e*)	" 101,	*P* = ·8715
	" 102,	*P* = ·6642
(*p*)	" 111,	*P* = ·6570
(*q*)	" 201,	*P* = ·4813

In crystals belonging to this system there are two variations which are always possible in the crystallographic elements: these are 1°. the pair of equal axes may be turned through half a right angle at the same time that they are diminished (or increased) in

Digitized by Google

the ratio $\sqrt{2} : 1$; and 2°. the ratio of each of the equal axes to the third axis may be increased or diminished in any simple proportion. In either case there will be corresponding changes in the symbols of the faces.

In the case of anatase if we take the octahedron indicated as (p) in the foregoing list to be the form 101, that is, take as equal axes lines in the plane of the square section bisecting the opposite edges of the octahedron instead of lines joining opposite angles, the ratio c/a will become 2·5132, and we shall have for the distances between the sets of planes parallel to the several faces above indicated and some others,

for (c)	001,	$P = 2{\cdot}5132$
(m)	100,	$P = 1{\cdot}0000$
(a)	110,	$P = {\cdot}7071$
(p)	101,	$P = {\cdot}9292$
(q)	111,	$P = {\cdot}6807$
	102,	$P = {\cdot}7825$
(e)	112,	$P = {\cdot}6162$
	201,	$P = {\cdot}4904$
	210,	$P = {\cdot}4472$
	211,	$P = {\cdot}4403$

Now the most common forms of anatase are those indicated as (c) and (p), and these are the directions of cleavage. The forms m, q, e are also frequent. It will be observed that whether we adopt the arrangement of molecules which corresponds to the first chosen axis or the other, the form (c) is that for which the condensation is greatest, while in the former case (a) in the latter (m) comes next in condensation. If we adopt the former position for the axes and suppose the molecules to occupy the centres of the faces of the parallelopipeds into which we suppose space to be divided, we shall get the condensation in the planes parallel to the faces as follows:

(c)	001,	$P = {\cdot}8885$
(p)	111,	$P = {\cdot}6570$
(a)	100,	$P = {\cdot}5000$
(z)	113,	$P = {\cdot}4541$
(e)	101,	$P = {\cdot}4358$
m	110,	$P = {\cdot}3535$
r	115,	$P = {\cdot}3176$
	112,	$P = {\cdot}2766$
	102,	$P = {\cdot}2214$
v	117,	$P = {\cdot}2389$
	211,	$P = {\cdot}2168$
u	105,	$P = {\cdot}1675$
q	201,	$P = {\cdot}1604$

Digitized by Google

This arrangement makes the two cleavages coincide with the planes of greatest condensation; and in other respects tallies with the habit of anatase in bringing comparatively high in the scale of condensation forms such as 115 and 117 which are not uncommon in this mineral but infrequent in others.

Example 4. Cassiterite.

We find for the distances between the planes parallel to the several faces

for (*a*)	100,	$P = 1{\cdot}0000$
(*m*)	110,	$P = {\cdot}7071$
(*c*)	001,	$P = {\cdot}6725$
(*e*)	101,	$P = {\cdot}5329$
(*s*)	111,	$P = {\cdot}4873$
(*h*)	210,	$P = {\cdot}4472$

and that for other faces the order in descending magnitude is 201, 211, 310, 221, 301, 311, 320, 212, 321, 410, 302, &c.

The cleavages are parallel to 100 and 110 the planes of greatest condensation, and the forms (*a*), (*m*), (*e*), (*s*) are of common occurrence.

Rhombohedral System.

For the rhombohedral system we may suppose space divided by three sets of parallel planes into equal rhombohedra and a molecule to be placed at every point of intersection of three planes. We shall have to take for origin of the axes one of the solid angles contained by three equal plane angles, and for axes the three straight lines containing those three plane angles. Then, as before, every plane which meets the three axes in points which are the positions of molecules will be a surface of maximum condensation. Also all the planes parallel to the faces of the form *hkl* will be planes of equal condensation, where the order of the symbols *h*, *k*, *l* is immaterial, provided that when they change sign they all change together. The distance between successive planes of the set parallel to the face *hkl* will in general, on account of the obliquity of the axes, be different from the distance between successive planes of the set parallel to $hk\bar{l}$, and these two sets belong to different crystallographic forms.

The calculation of the distances between successive planes of a set is not quite so simple with oblique axes as with rectangular axes, but if *A*, *B*, *C* (Fig. 3) be the places on the sphere of projection of the poles of the form 100, *X*, *Y*, *Z* the traces on the same sphere of the three axes, *O* the pole of the form 111, and *P* that of the face *hkl*, which is parallel to a plane which intersects the axes in x_p, y_q, and z_r respectively, the angles *PO* and *POA* can be found by formulæ given in Miller's crystallography, *OA* is the angular element of the crystal and *OX* can be found from the formula $\tan OX = 2 \cot OA$. We have then two sides *PO*, *OX* and the included angle *POX* of the spherical triangle *POX*,

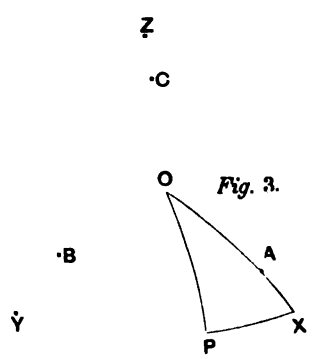


Fig. 3.

Digitized by Google

whence PX may be found, and the perpendicular distance from the origin upon the plane which passes through x_p, y_q, z_r is $p \cos PX$. Also between the origin and this plane there are (including this plane) in all M planes belonging to the set where M is the least common multiple of p, q and r. The distance between successive planes of the set is therefore $\frac{p}{M} \cos PX$ or $\frac{1}{h} \cos PX$.

Example 1. Calcite. Angular element 44°. 37′.

Distances between the planes of sets parallel to the faces

(*r*)	001	·9448	(*o*)	111	·4425
(*a*)	$01\bar{1}$	·7767		$21\bar{1}$	·4036
(*f*)	$\bar{1}11$	·5999	(*ϖ*)	210	·3844
(*e*)	011	·5951	(*m*)	$3\bar{1}\bar{1}$	·3260
(*v*)	$20\bar{1}$	·4748	(*u*)	211	·3222
(*b*)	$2\bar{1}\bar{1}$	·4484	(*ϵ*)	$\bar{1}22$	·3150

Here the cleavage form (*r*) 001 is that for which the condensation is far the greatest, and the forms which are high in this list are of very frequent occurrence.

Example 2. Chabasie. Angular element 50°. 45′.

Distances between the planes of sets parallel to the faces

(*r*)	001	·9908		$2\bar{1}0$	·4620
(*a*)	$01\bar{1}$	·7387	(*t*)	210	·4263
(*e*)	011	·6679		$2\bar{1}\bar{1}$	·4265
(*s*)	$\bar{1}11$	·5922		$2\bar{1}1$	·4114
(*o*)	111	·5220		211	·3743

In this crystal the cleavage is parallel to the faces 001 for which the condensation is greatest, and the most common forms and combinations are *r*, *er*, *ers*, *ersa*, *ersta*; all of which are amongst those for which the condensations are high maxima.

The two foregoing examples are crystals which generally assume rhombohedral forms and have easily obtained rhombohedral cleavages, and the rhombohedral arrangement of the molecules satisfies the observed facts. But there are many crystals of this system which generally assume a hexagonal character, and for them we may suppose a different arrangement of the molecules. These crystals may be referred, as is done by many crystallographers, to four axes, of which three are equal to one another and lie in one plane and are equally inclined to one another, while the fourth is at right angles to them, and may be unequal to them in magnitude. Now if we suppose space to be

Digitized by Google

divided into equal right triangular prisms by four sets of parallel planes, each set parallel to two of the axes (one of the sets being of course parallel to the *three* equal axes), and suppose a molecule placed in every point where three planes intersect, we shall have an arrangement which will satisfy the optical conditions observed in such crystals, for we shall have an arrangement which is symmetrical about the axis which is at right angles to the other three; and the surfaces of maximum condensation will, as before, be the planes which satisfy the law of indices.

There are always two positions possible for the three equal axes, one set of axes being inclined at 30° to the other set but in the same plane with them.

Also it should be observed that a crystal with the molecular arrangement here supposed may assume a rhombohedral character so far as its external shape is concerned; because the symmetrical omission of every alternate face of a hexagonal crystal will leave a form which will satisfy the conditions of equilibrium, for it will have no unbalanced edges or angles.

If OS, OT, OY (Fig. 4) be the directions of the three equal axes, OW that of the axis at right angles to them, and s, y, t, w be the positions of molecules in those lines at distances from O which are multiples, s, t, y, w, of the lengths of the respective axes, and ON be perpendicular to st, On perpendicular to Nw; then since sO and Ot are known, and the angle sOt is 120°, ON can be found by ordinary trigonometrical formulæ, and we have

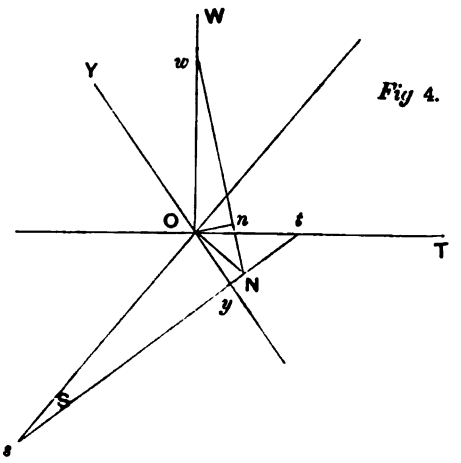


Fig 4.

$$On = wO \sin . \tan^{-1} \frac{ON}{wO}.$$

Then the perpendicular distance between the planes parallel to that through s, y, t, w will be On divided by M the least common multiple of s, t, y and w.

s/M, t/M, y/M, and w/M, in their lowest terms will be the indices of the face parallel to the plane $sytw$ in Weiss' notation reduced to fractions with unity as numerator.

Example 3. Apatite. Angular element 55°. 40′ in the rhombohedral system.

If we choose as equal axes on the hexagonal system the lines which bisect the opposite angles of the hexagonal prism of most common occurrence, to which Miller assigns the symbol $01\bar{1}$ (a), and take that form to have the symbol $\bar{1}010$ where the last index belongs to the vertical axis, and make the length of the vertical axis unity, the length of the other axes will be 1·3660, and the perpendicular distances between the planes of the sets parallel to the faces will be

Digitized by Google

Miller's notation (rhombohedral).		Weiss' indices reduced (hexagonal).	Distance.
a	$01\bar{1}$	$0\bar{1}10$	1·1829
o	111	0001	1·0000
x	120	$0\bar{1}11$	·7636
b	$21\bar{1}$	$\bar{1}2\bar{1}0$	·6830
r	100, $\bar{1}22$	$\bar{1}2\bar{1}1$	·5640
z	$13\bar{1}$	$0\bar{2}21$	·5090
i	231	$0\bar{1}12$	·4605
s	$\bar{1}11$, $5\bar{1}\bar{1}$	$\bar{2}4\bar{2}1$	·3231
		$0\bar{3}31$	·3667
		$0\bar{2}23$	·2904

This table includes the planes for which on the arrangement supposed the condensation is greatest, and the forms which are most common in apatite.

The cleavages are parallel to the first two, that parallel to *a* being more easily obtained than that parallel to *o*.

Example 4. Greenockite. Angular element on rhombohedral system 43°.37′. Taking as in the last example the equal axes to be the lines bisecting the opposite angles of the hexagonal prisms to which Miller assigns the symbol $0\bar{1}1$ (*a*), and assigning to it the symbol $0\bar{1}10$, and making the vertical axis unity, the length of the equal axes will be 1·2118, and the perpendicular distances between the planes of the sets parallel to the several faces will be

Miller's notation.		Weiss' indices reduced.	Distance.
a	$01\bar{1}$	$\bar{0}110$	1·0494
o	111	0001	1·0000
x	120	$0\bar{1}11$	·7240
b	$2\bar{1}\bar{1}$	$1\bar{2}10$	·6059
r	100	$1\bar{2}11$	·5183
z	$13\bar{1}$	$0\bar{2}\bar{2}1$	·4648
i	231	$0\bar{1}12$	·4514

These are the forms which commonly occur in this mineral, and they are those for which the condensation is greatest on the supposition here made as to the arrangement of molecules. The cleavages are parallel to the first two forms.

Prismatic System.

In the prismatic system there are three axes at right angles to each other and all unequal. We have to suppose, as before, space divided into rectangular parallelopipeds by three sets of parallel planes, the planes of each set parallel to two of the axes and at a distance apart equal to the length of the third axis. A molecule may be supposed placed in every intersection of three planes. If *a*, *b*, *c* be the lengths of the three axes

Digitized by Google

we shall have by reasoning precisely the same as is given above for the pyramidal system, P, the perpendicular distance between the successive planes parallel to a face of the crystal h, k, l, on Miller's notation, equal to the reciprocal of

$$\sqrt{\left(\frac{h}{a}\right)^2+\left(\frac{k}{b}\right)^2+\left(\frac{l}{c}\right)^2}.$$

Applying this to particular cases:

Example 1. Epsomite. $a : b : c = 1 : 0{\cdot}9901 : 0{\cdot}5709$.

The distances between the successive planes parallel to the faces are

Miller's notation.		Distance.	Miller's notation.		Distance.
a	100	1·0000	f	210	·4463
b	010	·9901		120	·4437
m	110	·7036	x	021	·3740
c	001	·5709	r	201	·3761
v	101	·4958	t	211	·3516
n	011	·4946		112	·2645
z	111	·4433	s	122	·2403

There is a very perfect cleavage in these crystals parallel to a, and a less perfect cleavage parallel to v. The forms most common are a, m, v, n, z. From the degree of condensation parallel to b on the supposition here made we should have expected a cleavage parallel to b if not a frequent occurrence of that form.

Example 2. Topaz.

In this case the easily obtained cleavage indicates the longest axis. We must therefore take the form which in Miller is designated (y) 201 to be 101, which is equivalent to doubling the length of the third axis, and is always crystallographically admissible.

We then get $a/b = \tan 62°.9'.30''$, $2b/c = \tan 29°.5'.30''$, and the distances between the successive planes of the sets parallel to the faces are

Symbol with Miller's axes.		Symbol with new axes.	Distance.
c	001	001	3·5945
a	100	100	1·8933
y	201	101	1·6752
n	101	102	1·3037
e	203	103	1·0125
b	010	010	1·0000
	021	011	·9636
w	401	201	·9155
i	011	012	·8738
m	110	110	·8842
	102	104	·8119
k	111	112	·7934
l	210	210	·6875
o	112	114	·6303
n	310	310	·5337

Digitized by Google

The forms *c, a, y, n, i, m, o* are very common: the cleavage parallel to *c* is very perfect, and there are imperfect cleavages parallel to *i* and *m*.

Example 3. Olivine. Here again the cleavage in one direction is easily obtained. If we take the cleavage plane for the plane perpendicular to the longest axis, we must take for the form 101 that to which Miller assigns the symbol (*h*) 102, which is equivalent to halving the longest of his axes.

We then get $a/b = \tan 47° . 1' . 24''$, $b/c = 2 \tan 38° . 27'$, and for the distances between the planes parallel to the several faces:

Symbol with Miller's axes.		Symbol with new axes.	Distance.
a	100	100	1·0732
b	010	010	1·0000
s	110	110	·7316
c	001	001	·6297
h	102	101	·5432
	012	011	·5329
z	210	210	·4728
n	120	120	·4532
k	101	201	·4084
d	011	021	·3915
	212	211	·3781
e	122	121	·3679
	310	310	·3368
	130	130	·3183
f	111	221	·3163

The faces *a, b, s, c, k, d, e* are frequent, and there is an easily obtained cleavage parallel to *a* and traces of cleavage parallel to *b*.

We have taken no account of the fact that the concentration of molecules in many of the planes is unequal in different directions; and it must be remembered that the distances given indicate only the average concentration.

Further there are other arrangements of molecules which will satisfy the optical conditions as well as that above supposed. For example we may suppose space to be divided by three sets of planes of which one set are parallel to two of the axes *a, b*, and the other two sets are each perpendicular to the first set, and cut them in rhombuses whose sides are equal to the diagonals of the rectangles on *a, b*, and are inclined at the same angles as those diagonals. We may suppose the molecules placed in the points where three planes intersect as before. The condensation will be the same as before in most of the planes, but will be doubled in the planes which are parallel to a face *hk2l*, when *h* and *k* are equal to one another or are both odd numbers, and this arrangement appears to accord with the observed forms and cleavages of some crystals better than the other. Thus

Digitized by Google

Example 4. Nitre.

Taking the axes as Miller takes them, we get for the distances between the planes of the several sets parallel to the faces, if the molecules in the planes parallel to that of the axes *a*, *c* lie in those diagonals of the rectangles on the axes which pass through the third axis,

k	101	$P = 1{\cdot}9425$		103	$P = {\cdot}7701$
a	100	1·6920		011	·7646
c	001	1·1861	*p*	111	·6967
	301	1·0187	*i*	201	·6888
b	010	1·0000		210	·6459
m	110	·8609	*x*	102	·5597

The forms *k*, *a*, *m* are most common and there is a perfect cleavage parallel to *k*, a less perfect cleavage parallel to *a*, and imperfect cleavage parallel to *m*.

Again, *Example* 5. Baryte.

Taking the axes as Miller takes them, and supposing the molecules in the planes parallel to that in which the axes *a*, *b* lie to be at the extremities of those diagonals of the rectangles on *a*, *b* which pass through *c*, we get for the distances between the planes parallel to the several faces

c	001	$P = 1{\cdot}6118$	*z*	111	$P = {\cdot}6986$
m	110	1·5506	*d*	012	·6274
a	100	1·2276		201	·5736
b	010	1·0000	*n*	210	·5231
o	101	·9765	*r*	112	1·1173
u	011	·8497	*q*	114	·7149
ϕ	102	·6734	*k*	118	·3899

There are perfect cleavages parallel to *c* and *m*, and the former more easily obtained than the latter, cleavage parallel to *a* less perfect, and parallel to *b* still less perfect; but the faces *r* and *q*, for which there is a considerable condensation, are not common and there is no cleavage parallel to them.

It may be noted that in the first example given above of a crystal of this system, namely epsomite, if we suppose the same sort of arrangement as that here suggested, with the molecules in the planes parallel to the axes *a*, *c* in the diagonals, the face *v* will come next to *a* in the order of concentration, and the face *s* will come high in the list. This is in agreement with observation since there is a cleavage parallel to *v* only less perfect than that parallel to *a*, and *s* is a form of not infrequent occurrence.

Crystals like karstenite with three cleavages at right angles to one another correspond to the first molecular arrangement. In fact in karstenite we have two axes very nearly equal and a very perfect cleavage at right angles to each of them; while the third axis is shorter than the other two and the cleavage perpendicular to it is less perfect though easily obtained.

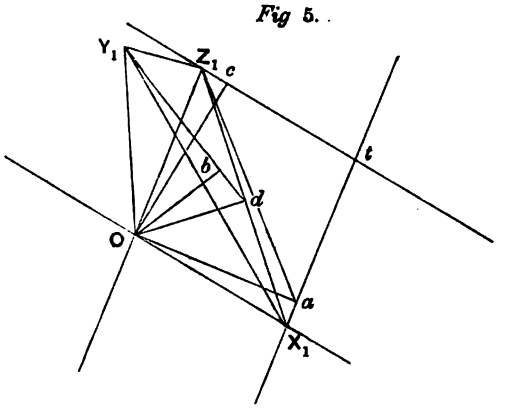
Digitized by Google

Oblique System.

Coming now to the oblique system we may suppose space divided as before by three sets of planes, parallel to the axes two and two, into equal right prisms on rhomboidal bases, the height of the prisms being the length of the vertical axis and the sides of the base equal to the lengths of the other axes. We may suppose the molecules placed in the points of intersection of three planes.

To find the perpendicular distance between the planes of any set, if OY_1 (Fig. 5) be the axis which is at right angles to the other two, OX_1, OZ_1 the other two axes, Z_1t, X_1t parallel to OX_1, OZ_1 respectively, OX_1tZ_1 will be the rhomboidal base of one of the prisms. Then if Oa, Oc be drawn perpendicular to X_1t, Z_1t respectively, and Od perpendicular to Z_1X_1, Ob perpendicular to Y_1d, the angles dOa, bOY_1 and dOc are the three angular elements of the crystal. And if we make the length of the axis OY_1 unity, $Od = \text{cotang}\, Y_1dO = \cot Y_1Ob$. Also since $dOa = dZ_1O$ we have $OZ_1 = Od \operatorname{cosec} dOa$, and so also $OX_1 = Od \operatorname{cosec} dOc$, which gives the lengths of the axes.

Fig 5.



Again $dOX_1 = 90° - dOc$, and therefore $aOX_1 = 90° - dOc - dOa$, and Oa the distance between the planes parallel to the form 100 is equal to $OX_1 \cos aOX_1$.

Also if a plane meet OX, OY and OZ in X_p, Y_q, Z_r, where OX_p is equal to p times OX_1 and so on, the length of the perpendicular OP from O on Z_rX_p can be found, since OX_p and OZ_r are known; and the two sides of the right-angled triangle Y_qOP being known the length of the perpendicular from O upon Y_qP may be found, and this length divided by the least common multiple of p, q, and r will be the distance between successive planes of the set parallel to the plane in question.

Applying this to particular cases.

Example 1. Gypsum. Angular elements 52°.16′, 28°.16′, and 71°.51′. If $OY_1 = 1$, $OX_1 = \cdot 69222$, $OZ_1 = \cdot 41450$, and the distances between the planes of the sets parallel to the several faces are:—

b	010	$P = 1\cdot 0000$		210	$P = \cdot 3231$
a	100	·6828		021	·3165
m	110	·5639	*l*	111	·3115
c	001	·4089	*x*	$\bar{1}21$	·3022
h	120	·4033		121	·2741
v	011	·3785	*s*	$\bar{1}31$	·2504
	$\bar{1}01$	·3793		012	·2003
n	$\bar{1}11$	·3547	*e*	$\bar{1}03$	·1381
d	101	·3278			

Digitized by Google

The cleavage parallel to *b* the plane of greatest maximum of concentration is very perfect and easily obtained, there is a less perfect cleavage parallel to *a*, and also an imperfect fibrous cleavage parallel to *n*. The most common forms are *b*, *l*, *m*, *h*, *n*, and *e*.

Example 2. Felspar. Angular elements 65°.47′, 50°.20′, and 63°.7′. Making $OY_1 = 1$, we get $OX_1 = \cdot 6586$ and $OZ_1 = \cdot 5560$.

We may however take the length of OZ_1 double this amount and make corresponding alterations in the angular elements. If we do this we find the following for the distances between the planes of the sets parallel to the several faces.

Symbol with Miller's axes.		Symbol with new axes.	Distance.
a	100	100	·5913
b	010	010	1·0000
c	001	001	·9983
n	021	011	·7065
o	111	112	·6553
y	201	101	·6493
u	221	111	·5446
m	110	110	·5090
x	101	102	·5070
	011	012	·4466
	120	120	·3818
q	203	103	·3672
	121	122	·3511
h	023	013	·3156
	211	212	·3088
z	130	130	·2903
	210	210	·2835
s	131	132	·2785
	102	104	·2779
g	112	114	·2678
	212	214	·2456
	012	014	·2422

There are perfect cleavages parallel to faces *b* and *c* which have the greatest condensation. Besides these faces, *x*, *m*, *o*, *y*, *n*, are of frequent occurrence and are all high in the list of condensation.

Anorthic System.

With regard to the anorthic system it does not seem necessary to say much. We may suppose space to be divided as before into parallelopipeds by three sets of parallel planes, each set being parallel to a plane containing two of the axes, and the edges of the parallelopipeds equal to the three axes respectively, and a molecule to be placed in each point of intersection of three planes. Faces which follow the law of indices will then be surfaces of maximum condensation; and we can generally so choose the axes that the cleavages shall be the planes of greatest maxima of condensation.

Digitized by Google

Inequalities of Growth.

The unequal development of crystals in different dimensions depends more upon the circumstances than upon the surface energy. When crystallization takes place more rapidly than diffusion the parts of the liquid where the crystals are forming become depleted, and the crystals can only grow into the more saturated parts of the solution. Hence when magnesium sulphate, or oxalic acid, crystallizes out of a hot solution which is rather quickly cooled the crystals run into long needles. At the same time the elongated faces are generally planes of a high degree of condensation. The like may be said of tabular crystals. Thus the tabular crystals of barium chloride have their broad faces perpendicular to the longest axis, the axes being in the ratio 0·9574 : 1 : 1·5778. The planes perpendicular to the longest axis are those of greatest condensation. In silver nitrate the axes are as 0·9429 : 1 : 1·3697 and the broad faces of the crystals are perpendicular to the longest axis. Ammonium citrate has axes in the ratio 0·5746 : 1 : 1·3749 and forms tabular crystals with the broad faces perpendicular to the longest axis. How the development of the crystal depends on circumstances is well seen in the case of the Iodo-sulphate of quinine, sometimes used instead of tourmalines as analysers of polarised light. When the crystals form rapidly from a hot solution which cools and becomes saturated through evaporation at the surface the crystals form broad plates which float on the liquid, but if the dish containing the solution is covered and cooled slowly the crystals form within the liquid in tolerably equally developed prisms with no tendency to a tabular form.

Hemihedral forms.

In regard to surface tension it is necessary to distinguish the two kinds of hemihedral forms, (*a*) those with parallel faces, and (*b*) those with inclined, or asymmetric, faces. In the hemihedral forms with parallel faces the stresses due to the surface tension at edges and angles are opposed to similar stresses on the opposite side of the crystal, and therefore in such forms the surface tensions are in equilibrium amongst themselves. In the hemihedral forms with inclined faces the surface tensions at edges and angles will in general give rise to stresses in the nature of couples which can be equilibrated only by opposite couples. It is this kind of asymmetric hemihedral formation which is connected as mentioned above with the properties of rotating the plane of polarisation of polarised light and of pyroelectricity.

In order to give this connexion a physical basis it is necessary to suppose that in the formation of crystals having these properties the solidification is attended with a certain internal strain. This strain we may imagine to arise in different ways. It may arise from some want of symmetry in the molecules which prevents their assuming the crystalline arrangement without some strain. This may well be the case where the molecules themselves seem to produce the rotation of the plane of polarisation as in tartaric acid and other compounds which possess rotatory power in solution and in the amorphous state. Or an internal strain may be left when crystallization occurs in a mixed mass of two or more substances which are not isomorphous, where the crystallization cannot take place with perfect freedom and where portions of one substance become entangled in the crystals of the other. Or a strain might be caused by some stress due to the electric field in

Digitized by Google

which the mass was situate at the time of crystallization. Whatever the origin of the strain, if it be one which is capable of being relieved in any degree by a counter stress due to surface energy it will become a factor in determining the development of the crystal. The condition of equilibrium will no longer be that the surface energy of the crystal shall be a minimum, but that this energy together with that of the strain shall be a minimum. In fact if an unsymmetrical development of the external form of the crystal would produce a strain opposite to that which would result from any of the other causes acting independently, such an unsymmetrical development will in general take place.

In so far as the surface tension counteracts the internal strain, the latter will be lessened and so will any physical property which arises directly from it. Hence we find that those substances which have rotatory power in the liquid state and form hemihedral crystals lose their rotatory power when crystallized. Sulphate of strychnia which has the rotatory power more strongly in the crystalline form than in solution forms holohedral crystals.

There is evidence of another kind that such a strain does exist in some cases. For if a substance, such as dextro-rotatory tartaric acid, can only crystallize with an internal strain, this will tend to impede its crystallization and it will crystallize more readily under circumstances which relieve the strain. Because the strain implies a supply, or a retention, of a certain amount of energy, and crystallization will be facilitated if the necessary energy is supplied from a source independent of the surface energy of the crystal. Now the asymmetry, whatever its nature be, in the molecules of dextro-tartaric acid has its exact counterpart in the molecules of lævo-tartaric acid, and we may reasonably suppose that the strain arising from this asymmetry in crystals of dextro-tartaric acid is opposite to that arising from the counter asymmetry in crystals of lævo-tartaric acid, and that consequently in crystals formed of a mixture of the two acids in equal proportions the stresses being equal and opposite would produce no torsional strain, and crystallization would be facilitated. This supposition is in fact confirmed; for concentrated solutions of dextro- and lævo-tartaric acid, which separately will not crystallize, when mixed form immediately crystals of the non-rotatory acid with an elevation of temperature.

When a substance exhibits rotatory power in the crystallized state but not in solution or in the amorphous form we have to look for a common cause for both the rotatory power and the hemihedral crystallization.

In sodium chlorate the cause may perhaps be traced as follows. The isomorphism of corresponding sodium and potassium salts is so well marked that we may be almost sure that sodium chlorate is capable of crystallizing in the same form as potassium chlorate, and vice versa, and as the former usually crystallizes in the cubic system and the latter in the oblique, that they are dimorphous, but that under ordinary circumstances the one crystallizes with greater facility in one form and the other in the other form. Now if the particles of sodium chlorate were aggregated partly according to the law of one form and partly according to the law of the other form there would probably result internal strains which would not be in equilibrium amongst themselves, and hence such an aggregation would not be possible unless there were some external stress to counteract them. That stress is supplied by the surface tension of the hemihedral form. This theory is borne out by the structure of boracite. For the examination of slices of boracite with

Digitized by Google

a polarising microscope shews that it is built up of particles which have some of the optical characters of a biaxial crystal. These particles, which have been called "parasite," may be boracite in a dimorphous state as supposed by Volger, or they may be parts of the regular crystal which derive their optical characters from a strain as supposed by Klein. Their general arrangement depends on the external form of the crystal, but they are not uniformly disposed throughout the whole mass, and corresponding to this want of uniformity we find that the hemihedral character of the crystals has not been uniformly maintained during their growth. For it seldom happens that the alternate faces of the octahedron are altogether absent, usually they are less developed than their fellows and the alternations of holohedral and hemihedral growth are indicated by roughness of the surface. At those stages of growth, perhaps dependent on temperature, when the parasite was formed the hemihedral crystallization was necessary to the equilibrium, but it would not be necessary at other stages, and then the growth would be holohedral.

If a pyroelectric crystal were placed in an electric field which would induce an electric state exactly the opposite of that which the crystal assumes when undergoing a change of temperature the strains due to the changing temperature would be relieved. And conversely if the material forming the crystal were placed during the growth of the crystal in such a field the regular crystalline arrangement of the molecules could not be assumed without strain, and crystallization would be impeded. But if the electric stress could be counteracted by the stress arising from a hemihedral formation, crystallization could occur in the hemihedral form. When a crystal so formed was removed from the electric field in which it had grown it would be left with a strain, namely the strain due to its hemihedral growth.

Action of solvents.

The explanation of the forms which crystals take on solidifying from solutions applies equally well to the development of faces on the surface of a crystalline mass which is undergoing slow solution. When the solvent is not nearly saturated differences of surface energy due to the form of the surface will have little effect, because there will always be a sensible increase of entropy resulting from the process of solution whatever the surface energy of the undissolved mass. But when the process of solution involves but little increase of entropy, as is the case when the solution is nearly saturated, those surfaces which have a minimum surface energy will sooner cease to be dissolved than those of greater energy, because there will be an increase of entropy by the transformation of the latter into surfaces of less energy through the process of solution.

There are many more details, at some of which I have already hinted, which remain to be worked out. These I hope to make the subject of a subsequent communication. I have presented the problem just as it unfolded itself in my own mind; but it might have been more logically attacked in the inverse way; I might have laid down the principle that cleavage planes and the faces of dominant forms must be surfaces of the greatest maxima of molecular condensation, and thence determined the arrangement of the molecules. This may now be done, and I hope it will be a step towards the solution of the question, What is the law of mutual action of the molecules which causes them to assume such a disposition?

Digitized by Google

XIV. *On Solution and Crystallization.* No. II. By G. D. Liveing, M.A., Professor of Chemistry in the University of Cambridge. (Plate VI.)

[*Read* Nov. 26, 1888.]

In my previous communication to the Society on this subject I said nothing about the physical causes which determine the arrangement of the molecules when passing from a fluid to a solid state, but only tried to shew that if the arrangements were such as I described the external form and the cleavage would, on well known mechanical principles, be such as we find them to be in fact. I purpose now to try and explain what these causes are. In accounting for the arrangement of the molecules I find it quite unnecessary to suppose that any special force is concerned; it is enough if the molecules attract each other, and so far as force goes the force of gravitation seems to be sufficient to produce the results. I have to suppose, in accordance with other physical properties of matter, that the parts of every molecule are in continual motion, but that in the solid state the excursions of the parts from the centre of mass of the molecule do not on the average exceed certain limited distances; that these distances are on the average the same for the same kind of matter at the same temperature and pressure, and determine what may be called the molecular volume. Further I assume that the average distribution of the mass of each molecule within the molecular volume is the same for the same substance under the same conditions, but may differ for molecules of different kinds of matter. I assume in fact that the molecules, even of the chemical elements, are, in the solid state at least, complex aggregates, and that they have motions of their own which are of the same sort and on the average of the same intensity for all the molecules of the same chemical substance.

To take first the simplest case: suppose that the excursions of the parts of the molecule extend on the average equally in all directions about the centre of mass of the molecule. The average space occupied by the molecule will be comprised in a sphere of which the centre will be the average position of the centre of mass of the molecule, and the radius the average distance to which the excursions of the parts from that centre extend. It is not intended to assume that parts of the molecule may never acquire such velocity as to go beyond the limits of this sphere, but that on the average, partly by collision with neighbouring molecules and partly by reason of their mutual attractions, they come up to but do not exceed this limit. If that be conceded, and if moreover the mutual action of the molecules is to attract one another, the problem of finding how

Digitized by Google

Fig. 1

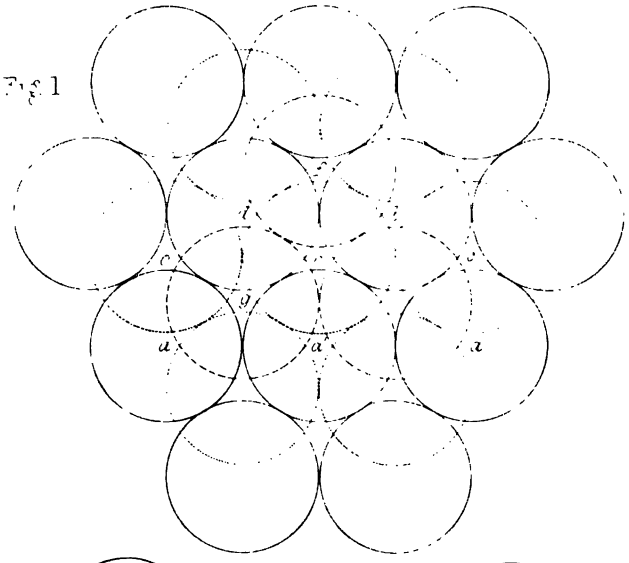


Fig. 2

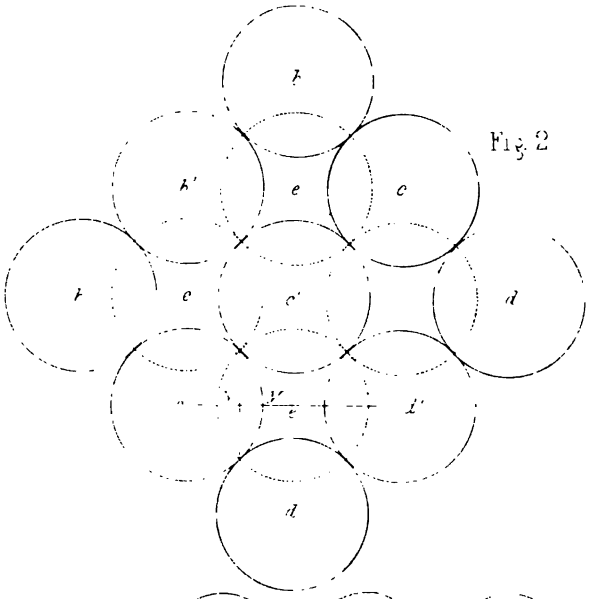


Fig. 3.

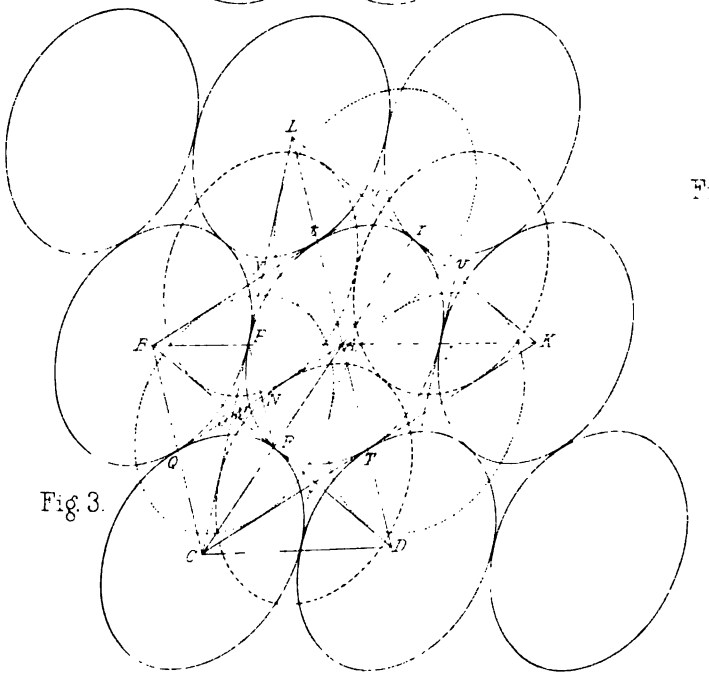


Fig. 5.

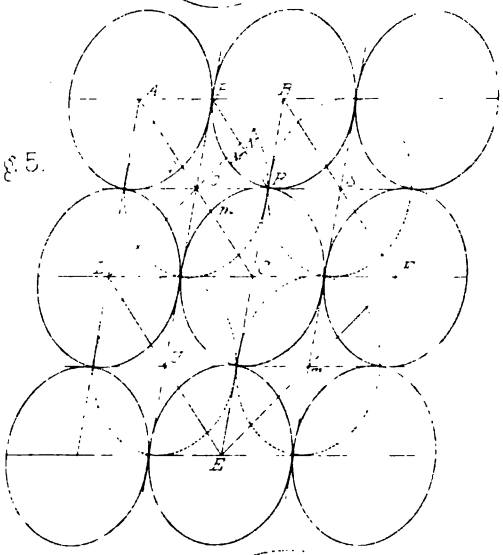


Fig. 6.

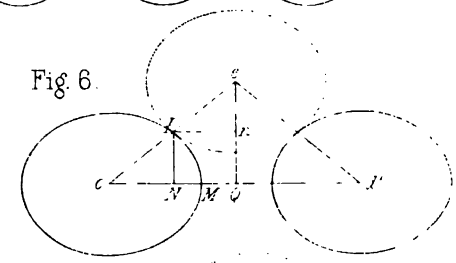


Fig. 4.

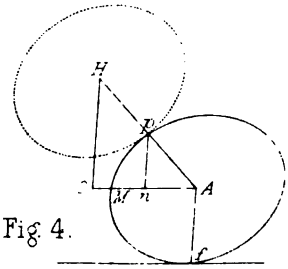


Fig. 7.

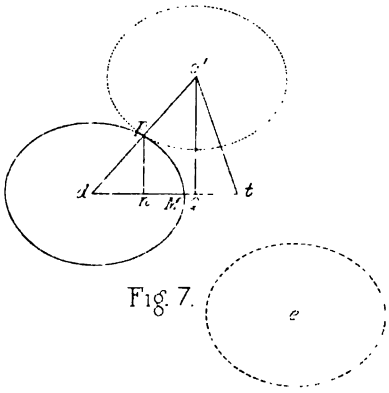
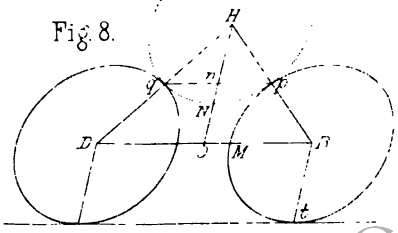


Fig. 8.



Digitized by Google

Digitized by Google

the molecules will arrange themselves in equilibrium becomes, by a well known mechanical principle, the problem of finding how the spheres representing the molecular volumes can be most closely packed in a given space.

Now if we arrange equal spheres with their centres in one plane as closely as possible they will be arranged as the circles of like outline are arranged in fig. 1, so that each sphere is touched by six others. Above them other equal spheres can be arranged in like order and so that each sphere of the upper set, indicated in dotted outline in the figure, touches three of the lower set. Again another set can be arranged above the second set in like order so that each of the third set touches three of the second, and so on.

If we call the radius of a sphere b, the area of the equilateral hexagon circumscribing one of the circles in fig. 1 will be $2\sqrt{3}b^2$, and there is one circle for every such hexagon so that the number of circles per unit area is $\frac{1}{2\sqrt{3}b^2}$. Also the distance between the plane in which the centres of the first set of spheres lie and that in which the centres of the second set lie, is the length of the perpendicular let fall from one angle of a regular tetrahedron with edges equal to $2b$ on the opposite side, and this is equal to $2\sqrt{\frac{2}{3}}b$. Hence the number of spheres per unit volume is $\frac{1}{4\sqrt{2}b^3}$. And this appears to be the maximum number which can be packed in unit volume.

We might have started with arranging the first set of spheres so that each one touches four others at points in one plane as in figure 2, and then arrange a second set in similar order above them, as indicated in dotted outline, in such wise that each sphere of the second set touches four of the first set, and so on with successive sets. On computing the number of spheres per unit volume we find it to be $\frac{1}{4\sqrt{2}b^3}$ as before. In fact the two arrangements are really identical, and in the figures we are merely looking at the same thing from different sides. The plane through the centres of the row a, a', a of the first set in fig. 1 which also passes through the centres of the row c, c', c of the second set, has in fact the spheres arranged as in fig. 2; while the plane through the centres of the row b, b', b of the first set in fig. 2, which also passes through the centres of the row e, e, of the second set, has the same arrangement as is shewn in fig. 1. Either arrangement is the same as if space were divided into equal cubes*, and a sphere placed with its centre at each angle, and also a sphere with its centre at the centre of each face, of every such cube. It is a combination of the two arrangements suggested for the molecules of a crystal of the cubic system. In this arrangement the relative condensations in planes parallel to the faces of the octahedron, cube and dodecahedron respectively are as $\frac{1}{\sqrt{3}} : \frac{1}{2} : \frac{1}{2\sqrt{2}}$. This agrees sufficiently with the observed facts in the greater part of the substances which crystallize in the cubic system. The octahedral cleavage is in general

* The square which forms the face of one cube in the plane of the figure is $b'cd'c$, and c' falls in the centre of the face. e, e fall in the centres of two faces at right angles to the plane of the figure.

Digitized by Google

the easiest to obtain, and the cubic next. There are a small number of types of chemical compounds forming crystals in which a cubic cleavage is most easily obtained. These are the haloid salts of the alkalies, such as sodium and potassium chloride, a few sulphides and selenides of the type of galena, and a few sulpharsenides such as cobaltine, to which pyrite may be added. There are also a still smaller number of substances such as blende in which the dodecahedral cleavage predominates. These are, however, the exceptions rather than the rule, and as such their consideration may be deferred.

The supposition above made for the sake of simplicity that the excursions of the parts of the molecule from its centre of mass are equal in all directions cannot be the general case. We should expect the excursions to be different in different directions on account of the different masses of the chemical atoms and their different relations to one another in the molecule. The average energy of the motion will however be the same in every direction, but the average molecular volume will be an ellipsoid with three unequal axes. Further we cannot in general assume that the matter is equally distributed on the average in all parts of the molecular volume. It may for example be more massed about one or other axis of the ellipsoid, or about one or other plane of principal section. Whatever the massing of the matter in the molecule may be, if it be, as the very definite characters of the chemical compounds lead us to suppose, the same in every molecule of the same compound, it will tend to place every molecule in the same position with reference to surrounding molecules. This is as much as to say that it will tend to give all the ellipsoids the same orientation of their axes so that they will be all similarly situated. This appears to be also the arrangement which admits of the packing of the greatest number of equal and similar ellipsoids into a given space.

Though I am not at present able to give a direct and complete mathematical demonstration of the solution of this problem of packing the greatest number of ellipsoids into unit space, I have no doubt what the solution is and can prove it indirectly. The arrangement required is similar to that already indicated for spheres.

By no arrangement can every ellipsoid be made to touch more than twelve others.

Suppose now the ellipsoids to be all similarly situated and to be arranged as in fig. 3 so that each one is touched by six others at points lying in one plane which is the plane of the figure. Then any one of these, as that with its centre at A, may be touched by three others which have their centres in a plane above the plane of the figure, as the three indicated in broken outline, or the three indicated in dotted outline, as well as by three more with their centres in corresponding positions below the plane of the figure. In this way every ellipsoid will be touched by twelve others.

Considering the ellipses which are the intersections of the plane of the paper with the ellipsoids which have their centres in that plane, since they are all equal, similar, and similarly situated, the lines joining the centres of any two which touch each other will pass through the point of contact. Also any such line as AC passing through the point of contact R will bisect the chord PT between the points where the two adjacent ellipses touch the ellipse having its centre at A. Hence the tangent at R will be parallel to PT and pass through the centres of the adjacent ellipses B, D. Also if Q be the point of contact of the ellipses of which B, C are the centres AQ will be in the line of the

Digitized by Google

diameter conjugate to AT, for it bisects the chord PR which is parallel to BC and AT. Now there is one ellipse for every hexagon such as *onsvwy* formed by the tangents at the six points of contact of the ellipse having its centre at A with the surrounding ellipses, and the hexagon circumscribing an ellipse will be a minimum when each side is bisected in the point of contact. For if the sides *oy*, *ns*, *vw* be produced they will meet in the centres of the ellipses C, K, L, and the ellipse having its centre at A and the triangle CKL will be the projections on the plane of the figure of a circle and its circumscribing triangle lying in a plane inclined at some angle to the plane of the paper. In the figure CA, KA and LA produced bisect the opposite sides of the triangle and hence CKL is the projection of the equilateral triangle circumscribing the circle of which the ellipse with centre A is the projection. Also the triangles *Lyw*, *Con*, *Kvs* are all similar to the triangle CKL, and therefore *onsvwy* is the projection of the equilateral hexagon circumscribing the circle. Such a hexagon is the smallest in area which can be drawn circumscribing the circle, and hence *onsvwy* is the smallest which can be drawn circumscribing the ellipse with centre A. Hence when the ellipses are all similarly situated the hexagon formed by the tangents at the points of contact is a minimum, and since such hexagons fill the whole area, and there is one ellipse for every such hexagon the number of ellipses per unit area is a maximum.

Now if we suppose space filled with the ellipsoids, and suppose the continuous, dotted, and broken-line ellipses in the figure to represent successive sets of ellipsoids with their centres in successive parallel planes, the plane through the centres of the broken-line ellipsoids y, v which also passes through the centre of the dotted ellipsoid w, and through L, will cut the ellipsoids in a series of ellipses of which each is touched by six others just as in fig. 3; so will the plane through the centres of broken-line ellipsoids y, n which also passes through o, B, C, and the plane through the centres of the broken-line ellipsoids n, v which also passes through s, K, D. Hence in each of the four planes, namely that of the figure and the three last mentioned planes, the number of ellipses will be a maximum per unit of area when the ellipses are all similar and similarly situated; and since in four different planes of which no two are parallel the concentration is a maximum under these conditions, I infer that the number of ellipsoids which can be packed into a given space is greatest when they are all similarly situated. If this were in doubt it becomes more certain when it is seen that, if the condition of similarity of situation be satisfied, the orientation of the axes is a matter of indifference.

To prove that statement, return to the hexagon *onsvwy*. The area of it will bear to the area of the ellipse the same ratio which the area of the circumscribing equilateral hexagon bears to the circle, which is $2\sqrt{3} : \varpi$. Hence the area of the hexagon is $2\sqrt{3}a_1b_1$, where a_1, b_1 are the semi-axes of the ellipse.

Again considering one of the ellipsoids, indicated in dotted outline in fig. 3, which touches the three ellipsoids with centres A, B, C, since all the ellipsoids are similar and similarly situated the lines joining the centre of the dotted ellipsoid with A, B, C will pass through the points of contact and be bisected at these points, which will therefore lie in one plane parallel to the plane ABC. Also the centre of the dotted ellipsoid will lie in the intersection of the three tangent planes at P, Q, R.

Digitized by Google

For by reasoning similar to that given above to shew that the tangent at Q passes through A it may be shewn that in the plane of B, C and the centre of the dotted ellipsoid the tangent at Q passes through the centre of the dotted ellipsoid, and so also it may be shewn that the tangent planes at P, R also pass through the centre of the dotted ellipsoid.

Let fig. 4 represent a section through this ellipsoid and that with centre A by the tangent plane through Q, and let H be the centre of the dotted ellipsoid and HO the intersection of the tangent planes through P, R; p the point of contact of the two ellipsoids, and f the point where the ellipsoid with centre A touches a plane parallel to the plane of fig. 3. Then Af will be parallel to HO for they both bisect the chords through p which are parallel to AO, and AM, Af are conjugate semi-diameters.

Draw the ordinate pn parallel to Af.

Then since AH is bisected in p, $HO = 2pn$, and $AO = 2An$.

Also in fig. 3 o, M, A are the same points as O, M, A in fig. 4.

And $$AN = \tfrac{1}{2}AQ = \tfrac{3}{4}Ao, \text{ and } Ao \,.\, AN = AM^2;$$

therefore $$3AO^2 = 4AM^2,$$

and (in fig. 4) $$3An^2 = AM^2.$$

But $$\frac{pn^2}{Af^2} + \frac{An^2}{AM^2} = 1;$$

therefore $$\frac{pn^2}{Af^2} = 1 - \tfrac{1}{3} = \tfrac{2}{3},$$

and $$8Af^2 = 12pn^2 = 3HO^2,$$

and $$2\sqrt{2}Af = \sqrt{3}HO.$$

Now let the inclination of Af or HO to the plane of fig. 3 be α, then $HO \sin\alpha$ is the perpendicular distance between the plane ABC and the parallel plane through H. And the number of ellipsoids per unit volume is

$$\frac{1}{2\sqrt{3}AM \,.\, BQ} \times \frac{\sqrt{3}}{2\sqrt{2}Af \sin\alpha} \text{ or } \frac{1}{4\sqrt{2}AM \,.\, BQ \,.\, Af \sin\alpha}.$$

But $8AM \,.\, BQ \,.\, Af \sin\alpha$ is the volume of the parallelopiped whose sides touch the ellipsoid at the extremities of three conjugate diameters, which is equal to $8abc$, if a, b, c be the semi-axes of the ellipsoid.

Hence the number of ellipsoids per unit volume is on this arrangement, which gives the maximum number,

$$\frac{\sqrt{2}}{8abc} \text{ or } \frac{1}{4\sqrt{2}abc},$$

which is independent of the orientation of the axes.

We might have started with the arrangement indicated in fig. 5, in which each ellipsoid with its centre in the plane of the paper is touched at points in that plane by four others, and is also touched by four others, indicated in dotted outline, which have their centres in

Digitized by Google

a parallel plane above the plane of the paper, and also by other four in a similar position below the plane of the paper, twelve in all.

The four ellipsoids with centres in the plane of fig. 5 which touch any one with centre C must touch it at the extremities of conjugate diameters, if each (dotted) ellipsoid, with its centre in the plane above the figure, is to touch simultaneously four below it. The tangents at P, R will then be parallel to BC, AB respectively, and will meet in O the point of intersection of AC, BD, and we shall have

$$BO^2 + CO^2 = 2(OR^2 + BR^2) = 2(BM^2 + Cm^2).$$

Also, since $CO = 2PN$, and $BO = 2BN$, and $\frac{PN^2}{Cm^2} + \frac{BN^2}{BM^2} = 1$,

$$\frac{CO^2}{Cm^2} + \frac{BO^2}{BM^2} = 4;$$

and because the circumscribing parallelogram is double the inscribed parallelogram,

$$Cm \,.\, BM = 2PN \,.\, NB;$$

therefore $$\frac{PN}{Cm} = \frac{BM}{2NB} = \frac{BM}{BO}, \text{ and } \frac{4BM^2}{BO^2} + \frac{BO^2}{BM^2} = 4;$$

therefore $$\frac{2BM^2}{BO^2} = 1, \text{ and } \frac{BM}{BO} = \frac{1}{\sqrt{2}},$$

and by similar reasoning $$\frac{Cm}{CO} = \frac{1}{\sqrt{2}}.$$

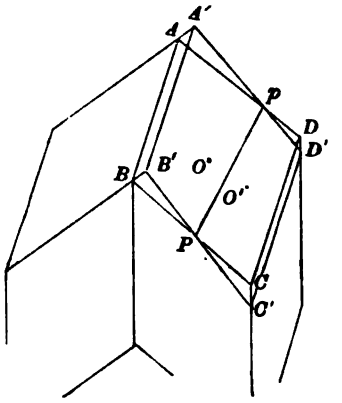
If now fig. 8 represent a section through the centre H of the ellipsoid which touches each of the four with centres A, B, C, D, the straight lines joining HB and HD will pass through the points of contact p, q, and be bisected in those points; and HO, which bisects BD, also bisects qp, and is the diameter to which qp is ordinate, and is parallel to Bt the diameter conjugate to BM. And by reasoning similar to that above given it may be shewn that $HO = \sqrt{2}Bt$. The same value for HO would have been found if we had started with supposing the dotted ellipsoid to touch these with centres A and C. Hence it will touch all four ellipsoids with centres A, B, C, D simultaneously.

This arrangement is really the same as before and will lead to the same result. In fact the plane through the centres of any two adjacent dotted ellipsoids in fig. 3 as o, s, which also passes through C, D, cuts the ellipsoids in ellipses arranged as in fig. 5; so does the plane through the centres of dotted ellipses o, w, and through B, L; and so on. While in fig. 5 the plane through the centres of the dotted ellipsoids O, S, which also passes through B cuts the ellipsoids in the way indicated in fig. 3; so does the plane through the centres of the dotted ellipses S, T, which passes through C; and so on.

Further, the hexagon in fig. 3 is the intersection of the plane of the figure with the tangent planes at the points of contact between the ellipsoids. These tangent planes form dodecahedrons circumscribing the ellipsoids; and since each side of the hexagon in fig. 3 is bisected in the point where it touches an ellipse, and the same is

Digitized by Google

true of the hexagons in the other three planes of section above mentioned, it appears that the points at which the faces of the dodecahedrons touch the ellipsoids are the centres of those faces. Now when this condition of touching the ellipsoid at the centres of the faces is satisfied the volume of the dodecahedron will be a minimum, inasmuch as any small shift of one of the planes forming the dodecahedron by rolling it on the surface of the ellipsoid must increase the volume of the figure. Thus, if *ABCD* be one of the faces of the dodecahedron, touching an ellipsoid at its centre *O*, and it be turned so as to touch the ellipsoid at *O'*, and the new position of the face *A'B'C'D'* intersect the old in *Pp*, *Pp* will lie between *O* and *O'*, and the area *ABPp* will be greater than the area *CPpD*, and the wedge *AB'p* will be greater than the wedge *C'PD* if the distance *OO'* be finite. Hence the circumscribing dodecahedron will have minimum volume when the ellipsoids are all similarly situated. And, since the dodecahedra fill space, it follows that there will be the greatest number of them in unit volume, and therefore the greatest number of ellipsoids in unit volume, when they are all similarly situated. Such an arrangement will therefore be assumed by the crystallizing molecules in consequence of their mutual attractions.



Now if we transfer our thoughts to the centres of the ellipsoids, we shall find that their arrangement, when the ellipsoids are situated as above described, will present different degrees of symmetry depending on the relative magnitudes of the axes of the ellipsoids and upon their orientation. For consider the ellipsoids indicated in continuous outline in fig. 5 as having their centres in the plane of the paper, those indicated in dotted outline with their centres in a plane above that of the paper, and again a third set with their centres, which we may call *b, c, d, e, f* &c. corresponding to *B, C, D, E, F* &c., but in a plane above the dotted set, then the plane of the paper and the parallel plane *bdef* together with the planes *BbdD*, *EefF*, *BbfF* and *DdeE*, will form a parallelopiped. And the whole space may be divided by parallel planes into similar and equal parallelopipeds which have the centre of an ellipsoid in every angular point and one in the centre of every face. Also every ellipsoid in the space will occupy one or other of these positions.

Also if we suppose *wos* in fig. 3 to be the centres of the dotted ellipsoids lying in a plane below that of the figure, *ynv* to be the centres of the broken line ellipsoids lying in a plane below *wos*, and *A'* to be the centre of the ellipsoid in a plane still lower which touches the three broken line ellipsoids: then the six planes *Aoyw*, *A'vsn*, *Awvs*, *A'yon*, *Asno*, *A'ywv* will form a parallelopiped. And the whole space can be divided by planes parallel to these into similar and equal parallelopipeds which have the centre of an ellipsoid in every angular point.

The characters of these parallelopipeds will differ according to the orientation of the axes of the ellipsoid, and according to the relative lengths of the axes.

In the most general case represented in figs. 3 and 5 each face of the parallelopiped

Digitized by Google

will be equal and similar to its opposite face but to no other, and the crystal with its molecules arranged in the same way as the ellipsoids will be anorthic.

If the plane of fig. 5 be a plane of principal section of the ellipsoids the edges *Bb*, *Dd*, *Ee*, *Ff* of the parallelopiped will be parallel to an axis and perpendicular to the plane *BDEF*. The crystal will in this case belong to the oblique system, if the points of contact in the plane *BDEF* be not the extremities of axes. But if the points of contact *P*, *R*, are the extremities of axes, the crystal having such a molecular arrangement will belong to the right prismatic system.

Also in the parallelopiped of which *A* in fig. 3 is one angular point, the edge *As* will for certain orientations of the axes of the ellipsoid be perpendicular to the plane *Aoyw*, and the crystal will be oblique. Further for certain orientations of the axes the edges *Ao*, *Aw* will also be equal to one another, or all three edges may be at right angles to one another, and the crystal will be right prismatic.

If two of the axes of the ellipsoids be equal to one another, so that the ellipsoids become spheroids, and if the axis of revolution be perpendicular to the plane of fig. 5, the arrangement of molecules will correspond to the pyramidal system, the axis of the spheroids being the axis of symmetry.

If the axis of the spheroids be perpendicular to a plane such as *Dbs* through the diagonals of three adjacent faces of the parallelopiped, the parallelopiped is a rhombohedron, and the crystal will belong to the rhombohedral system. The axis of revolution is in this case perpendicular to the plane of fig. 3.

In each of the last two cases there are certain relations between the length of the axis of the spheroid and the diameter of the principal circular section which give a cubic arrangement.

If in fig. 5 the ellipsoids become spheroids with their axes of revolution perpendicular to the plane of the figure, figure 2 will represent them, and consider the four with centres *c'*, *c*, *d*, *d'* together with that which touches them, indicated in dotted outline, with centre *e*, and let fig. 6 represent a section of them through *ced'*, which will be in a plane perpendicular to the plane of fig. 2. Then if the radius of the principal circular section be *a*, and *b* be the semi-axis of revolution, and we take $a = \sqrt{2}b$, we have in fig. 6,

$$cN = \frac{cd'}{4} = \frac{a}{\sqrt{2}}, \text{ and } \frac{pN^2}{b^2} + \frac{cN^2}{a^2} = 1;$$

therefore
$$\frac{pN^2}{b^2} = 1 - \tfrac{1}{2} = \tfrac{1}{2},$$

and
$$\sqrt{2}pN = b = \frac{a}{\sqrt{2}},$$

and
$$pN = \frac{a}{2},$$

and
$$eQ = 2pN = a,$$

and *e* is the centre of the cube described upon the square *c'cdd'*.

Digitized by Google

And if the whole space be divided into such cubes there will be one spheroid at every corner of each cube and one at the centre of each cube and none in any other position. Hence the crystal will have its external form in the cubic system, and in this arrangement the planes of the rhombic dodecahedron will be those of greatest condensation of molecules and the crystal will cleave most easily in those directions. It may be doubted whether such a crystal would be truly isotropic, but the want of that character in the molecules themselves need not, and for reasons given below I think it does not, influence light passing through the crystals so as to produce double refraction.

Again, if in fig. 1 the different circles represent spheroids with their centres in different planes; those in continuous outline with their centres in the plane of the paper; those in dotted outline next above them; those in broken outline next below those in continuous outline; then the planes through the centres $c'da'g$, $c'a'be$, and $c'bdf$ will each be at right angles to the other two if the radius of the principal circular section be double the semi-axis of revolution. For let fig. 7 represent a section through $c'de$, and let a be the radius dM of the principal circular section, b the semi-axis of revolution, $c'Q$ and pn perpendiculars on dM. Then if the three planes through c' intersect one another at right angles we shall have $dc'^2 = 2a^2$, also Q will be the centre of the equilateral triangle $da'b$, with sides $2a$, and $dQ = \frac{2}{\sqrt{3}}a$. Also $c'd$ is bisected in the point of contact p, and dp, dn are halves of dc', dQ respectively, and

$$pn^2 = dp^2 - dn^2 = \frac{a^2}{2} - \frac{a^2}{3} = \frac{a^2}{6}.$$

Also $\frac{pn^2}{b^2} + \frac{dn^2}{a^2} = 1$, and therefore $\frac{pn^2}{b^2} = 1 - \frac{1}{3} = \frac{2}{3}$, and $\frac{a^2}{6} = \frac{2b^2}{3}$ or $a^2 = 4b^2$, and $a = 2b$.

Also if space be everywhere divided by planes parallel to the three planes above mentioned passing through the centres of the spheroids, it will be divided into cubes with the centre of a spheroid at every angle. The external form of a crystal with such molecules so aggregated will be of the cubic system, and the most easily obtained cleavage will be cubic. It may be doubted whether it would be truly isotropic on account of the want of this quality in the molecules individually. It is however by no means certain that the optical characters of crystals depend directly on those of the molecules. For double refraction wholly disappears when the crystals are fused or dissolved in menstrua. Moreover the motions which give the molecular volume are those which produce expansion when the substance is heated, and are probably distinct from those vibrations which produce radiation in solids as well as in gases. It is therefore quite possible that these two cases in which the arrangement is such as to make the crystal belong externally to the cubic system may be represented by blende with its predominant dodecahedral cleavage, and by rock salt and galena with predominant cubic cleavages.

The ordinary laws of mechanics require that the molecules should, when in the most stable equilibrium, be packed as closely as possible, and this appears to give the arrangement above indicated and to determine that all the molecules shall be similarly situated. But it does not determine the orientation of the axes of the ellipsoids which represent the

Digitized by Google

molecular volumes, on which the system of crystallization depends. That must be due to the mutual action of the molecules, and, as I have said above, I regard it as dependent on the massing of the matter in the molecular volume. If the mass is chiefly aggregated about any particular diametral plane one would expect that plane to be the plane of fig. 3 in which each molecule meets six others while it meets only three in a plane above and three in a plane below that of the figure. While if the mass is aggregated chiefly about any one diametral line we should expect the diametral plane conjugate to it to assume the position of the plane of fig. 5, since the molecules are symmetrically arranged about a diameter conjugate to the diametral planes represented in the figure, and the position is that which will bring the masses as close as possible. I assume that the influence of the immediately contiguous molecules is much greater than that of those more distant, as we know experimentally that this is the case in capillary actions.

To find the relation between the length of the axis of revolution and the diameter of the principal section in any given crystal of the rhombohedral system, if $c't$ in fig. 7 be the trace of the plane through the centres $c'a'b$ in fig. 1 the angle $c'tQ$ will be the angle between the axis of the rhombohedron and the normal to one of the adjacent faces, that is the angular element of the crystal; and if we call this angle D,

$$\tan D = \frac{c'Q}{Qt} = \frac{2pn}{dn} = \frac{2\sqrt{2}b}{a}.$$

Also the perpendicular distance between successive planes passing through the centres of spheroids and parallel to a face of the rhombohedron will be

$$dt \sin D, \quad \text{or} \quad \frac{2\sqrt{2}\sqrt{3}ab}{\sqrt{8b^2+a^2}}.$$

The distance between successive planes perpendicular to the axis of the rhombohedron, $c'Q$, is $\frac{2\sqrt{2}b}{\sqrt{3}}$.

And the distance between successive planes parallel to the axis is Qt, or $\frac{a}{\sqrt{3}}$.

The first of these three distances will be greatest if a is greater than b but not greater than $8b$, and in this case the principal cleavage will be rhombohedral.

The second of the distances will be greatest when b is greater than a, in which case the principal cleavage will be perpendicular to the axis.

The third distance will be greatest when a is greater than $8b$, in which case the principal cleavage will be parallel to the axis.

It is obvious that the value of D, and consequently the ratio between a and b, will depend on the choice of the rhombohedron which is taken as the fundamental one.

For calcite we have for the cleavage rhombohedron

$$D = 44^\circ . 36' . 36'' \quad \text{and} \quad a : b = 1 : \cdot 34877.$$

Digitized by Google

For quartz, taking the usual value for D, namely 51°. 47′, we find $a : b = 1 : 4490$, which makes the fundamental rhombohedron the form of least surface tension, and the six-sided prism $(01\bar{1})$ comes next to it.

I have now, I hope, sufficiently explained the principles which determine the regular aggregation of the molecules in assuming the solid from the liquid state, and how this leads up to the system of crystallization and the predominant forms and cleavages.

There is, however, one arrangement of molecules, suggested in the first part of this communication to explain the forms of hexagonal crystals, which is not accounted for by the principle here laid down. This is the arrangement at the angular points of right triangular prisms. If there were any good reasons for thinking that the molecules would in consequence of their mutual action so place themselves, the facts of hexagonal crystallization would be adequately accounted for. But such an arrangement does not appear to me necessary for the explanation of those facts. Whenever there is no great difference between the surface tension of faces of different forms, there must always be a tendency for the crystal to assume such a combination as will approach to a globular form, as we see in hexagonal crystals. And there is no instance, so far as I am aware, in which a crystal cleaves into a six-sided pyramid. The cleavages in this system are either rhombohedral, or are perpendicular to, or parallel to, the axis, and are adequately accounted for by the rhombohedral arrangement of molecules.

It appears from this that dimorphous substances in their different forms differ in the relative dimensions, or in the orientation, of their molecular volumes. For instance the spheroidal molecules of calcite either have the direction of their axes altered, or become ellipsoidal when the material takes the form of aragonite. Now of all the circumstances which help to determine which of two crystalline forms a dimorphous substance shall assume temperature seems the most important. A change of temperature may alter the relative as well as the absolute distance of the excursions of the parts of the molecule from the centre of mass, and thus alter the relative dimensions of the molecular volume. It may also affect the average distribution of the mass in the molecular volume and so alter the orientation, of the axes of the ellipsoid which represents it. But changes of this sort must, except for the constraint of the solid state, be reversed when the temperature comes back to its former degree, and in such a case the altered crystalline form must be an unstable one. This is no doubt the case with some substances like sulphur which take one form when crystallized from fusion and a different form when crystallized from solution at a lower temperature. The former crystals are always unstable and become split and cracked by the change which the molecules undergo when cooled. Something more seems wanted to account for such permanent difference as we find between aragonite and calcite. For this I look to the aggregation of matter in the individual molecule. Increase of temperature tends to multiply the number of molecules into which a given mass is divided. We know that it is so in gases, and it may be so in solutions. The number of particles of calcium carbonate each containing one chemical atom of carbon may be greater in a molecule of aragonite than in a molecule of calcite. Each molecule is almost certainly a complicated aggregate containing many chemical atoms of calcium and carbon, and it is

Digitized by Google

not unreasonable to suppose that the complication depends on the temperature. There is however no reason to think that this aggregation can be at all readily changed when the substance is solid, so that the crystals formed at one temperature will be permanent at other temperatures within certain limits. The change of the specific gravity of aragonite to that of calcite by a red heat seems to favour this view, though calcium carbonate crystallizes as calcite from cold solutions as well as from fusion, while it takes the form of aragonite when crystallizing at some intermediate temperatures.

It seems now possible to acquire a knowledge of the form of the average molecular volume in the case of those substances of which the crystallographic elements are known. To work out the details will be a matter of time, but I think it important that this should be done as it seems to take us a step further in our knowledge of the nature of molecules and chemical combination.

APPENDIX.

HISTORICAL NOTE.

M. P. CURIE, in a short paper published in the *Bull. de la Société Minéralogique de France*, Vol. VIII. p. 145, points out the influence of capillarity on the growth of crystals, and lays down, as I have done, the principle that the development, in order to be stable, must be such that the surface tension is on the whole a minimum. He applies the principle only to the determination of the relative magnitudes of the faces of a crystal which have different constants of capillarity. There is enough in the paper to have suggested the line of thought pursued in the first part of this communication. I had not, however, seen or heard of M. Curie's paper when my own was written, or I should have mentioned it in connexion with my investigation.

M. Bravais long ago investigated the geometrical properties of a system of points regularly distributed in space, and the degrees of symmetry belonging to several arrangements, and supposed that in a crystal of given symmetry the molecules must be placed in one or other of the arrangements which have the same degree of symmetry as the crystal. I ought, no doubt, to have been acquainted with M. Bravais' investigation. He has anticipated me in some points. The arrangements of molecules which I have suggested are all, I think, to be found amongst those he has described. He has also pointed

Digitized by Google

out that the density of the molecules in any plane will be inversely proportional to the perpendicular distance between that plane and the next parallel planes of the set; and concluded that cleavage will be most easily obtained in planes in which the molecular density is greatest. His reason for this conclusion differs from mine. He thinks that cohesion in a direction normal to a set of parallel planes will be less when the perpendicular distance between successive planes of the set is greater. Whether that is so or not depends on the law of force according to which the molecules attract one another. It is probable that it is so, and experiments have shewn that cohesion in rock-salt and some other crystals is less in a direction normal to a plane of cleavage than in other directions. The experiments however do not seem to me conclusive, because in rock-salt a direction normal to one plane of cleavage is parallel to two other planes of cleavage in which, according to M. Bravais, the cohesion should be a maximum. The cohesion, per unit of area, between two planes will increase when the number of molecules per unit of area is increased. And my view of the importance of surface tension in relation to cleavage is borne out by observation that when an amorphous solid is broken by a blow, or crushed, the direction of fracture is approximately that of compression. I have lately met with instances of this in the fracture of the conical quartz stoppers used by Prof. Dewar and myself to close the ends of tubes into which gases were compressed. These stoppers were fitted into conical openings in the gun-metal ends of the tubes, and for want of exact fitting sometimes obtained a bearing on only a narrow ring of the hollow metallic cone. When the tube came to be filled with gas at a pressure of 200 or 300 atmospheres, the pressure on the quartz in the plane of the ring became enormous, and when the quartz has cracked it has done so in the plane of the ring approximately. M. Bravais' idea of hemihedry depending on the polyhedral form of the molecules is essentially different from the dynamical notions which I have advanced on that subject. M. Bravais also supposed that the internal movements which are going on when a crystal is growing play somewhat the same sort of part in regard to the growth of faces which external forces play in regard to cleavage, and tend to develope faces parallel to planes in which the molecular density is greatest. Professor Sohncke has extended M. Bravais' investigations of the geometrical properties of systems of points regularly disposed in space, so as to include the more complicated cases arising from the superposition of several simple systems; but neither of these investigators have, I believe, attempted, as I have done, to explain on mechanical grounds why the molecules should assume any such arrangement. I am unable to find any mechanical reasons why the molecules should assume the more complicated arrangements. Sohncke has pointed out (*Zeitschrift für Krystallographie*, 1888, p. 218) that in some of the systems of regularly disposed points (not in the most simple) the greatest distance between the successive planes of a set of parallel planes does not always correspond to the greatest molecular density in those planes, and that this will introduce some modification of Bravais' laws as to the growth of faces and as to cleavages. He also refers to the influence of capillarity as affecting those laws; yet he does not appear to notice the influence of capillarity in regard to cleavage, but only in the development of the natural faces of crystals. He does not notice that the constant of capillarity of any plane surface will depend, not only on the closeness with which the molecules are set in

Digitized by Google

that plane, but also on their distances from the molecules in adjoining parallel planes. I regard it as only a first approximation to say that the constant of capillarity will decrease with increasing molecular density in the surface; and it should be observed that the constant will be different in a solution in which the crystal may be growing from what it will be in air.

The notion that the molecules might be represented by ellipsoids is not new. Brewster has suggested something of the same kind, and Dana in his *Manual of Mineralogy*, 3rd edition, has pointed out that Haüy's parallelopipeds might be replaced by the inscribed ellipsoids. My idea of the molecules is essentially a kinetic one, in which the ellipsoid represents the average molecular volume, not a solid mass. Moreover Dana's arrangement of ellipsoids, each touched by only six others, will not satisfy the mechanical requirements which I have formulated. At the same time it may be said that there is a certain resemblance between his notion of polar forces acting chiefly at the extremities of conjugate diameters and determining the similarity of position of all the ellipsoids, and my notion of the distribution of the matter in the molecule determining the orientation of the axes of the ellipsoids though not the similarity of their situation.

Although I have been anticipated in many points I hope I have added something to the mechanical theory of crystals which may lead to further development on that side of the subject.

CAMBRIDGE: PRINTED BY C. J. CLAY, M.A. & SONS, AT THE UNIVERSITY PRESS.

Digitized by Google

Digitized by Google

XV. *Systems of Quaternariants that are algebraically complete.* By A. R. Forsyth, M.A., F.R.S., Fellow of Trinity College, Cambridge.

[Read *February* 11, 1889.]

The aim of this memoir is to obtain for certain quaternary quantics the respective systems of concomitants which are algebraically complete, that is, are such that every concomitant of a quantic can be expressed as an algebraical (but not necessarily nor generally an integral) function of the members of the system appertaining to that quantic. The method and the course of development are similar to those in the corresponding investigations relating to ternariants*, in the present case complicated by the presence of the six (non-independent) line-variables.

It is shewn that the characteristic equations satisfied by quaternariants† can be reduced to twelve linear partial differential equations of the first order, which are independent of one another in form; and that these twelve can be reduced to six of them, properly chosen and absolutely independent of one another. The leading coefficient of a quaternariant satisfies three linear partial differential equations, also of the first order; and, when obtained, it uniquely determines the quaternariant by being symbolised into the umbral elements of the coefficients of the quantic—this result being a consequence of a theorem proved that every quaternariant is expressible as an aggregate of symbolic factors of some of five forms.

The number of quaternariants in an algebraically complete system is $N-5$, where N is the number of coefficients in the most general form of the quantic (or of the set of simultaneous quantics) with which the system is associated. A method is indicated by which the leading coefficients of these $N-5$ quaternariants can be obtained as combinations of binariants, which belong to binary quantics derivable from the original quantic. And for those quaternariants, which do not involve line-variables and which belong to unipartite quantics in point-variables, it is shewn that their leading coefficients can be expressed as contravariants (and invariants) of ternary quantics derivable from the original quantic.

The general theory thus indicated is applied to obtain the special results for the following cases: (i) a quadratic, and (ii) two quadratics, in point-variables; (iii) a quantic lineo-linear in point- and plane-variables; (iv) a linear complex; (v) a congruence of

* "Systems of Ternariants that are algebraically complete", *Amer. Journ. of Math.*, vol. xii. (1889).

† I have used the word quaternariant to denote the generic concomitant of the quaternary quantic instead of its leading coefficient, as suggested by Sylvester, *Amer. Journ. of Math.*, vol. v., p. 81.

Digitized by Google

two linear complexes; (vi) a regulus of three linear complexes; and (vii) a quadratic complex.

In regard to quaternariants which are pure invariants or pure covariants Salmon's *Geometry of Three Dimensions* may be consulted, in which there are frequent references to Sylvester and Clebsch; but except the quaternariant which is the complex of lines touching a surface there are hitherto, I think, few cases of quaternariants which involve line-variables. The bibliography of the lineo-linear quantic is given elsewhere*. For complexes the most important references, in addition to the investigations of Cayley and Plücker, are to be found at the end of Klein's memoir (all cited in the note to § 1 of this memoir); most of the results, so far as they concern the theory of forms, relate to the canonisation of the complex of the second grade.

GENERAL THEORY.

The Differential Equations of the Concomitants.

1. In the discussion of the concomitants of quaternary quantics the ordinary (point) variables will be denoted by x_1, x_2, x_3, x_4; the linear transformations to which they are subject will be taken to be

$$x_1, x_2, x_3, x_4 = \left(\begin{array}{cccc} 1, & l_2, & l_3, & l_4 \\ m_1, & 1, & m_3, & m_4 \\ n_1, & n_2, & 1, & n_4 \\ k_1, & k_2, & k_3, & 1 \end{array} \right) (X_1, X_2, X_3, X_4).$$

The variables (plane) contragredient to these will be denoted by u_1, u_2, u_3, u_4; and they are subject to the transformations

$$U_1, U_2, U_3, U_4 = \left(\begin{array}{cccc} 1, & m_1, & n_1, & k_1 \\ l_2, & 1, & n_2, & k_2 \\ l_3, & m_3, & 1, & k_3 \\ l_4, & m_4, & n_4, & 1 \end{array} \right) (u_1, u_2, u_3, u_4).$$

The coefficients in the unit power of the symbolical quantity $a_x = (a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4)$ are transformed by the same relations as the quantities u.

The six line-variables will be denoted by $p_1, p_2, p_3, p_4, p_5, p_6$. If y_1, y_2, y_3, y_4; z_1, z_2, z_3, z_4 be two sets of point-variables, that is, variables cogredient with the variables x, the line-variables are defined by the equations

$$\begin{aligned} p_1 &= y_2z_3 - y_3z_2, & p_6 &= y_1z_4 - y_4z_1, \\ p_2 &= y_3z_1 - y_1z_3, & p_5 &= y_2z_4 - y_4z_2, \\ p_3 &= y_1z_2 - y_2z_1, & p_4 &= y_3z_4 - y_4z_3, \end{aligned}$$

* "Systems of Ternariants" (already cited), § 60.

Digitized by Google

connected by a permanent relation

$$p_1 p_6 + p_2 p_5 + p_3 p_4 = 0\,^*.$$

The transformations, to which the line-variables are subject in consequence of the variations of the point-variables, are not linear, being

$$p_1, p_2, p_3, p_4, p_5, p_6 =$$

$$\left(\begin{array}{cccccc} 1-m_3n_3, & -m_1+m_3n_1, & -n_1+m_1n_3, & -m_4+m_3n_4, & n_4-m_4n_3, & m_1n_4-n_1m_4 \\ -l_3+l_3n_3, & 1-l_3n_1, & -n_3+n_1l_3, & l_4-n_4l_3, & n_3l_4-n_4l_3, & -n_4+n_1l_4 \\ -l_3+l_2m_3, & -m_3+l_3m_1, & 1-m_1l_2, & l_3m_4-l_4m_3, & -l_4+l_2m_4, & m_4-m_1l_4 \\ -k_2+n_2k_3, & k_1-n_1k_3, & n_1k_2-n_2k_1, & 1-n_4k_3, & n_3-n_4k_3, & n_1-n_4k_1 \\ k_3-m_3k_2, & k_1m_3-m_1k_3, & -k_1+m_1k_3, & m_3-k_3m_4, & 1-m_4k_3, & m_1-k_1m_4 \\ l_2k_3-l_3k_2, & -k_3+l_3k_1, & k_3-l_2k_1, & l_3-l_4k_3, & l_3-l_4k_3, & 1-l_4k_1 \end{array}\right)(P_1, P_2, P_3, P_4, P_5, P_6).$$

This is not the only definition of the line-variables; for they can be constructed from two sets of plane-variables v_1, v_2, v_3, v_4 and w_1, w_2, w_3, w_4 in the forms

$$\begin{aligned} p_1 &= v_1w_4 - v_4w_1, & p_6 &= v_2w_3 - v_3w_2, \\ p_2 &= v_2w_4 - v_4w_2, & p_5 &= v_3w_1 - v_1w_3, \\ p_3 &= v_3w_4 - v_4w_3, & p_4 &= v_1w_2 - v_2w_1, \end{aligned}$$

these being connected by the same permanent relation as before.

* The following table gives a comparison between some of the notations most frequently used for line variables:—

p	q	r	u	t	s	Cayley, *Quart. Journ. Math.*, vol. iii., pp. 225—236.
l	m	n	r	q	p	Cayley, *Proc. Lond. Math. Soc.*, vol. iv., p. 16.
a	b	c	h	g	f	Cayley, *Camb. Phil. Trans.*, vol. xi., pp. 290—323.
x_6	$-x_2$	x_1	x_4	$-x_5$	x_3	Lüroth, *Crelle*, t. lxvii., pp. 130—152.
p_{23}	$-p_{13}$	p_{12}	p_{34}	$-p_{42}$	p_{14}	Weiler, *Math. Ann.*, t. vii., pp. 146—207.
p_6	$-p_2$	p_1	p_4	$-p_3$	p_5	Voss, *Math. Ann.*, t. x., p. 169.
p_6	$-p_2$	p_1	p_4	$-p_5$	p_3	Klein, *Math. Ann.*, t. xxiii., pp. 539—578.
p_1	p_2	p_3	p_4	p_5	p_6	Notation used in this paper.

Since only the ratios of the six coordinates are necessary to determine any line, the given definitions imply the existence of an unspecified factor common to all the coordinates. Instead of six, Plücker (in his "New Geometry of Space", *Phil. Trans.* 1865, pp. 725—791, and his "Neue Geometrie des Raumes" 1868) takes the quantities

$$(\sigma = -p_1 \div,\ \rho = p_2 \div,\ s = p_5 \div,\ r = p_6 \div,\ \eta = p_3 \div)\, p_4$$

the common denominator being p_4; and the permanent relation is $\eta = r\sigma - s\rho$.

The introduction of Plücker's quantities into concomitants would leave only a partial symmetry and would destroy the homogeneity: so that for the present purpose it is preferable to retain some set of six quantities. I should have adopted the set (a, b, c, f, g, h) as these occur very frequently: but, as there is the disadvantage that all these symbols occur in the usual notation for quadratics, I adopted the system $(p_1, p_2, p_3, p_6, p_5, p_4$, the permanent relation being homogeneous in subscript indices) in the text, so as to have the leading term of a concomitant that which involves x_1, u_1, and p_1.

Digitized by Google

Strictly, only the ratios of the last six quantities are equal to the same ratios of the first six quantities. But, as will be seen in § 5, we only retain in the variational equations terms which are of the first order in the coefficients of transformation; and as, up to this order, the six quantities in the first definition are equal to the six quantities in the second save as to a factor which is unvarying, we are justified for the present purpose in making the corresponding quantities equal, instead of merely proportional, to one another.

2. The fundamental quantic, when taken to be unipartite and of order μ in the point-variables, may be represented in the symbolical form a_x^μ or in the explicit form

$$\Sigma \frac{\mu!}{q!\,r!\,s!\,t!} a_{qrst} x_1^q x_2^r x_3^s x_4^t,$$

with the condition $q+r+s+t=\mu$; and the real coefficients are

$$a_{qrst} = a_1^q a_2^r a_3^s a_4^t$$

when expressed in terms of the umbral coefficients. From the last equation the coefficients of the new quantic are deduced when the linear transformations in the variables are applied.

3. The most general concomitant possible is a function ϕ of the coefficients of the quantic and of the different kinds of variables such that, when the same function Φ is formed in the transformed coefficients and in the transformed variables, the equation

$$\Phi(A, X, P, U) = \Delta^\theta \phi(a, x, p, u)$$

is satisfied, Δ being the determinant of transformation and θ an integer which is the index of the concomitant.

When in this equation we substitute for A and U in terms of a and u and for x and p in terms of X and P, it comes to be an identity; and, when the two sides are expanded in the coefficients of the transformation, the terms involving the same combinations of those coefficients on the two sides are equal to one another. Using the expansion by Taylor's theorem, we thus obtain from each such combination a differential equation satisfied by the function ϕ of a, X, P, u identically; after it has been obtained, we may replace X and P by x and p respectively. Also since

$$\Delta = \begin{vmatrix} 1, & l_2, & l_3, & l_4 \\ m_1, & 1, & m_3, & m_4 \\ n_1, & n_2, & 1, & n_4 \\ k_1, & k_2, & k_3, & 1 \end{vmatrix},$$

there are no terms in Δ of the first order in the coefficients of the linear transformation, and the only terms of the second order are those which involve the combinations m_3n_2, l_3n_1, m_1l_2, n_4k_3, m_4k_2, l_4k_1.

4. Taking then the coefficients of the various combinations of the coefficients in the covariantive equation, we have $\phi = \phi$ as derived from the terms which are independent

Digitized by Google

of those coefficients; and from the terms which are of the first order we have the following twelve characteristic equations:

From the coefficient of

$$\left.\begin{array}{lll}
l_2 & \text{the equation} & L_2 + u_1 \dfrac{\partial}{\partial u_2} = x_2 \dfrac{\partial}{\partial x_1} - p_1 \dfrac{\partial}{\partial p_2} + p_5 \dfrac{\partial}{\partial p_6} \\
l_3 & \text{.....} & L_3 + u_1 \dfrac{\partial}{\partial u_3} = x_3 \dfrac{\partial}{\partial x_1} - p_1 \dfrac{\partial}{\partial p_3} + p_4 \dfrac{\partial}{\partial p_6} \\
l_4 & \text{.....} & L_4 + u_1 \dfrac{\partial}{\partial u_4} = x_4 \dfrac{\partial}{\partial x_1} - p_5 \dfrac{\partial}{\partial p_3} + p_4 \dfrac{\partial}{\partial p_2} \\
m_1 & \text{.....} & M_1 + u_2 \dfrac{\partial}{\partial u_1} = x_1 \dfrac{\partial}{\partial x_2} - p_2 \dfrac{\partial}{\partial p_1} + p_6 \dfrac{\partial}{\partial p_5} \\
m_3 & \text{.....} & M_3 + u_2 \dfrac{\partial}{\partial u_3} = x_3 \dfrac{\partial}{\partial x_2} - p_3 \dfrac{\partial}{\partial p_3} + p_4 \dfrac{\partial}{\partial p_5} \\
m_4 & \text{.....} & M_4 + u_2 \dfrac{\partial}{\partial u_4} = x_4 \dfrac{\partial}{\partial x_2} - p_4 \dfrac{\partial}{\partial p_1} + p_6 \dfrac{\partial}{\partial p_3} \\
n_1 & \text{.....} & N_1 + u_3 \dfrac{\partial}{\partial u_1} = x_1 \dfrac{\partial}{\partial x_3} - p_3 \dfrac{\partial}{\partial p_1} + p_6 \dfrac{\partial}{\partial p_4} \\
n_2 & \text{.....} & N_2 + u_3 \dfrac{\partial}{\partial u_2} = x_2 \dfrac{\partial}{\partial x_3} - p_3 \dfrac{\partial}{\partial p_2} + p_5 \dfrac{\partial}{\partial p_4} \\
n_4 & \text{.....} & N_4 + u_3 \dfrac{\partial}{\partial u_4} = x_4 \dfrac{\partial}{\partial x_3} - p_6 \dfrac{\partial}{\partial p_2} + p_5 \dfrac{\partial}{\partial p_1} \\
k_1 & \text{.....} & K_1 + u_4 \dfrac{\partial}{\partial u_1} = x_1 \dfrac{\partial}{\partial x_4} - p_3 \dfrac{\partial}{\partial p_5} + p_2 \dfrac{\partial}{\partial p_4} \\
k_2 & \text{.....} & K_2 + u_4 \dfrac{\partial}{\partial u_2} = x_2 \dfrac{\partial}{\partial x_4} - p_1 \dfrac{\partial}{\partial p_4} + p_3 \dfrac{\partial}{\partial p_6} \\
k_3 & \text{.....} & K_3 + u_4 \dfrac{\partial}{\partial u_3} = x_3 \dfrac{\partial}{\partial x_4} - p_2 \dfrac{\partial}{\partial p_6} + p_1 \dfrac{\partial}{\partial p_5}
\end{array}\right\} \text{.....(I)},$$

the literal operators L, M, N, K being

$$\left.\begin{array}{l}
L_2 = \Sigma r a_{q+1,\, r-1,\, s,\, t} \dfrac{\partial}{\partial a_{qrst}} \\
L_3 = \Sigma s a_{q+1,\, r,\, s-1,\, t} \dfrac{\partial}{\partial a_{qrst}} \\
L_4 = \Sigma t a_{q+1,\, r,\, s,\, t-1} \dfrac{\partial}{\partial a_{qrst}}
\end{array}\right\}, \qquad
\left.\begin{array}{l}
M_1 = \Sigma q a_{q-1,\, r+1,\, s,\, t} \dfrac{\partial}{\partial a_{qrst}} \\
M_3 = \Sigma s a_{q,\, r+1,\, s-1,\, t} \dfrac{\partial}{\partial a_{qrst}} \\
M_4 = \Sigma t a_{q,\, r+1,\, s,\, t-1} \dfrac{\partial}{\partial a_{qrst}}
\end{array}\right\},$$

$$\left.\begin{array}{l}
N_1 = \Sigma q a_{q-1,\, r,\, s+1,\, t} \dfrac{\partial}{\partial a_{qrst}} \\
N_2 = \Sigma r a_{q,\, r-1,\, s+1,\, t} \dfrac{\partial}{\partial a_{qrst}} \\
N_4 = \Sigma t a_{q,\, r,\, s+1,\, t-1} \dfrac{\partial}{\partial a_{qrst}}
\end{array}\right\}, \qquad
\left.\begin{array}{l}
K_1 = \Sigma q a_{q-1,\, r,\, s,\, t+1} \dfrac{\partial}{\partial a_{qrst}} \\
K_2 = \Sigma r a_{q,\, r-1,\, s,\, t+1} \dfrac{\partial}{\partial a_{qrst}} \\
K_3 = \Sigma s a_{q,\, r,\, s-1,\, t+1} \dfrac{\partial}{\partial a_{qrst}}
\end{array}\right\},$$

in which the summations extend to all integral combinations such that $q + r + s + t = \mu$, the order of the fundamental quantic.

Digitized by Google

5. Since the transformations which are applied to the x-variables are linear, it is to be expected that the foregoing twelve equations arising from the linear terms in the covariantive relation will be sufficient for the determination of the concomitant as a function of x (and also of u). But because the transformations which are applied to the p-variables are not linear, it at first appears as if the foregoing equations would not be sufficient for the determination of the concomitant as a function of p; for the full variation of the variables p is not given by the linear coefficient-combinations. This inference as to insufficiency is not however justified; for, though those linear coefficient-combinations do not *give* the full variation of the quantities p, they *determine* it uniquely. And, when the equations derived from a comparison of the terms in coefficient-combinations of the second order are formed, they are found to be either quasi-homogeneous equations—satisfied in virtue of the isobaric homogeneity of the concomitants—or equations which are derivable from the twelve characteristic equations earlier obtained. The verification of these statements is not difficult: thus for instance the equation which arises from a comparison of the terms involving $m_1 n_2$ can easily be derived from the two equations which arise from the terms involving m_1 and n_2.

The general result therefore is that *the twelve equations arising from the terms of the first order in the coefficients of transformation in the covariantive relation form an implicitly complete set of characteristic equations* *.

6. Though all the characteristic equations can be derived from the set of twelve retained, it does not follow that these twelve are independent of one another. As a matter of fact, they can easily be reduced to six equations in the following manner.

Let any one of the equations be written in the form $P = P'$, so that for instance

$$L_2' = x_2 \frac{\partial}{\partial x_1} - u_1 \frac{\partial}{\partial u_2} - p_1 \frac{\partial}{\partial p_2} + p_5 \frac{\partial}{\partial p_6};$$

and let any other be written in the form $Q = Q'$, it being evident that any one of the twelve literal operators is commutative with any one of the twelve variable-operators. Thus for any function ϕ which satisfies both equations, we have

$$P\phi = P'\phi, \qquad Q\phi = Q'\phi.$$

Hence
$$\begin{aligned}(QP - PQ)\,\phi &= (QP' - PQ')\,\phi \\ &= (P'Q - Q'P)\,\phi \\ &= (P'Q' - Q'P')\,\phi.\end{aligned}$$

In every case therefore in which P and Q are not commutative operators, this last equation is different in form from either of the preceding equations and is not evanescent; and it is satisfied in virtue of those two equations.

On forming these equations for the different combinations, it appears that the operators L_3, L_4, N_2, N_4, K_2, K_3 are all commutative with L_2 and similarly for the variable-operators; the corresponding derived equations are therefore evanescent. The equation constructed from

* The equations agree with those given in a paper by the author in the *Proc. Lond. Math. Soc.*, vol. xix. (1887), p. 42.

Digitized by Google

$L_2 = L_2'$, $M_1 = M_1'$ is a quasi-homogeneous equation satisfied in virtue of the isobaric homogeneity of the concomitant. But for the others

$$\begin{aligned} L_2M_3 - M_2L_3 &= L_3, & M_3'L_2' - L_2'M_3' &= L_3'; \\ L_2M_4 - M_4L_2 &= L_4, & M_4'L_2' - L_2'M_4' &= L_4'; \\ N_1L_2 - L_2N_1 &= N_2, & L_2'N_1' - N_1'L_2' &= N_2'; \\ K_1L_2 - L_2K_1 &= K_2, & L_2'K_1' - K_1'L_2' &= K_2'; \end{aligned}$$

so that four of the equations can be derived by combining in the above manner the equation $L_2 = L_2'$ with others of the equations.

The aggregate of these relations* is:

$$\left.\begin{aligned}
L_2 &= L_2N_2 - N_2L_2 = L_4K_3 - K_3L_4 \\
L_3 &= L_2M_3 - M_3L_2 = L_4K_3 - K_3L_4 \\
L_4 &= L_2M_4 - M_4L_2 = L_3N_4 - N_4L_3 \\
M_1 &= M_2N_1 - N_1M_2 = M_4K_1 - K_1M_4 \\
M_3 &= M_1L_3 - L_3M_1 = M_4K_3 - K_3M_4 \\
M_4 &= M_3N_4 - N_4M_3 = M_1L_4 - L_4M_1 \\
N_1 &= N_2M_1 - M_1N_2 = N_4K_1 - K_1N_4 \\
N_3 &= N_1L_3 - L_3N_1 = N_4K_3 - K_3N_4 \\
N_4 &= N_1L_4 - L_4N_1 = N_3M_4 - M_4N_3 \\
K_1 &= K_2M_1 - M_1K_2 = K_3N_1 - N_1K_3 \\
K_2 &= K_1L_2 - L_2K_1 = K_3N_2 - N_2K_3 \\
K_3 &= K_1L_3 - L_3K_1 = K_2M_3 - M_3K_2
\end{aligned}\right\}
\qquad
\left.\begin{aligned}
L_2' &= N_2'L_3' - L_3'N_2' = K_2'L_4' - L_4'K_2' \\
L_3' &= M_3'L_2' - L_2'M_3' = K_3'L_4' - L_4'K_3' \\
L_4' &= M_4'L_2' - L_2'N_4' = N_4'L_3' - L_3'N_4' \\
M_1' &= N_1'M_2' - M_2'N_1' = K_1'M_4' - M_4'K_1' \\
M_3' &= L_3'M_1' - M_1'L_3' = K_3'M_4' - M_4'K_3' \\
M_4' &= N_4'M_3' - M_3'N_4' = L_4'M_1' - M_1'L_4' \\
N_1' &= M_1'N_2' - N_2'M_1' = K_1'N_4' - N_4'K_1' \\
N_2' &= L_3'N_1' - N_1'L_3' = K_2'N_4' - N_4'K_2' \\
N_4' &= L_4'N_1' - N_1'L_4' = M_4'N_2' - N_2'M_4' \\
K_1' &= M_1'K_2' - K_2'M_1' = N_1'K_3' - K_3'N_1' \\
K_2' &= L_2'K_1' - K_1'L_2' = N_2'K_3' - K_3'N_2' \\
K_3' &= L_3'K_1' - K_1'L_3' = M_3'K_2' - K_2'M_3'
\end{aligned}\right\}.$$

From these it follows that the twelve equations can be reduced to six, for example the set of six constituted by

$$\begin{aligned} L_2 &= L_2', & M_1 &= M_1', \\ L_3 &= L_3', & N_1 &= N_1', \\ L_4 &= L_4', & K_1 &= K_1'. \end{aligned}$$

It is not difficult to see the analytical origin of such a result. The six equations which remain correspond to the changes in the variables caused by the transformations

$$\begin{aligned} x_1 &= X_1 + l_2X_2 + l_3X_3 + l_4X_4, \\ x_2 &= m_1X_1 + X_2, \\ x_3 &= n_1X_1 + X_3, \\ x_4 &= k_1X_1 + X_4. \end{aligned}$$

A combination of any two of these transformations, as for example

$$x_2 = m_1X_1 + (1 + l_2m_1)\,X_2 + m_1l_3X_3 + m_1l_4X_4,$$

introduces into x_2 terms corresponding to m_3X_3 and m_4X_4 which in the covariantive relation

* The other similar combinations of the twelve operators do not lead to any new forms, the operators in such cases being lineo-commutative; thus for instance

$$L_2L_3 = L_3L_2.$$

Those of the operators, which are not lineo-commutative are quadrato-commutative according to laws of the form

$$L_2^2\,M_3 - 2L_2\,M_3\,L_2 + M_3\,L_2^2 = 0,$$
$$L_2\,M_3^2 - 2M_3\,L_2\,M_3 + M_3^2\,L_2 = 0.$$

Digitized by Google

lead to two characteristic equations, $M_3 = M_3'$ and $M_4 = M_4'$ shewn to be dependent respectively upon $M_1 = M_1'$ and $L_3 = L_3'$, and upon $M_1 = M_1'$ and $L_4 = L_4'$.

7. This dependence of six equations upon the other six is however of a different kind from the dependence of the characteristic equations derived through terms of the second order in the coefficients of transformation. There, the characteristic equations arise merely from successive application of the operators which occur in the set of twelve; and they furnish no equation new in form. Here, the six dependent equations arise from a functional combination of the operators in the other six; they are similar equations, but are new in form.

Since then by means of the twelve all other characteristic equations can be immediately deduced, merely by the successive application of the operators they contain and without the introduction of any new operator through combinations, it is desirable to retain the full system of twelve. It will appear immediately that they suffice for the complete explicit development of a concomitant in point-variables and in plane-variables; but for the full development by this method in line-variables certain combinations will need to be taken. Such combinations are not however here given as the present purpose does not hold in special view the tabulation of the concomitants.

8. The Jacobian conditions that the coexistent equations shall possess common solutions are all satisfied either (i) identically or (ii) in virtue of equations in the set of twelve or (iii) in virtue of quasi-homogeneous equations.

Symbolical Representation of Concomitants.

9. Before proceeding further it is necessary to shew that every quaternariant can be exhibited in a symbolical form. The following is an outline of the proof, being similar to the one ordinarily applying in the case of binariants and to that given by Clebsch* as applied to ternariants.

(i) Denoting the quaternariant by $\phi(a, u, x, p)$ and introducing five new linear forms v_x, w_x, ξ_x, η_x, ζ_x such that (§ 1) we may take

$$\left.\begin{array}{ll} p_1 = v_1 w_4 - v_4 w_1, & p_6 = v_3 w_2 - v_2 w_3 \\ p_2 = v_2 w_4 - v_4 w_2, & p_5 = v_3 w_1 - v_1 w_3 \\ p_3 = v_3 w_4 - v_4 w_3, & p_4 = v_1 w_2 - v_2 w_1 \end{array}\right\}$$

from the coefficients of two of them v_x, w_x; and from the coefficients of the other three ξ_x, η_x, ζ_x the relations

$$\frac{x_1}{\Sigma \pm \xi_2 \eta_3 \zeta_4} = \frac{-x_2}{\Sigma \pm \xi_1 \eta_3 \zeta_4} = \frac{x_3}{\Sigma \pm \xi_1 \eta_2 \zeta_4} = \frac{-x_4}{\Sigma \pm \xi_1 \eta_2 \zeta_3},$$

the concomitant changes into $\phi(a, u, \xi, \eta, \zeta, v, w)$, that is, into an invariant of the simultaneous system a_x, u_x, ξ_x, η_x, ζ_x, v_x, w_x.

* Crelle, t. lviii., p. 118; see also Crelle, t. lix., pp. 1—62.

Digitized by Google

(ii) Every invariant of this system is an aggregate of products of the determinants of every four of them, the member a_x being repeated any number of times.

(iii) The different forms of determinant that may arise are:—

(1) $(abcd)$;

(2) (i) $(abcu)$,

(ii) $(abcv)$, which is merely a 'polar' of $(abcu)$ and so is not different in nature, and similarly for all the others involving only one of the six sets of quantities $u, \xi, \eta, \zeta, v, w$;

(3) (i) $(abvw)$,

(ii) $(abuv)$, which is merely a 'polar' of $(abvw)$ and so is not different in nature: similarly for all the others involving two of the six sets of quantities;

(4) (i) $(a\xi\eta\zeta)$,

(ii) $(au\eta\zeta)$, which is merely a 'polar' of $(a\xi\eta\zeta)$ and so is not different in nature: similarly for all the others involving three of the six sets of quantities;

(5) (i) $(u\xi\eta\zeta)$,

(ii) $(v\xi\eta\zeta)$, which is merely a 'polar' of $(u\xi\eta\zeta)$ and so is not different in nature: similarly for all the others involving four of the six sets of quantities.

Hence every invariant is an aggregate of products of determinants of the five different natures given by $(abcd)$, $(abcu)$, $(abvw)$, $(a\xi\eta\zeta)$ and $(u\xi\eta\zeta)$, and of polars of these determinants.

(iv) When we replace the coefficients in these invariants by their values in terms of the variables as already substituted, the invariants are changed into concomitants: and hence concomitants are sums of products of the transformed values of the five different kinds of factors. These factors now are:

(1) $(abcd)$;

(2) $(abcu)$;

(3) $(a_2b_3 - a_3b_2)\,p_1 + (a_3b_1 - a_1b_3)\,p_2 + (a_1b_2 - a_2b_1)\,p_3 + (a_1b_4 - a_4b_1)\,p_5$
$+ (a_2b_4 - a_4b_2)\,p_6 + (a_3b_4 - a_4b_3)\,p_4$

which we shall denote by (abp): it is also the equivalent of $a_yb_z - a_zb_y$ where the variables y and z are cogredient with x;

(4) a_x;

(5) u_x, the universal concomitant.

Digitized by Google

*[Hence the *five different kinds of factors which enter into the symbolical expression of a concomitant are of the forms* u_x, a_x, (abp), $(abcu)$ *and* $(abcd)$.

(v) It is now necessary to consider the polars of these.

(*a*) Some of the polar variables which occur, as in (iii) (4), can be expressed in terms of the variables x, p, u, and therefore must be retained.

Thus one of the determinants is

$$(auvw) = a_{x'},$$

where

$$\begin{aligned} x_1' &= (u_2v_3w_4) = \quad u_2p_3 - u_3p_2 + u_4p_6, \\ x_2' &= -(u_1v_3w_4) = -u_1p_3 \quad + u_3p_1 + u_4p_5, \\ x_3' &= (u_1v_2w_4) = u_1p_2 - u_2p_1 \quad + u_4p_4, \\ x_4' &= -(u_1v_2w_3) = -u_1p_6 - u_2p_5 - u_3p_4, \end{aligned}$$

a new set of point-variables, being those of the intersection of the line p and the plane u. Hence, when we have a concomitant which involves the point-variables x, other concomitants can be derived by means of the polar operator

$$\partial_x = x_1' \frac{\partial}{\partial x_1} + x_2' \frac{\partial}{\partial x_2} + x_3' \frac{\partial}{\partial x_3} + x_4' \frac{\partial}{\partial x_4},$$

repeated any number of times. Thus for instance if U be a quadratic, $U=0$ is the condition that the point x lies on a quadric surface; $\partial_x U = 0$ is the condition that the points x and x' are conjugate with regard to the quadric surface; and $\frac{1}{2}\partial_x^2 U = 0 = U'$ is the condition that the point x' lies on the quadric surface.

The preceding proof, given in outline, is limited to a quantic of the form a_x^n; but it is similar in the case of a quantic of the form u_a^m, as well as in the case of a quantic in line-variables which is separately discussed in § 51. In the former of these there arises similarly a set of plane-variables which must be retained: they are

$$\begin{aligned} u_1' &= (x_2y_3z_4) = \quad x_2p_4 - x_3p_5 + x_4p_1, \\ u_2' &= -(x_1y_3z_4) = -x_1p_4 \quad + x_3p_6 + x_4p_2, \\ u_3' &= (x_1y_2z_4) = x_1p_5 - x_2p_6 + \quad + x_4p_3, \\ u_4' &= -(x_1y_2z_3) = -x_1p_1 - x_2p_2 - x_3p_3, \end{aligned}$$

being the coordinates of the plane which passes through the point x and the line p. Hence when we have a concomitant which involves the plane-variables u, other concomitants can be derived by means of the polar operator

$$\partial_u = u_1' \frac{\partial}{\partial u_1} + u_2' \frac{\partial}{\partial u_2} + u_3' \frac{\partial}{\partial u_3} + u_4' \frac{\partial}{\partial u_4},$$

repeated any number of times. Thus, for instance, in the case of the same quadratic as before, there is (§ 20) a concomitant Σ such that $\Sigma = 0$ is the condition that the

* (11 March, 1889.) The remainder of this article and § 10 have, in consequence of some remarks kindly made to me by Professor Cayley, been considerably amplified from the form in which they existed at the time the paper was read, with the purpose of rendering more explicit the discussion of the polar variables.

Digitized by Google

plane u touches the former quadric surface; $\partial_u \Sigma = 0$ is the condition that the planes u and u' are conjugate with respect to the quadric surface; and $\frac{1}{2}\partial_u^2 \Sigma = 0 = \Sigma'$ is the condition that the plane u' touches the quadric surface.

Similarly, because we have two points x and x' we have a line p' other than p such that its coordinates are

$$\begin{aligned} p_1' &= x_2x_3' - x_3x_2', & p_6' &= x_1x_4' - x_1'x_4, \\ p_2' &= x_3x_1' - x_1x_3', & p_5' &= x_2x_4' - x_2'x_4, \\ p_3' &= x_1x_2' - x_2x_1', & p_4' &= x_3x_4' - x_3'x_4, \end{aligned}$$

being the line joining x and x'. And because we have two planes u and u' we have a line p'' other than p and p' such that its coordinates are

$$\begin{aligned} p_1'' &= u_1'u_4 - u_1u_4', & p_6'' &= u_2'u_3 - u_2u_3', \\ p_2'' &= u_2'u_4 - u_2u_4', & p_5'' &= u_3'u_1 - u_3u_1', \\ p_3'' &= u_3'u_4 - u_3u_4', & p_4'' &= u_1'u_2 - u_1u_2', \end{aligned}$$

being the intersection of the planes u and u'. But it is easy to prove that

$$\frac{p_1'' - p_1'}{p_1} = \ldots\ldots = \frac{p_6'' - p_6'}{p_6} = u_x;$$

so that, even from the point of view of asyzygetic concomitants, it is sufficient to retain only the coordinates of the line p'. Hence, when we have a concomitant which involves the line-variables p, other concomitants can be derived by means of the polar operator

$$\partial_p = p_1' \frac{\partial}{\partial p_1} + p_2' \frac{\partial}{\partial p_2} + p_3' \frac{\partial}{\partial p_3} + p_4' \frac{\partial}{\partial p_4} + p_5' \frac{\partial}{\partial p_5} + p_6' \frac{\partial}{\partial p_6},$$

repeated any number of times.

Thus for instance in the case of the same quadratic as before there is a concomitant T such that $T=0$ is the condition that the line p touches the former quadric surface; $\partial_p T = 0$ is the condition that the lines p and p' are conjugate with respect to the quadric surface; and $\frac{1}{2}\partial_p^2 T = 0 = T'$ is the condition that the line p' touches the quadric surface.

(β) Some of the polar variables which occur, as in (2), (3), (4), cannot be expressed in terms of the sets of given variables, and therefore are not to be retained. Thus, for instance, if y be the point which is the intersection of the planes u, η, ζ, and if q be the line which joins x and y, the value of the concomitant

$$a_x (abp)(bu\eta\zeta)$$

of the former quadratic is

$$\begin{aligned} &(abp)\, a_x b_y \\ &= \tfrac{1}{2}(abp)(a_x b_y - a_y b_x) \\ &= \tfrac{1}{2}(abp)(abq), \end{aligned}$$

a polar of T by means of variables extraneous to the given system.

Digitized by Google

Hence *all the polar variables that require to be retained are those of the point x', the line p', and the plane u'.*

(vi) All functions derived from concomitants, which involve only the variables of the point x, the line p and the plane u as given in (iv.), by means of any combinations of the polar operators ∂_x, ∂_p, ∂_u we call *polar derivatives*; and evidently

$$\partial_x \cdot a_x = a_{x'},$$

$$\partial_p (abp) = (abp'),$$

$$\partial_u (abcu) = (abcu').$$

Hence we have as the general theorem:

Every concomitant of the quantic a_x^n can be represented as an aggregate of products of factors of the forms a_x, a_x'; (abp), (abp'); $(abcu)$, $(abcu')$; $(abcd)$, together with u_x.

(vii) But conversely when we have obtained a concomitant which involves not only the variables of the original system x, u, p, but also the derived polar variables x', u', p', we can pass, uniquely and by a reversible process, to a concomitant which involves only the variables of the original system. For this purpose we merely need to employ the operators which are reverse to ∂_x, ∂_p, ∂_u, viz.

$$\partial_x' = x_1 \frac{\partial}{\partial x_1'} + x_2 \frac{\partial}{\partial x_2'} + x_3 \frac{\partial}{\partial x_3'} + x_4 \frac{\partial}{\partial x_4'},$$

$$\partial_p' = p_1 \frac{\partial}{\partial p_1'} + p_2 \frac{\partial}{\partial p_2'} + p_3 \frac{\partial}{\partial p_3'} + p_4 \frac{\partial}{\partial p_4'} + p_5 \frac{\partial}{\partial p_5'} + p_6 \frac{\partial}{\partial p_6'},$$

$$\partial_u' = u_1 \frac{\partial}{\partial u_1'} + u_2 \frac{\partial}{\partial u_2'} + u_3 \frac{\partial}{\partial u_3'} + u_4 \frac{\partial}{\partial u_4'},$$

each operating as many times as the variables x', u', p' occur respectively; and we obtain a concomitant, the same in literal coefficients and involving only the variables x, u, p alone.

If, then, we have obtained all the concomitants which can be represented as aggregates of products of factors of the form a_x, (abp), $(abcu)$, $(abcd)$, u_x, all the concomitants of the system associated with the quantic under consideration can be derived from them as polar derivatives by means of the operators ∂_x, ∂_u, ∂_p; and all concomitants which have the same literal coefficients can be derived from a concomitant with those leading coefficients and involving in its symbolical expression only factors of the form

$$u_x,\ a_x,\ (abp),\ (abcu),\ (abcd).$$

We shall therefore speak of *these concomitants as the system of fundamental concomitants appertaining to the quantic; and it is to be understood that with the system must always be associated all those polar derivatives which are derivable by means of any combinations of the operators ∂_x, ∂_p, ∂_u.*

Digitized by Google

10. It follows, then, from the preceding investigation, that the fundamental concomitants of the system appertaining to a quantic are unique in their expression in terms of the symbolic factors; and therefore that, when these factors are expanded in powers of the variables, the leading coefficient is unique, the leading coefficient being the coefficient of the term which involves the highest power of x_1, the highest power of p_1 and the highest power of u_1. And it follows also, from the preceding investigation, that no variables other than x_1, u_1, p_1 occur in that leading term of a fundamental concomitant.

Conversely, also, when the coefficient of the leading term of a fundamental concomitant is known as a function of the coefficients of the quantic, we have one simple method of obtaining the concomitant by expressing that leading coefficient in terms of the umbral elements of the quantic and completing the umbral expression by proper association of the variables. When the fundamental concomitant, thus determined by a leading coefficient, is known, all the polar derivatives can be obtained.]

11. When the symbolisation of the leading coefficient has been effected, the degree of the concomitant in x—say its order, its degree in p—say its grade, and its degree in u—say its class, are immediately evident; a partial verification of these can be derived from the isobaric property of the concomitant.

We assign the following weights:—

to x_1 the weight $\sigma + \rho + 1$,
x_2 ..... $\rho + 1$,
x_3 1,
x_4 ..... 0;

and therefore, since a_s and u_s are isobaric,

to a_1 and u_1 the weight 0,
a_2 and u_2 ..... σ,
a_3 and u_3 ..... $\sigma + \rho$,
a_4 and u_4 ..... $\sigma + \rho + 1$;

and the weights to be assigned to the quantities p are:

to p_1 the weight $2 + \rho$,
p_2 ..... $2 + \rho + \sigma$,
p_3 ..... $2 + 2\rho + \sigma$,
p_4 1,
p_5 ..... $1 + \rho$,
p_6 ..... $1 + \rho + \sigma$.

Digitized by Google

Here ρ and σ may be any magnitudes we please. And it may be noted that, if desirable, an arbitrary magnitude (the same for all) may be added to the weights of the variables u; and similarly for the variables p. For the present purpose there is no advantage in this, as we shall be concerned only with differences of weights.

Let the leading term of a concomitant be $x_1^n p_1^m u_1^l A$; then another term must be $\pm x_2^n p_2^m u_2^l A'$, where A' is the value of A when the substitutions $x_1 = X_2$, $x_2 = X_1$ are effected on the quantic; and another term must be $\pm x_4^n p_4^m u_4^l A''$, where A'' is the value of A when the substitutions $x_1 = X_4$, $x_4 = X_1$ are effected on the quantic. Let W, W', W'' be the respective weights of A, A', A'', which can be determined by the inspection of any term in each of them. Because the concomitant is isobaric, we have the common weight

$$\begin{aligned} &= n(\sigma+\rho+1) + m(2+\rho) + W \\ &= n(\rho+1) + m(2+\rho+\sigma) + l\sigma + W' \\ &= m(2+\rho) + l(\sigma+\rho+1) + W''; \end{aligned}$$

and therefore

$$n - m - l = \frac{W' - W}{\sigma},$$

$$n - l = \frac{W'' - W}{\sigma+\rho+1},$$

two equations determining m and $n-l$, the values of which should agree with the values derived from the symbolical expression.

Equations which determine Leading Coefficients.

12. In the equations (I) let (L_2) denote $L_2 + p_1 \dfrac{\partial}{\partial p_2} - p_5 \dfrac{\partial}{\partial p_6}$ and so for the others; let $(L_2) + u_1 \dfrac{\partial}{\partial u_2}$ be denoted by $\{L_2\}$, and so for the others; thus the equations can be written in the form

$$\{J\} = x_\rho \frac{\partial}{\partial x_\sigma}.$$

Let ϕ be any fundamental concomitant of order n, grade m, class l; and consider first its expression in powers of x, so that we may write

$$\phi = x_1^n \phi_{0,0,0} + \dots + \frac{x_1^{n-r-s-t}\, x_2^r\, x_3^s\, x_4^t}{r!\, s!\, t!}\, \phi_{r,s,t} + \dots.$$

When this value is substituted in the equations (I) we have

$$\{M_1\}\, \phi_{r,s,t} = \phi_{r+1,s,t},$$

$$\{N_1\}\, \phi_{r,s,t} = \phi_{r,s+1,t},$$

$$\{K_1\}\, \phi_{r,s,t} = \phi_{r,s,t+1},$$

Digitized by Google

from three of them. Hence if $\phi_{0,0,0}$ and n be known, the full development of the concomitant can be obtained; for

$$\phi_{r,s,t} = \{M_1\}^r \{N_1\}^s \{K_1\}^t \phi_{0,0,0},$$

and the value of n can be obtained from any of the equations

$$\{M_1\}^{n+1} \phi_{0,0,0} = 0 = \{N_1\}^{n+1} \phi_{0,0,0} = \{K_1\}^{n+1} \phi_{0,0,0}.$$

The remaining nine equations give

$$\left.\begin{aligned} \{L_2\}\,\phi_{r,s,t} &= (n-r-s-t+1)\,r\phi_{r-1,s,t} \\ \{L_3\}\,\phi_{r,s,t} &= (n-r-s-t+1)\,s\phi_{r,s-1,t} \\ \{L_4\}\,\phi_{r,s,t} &= (n-r-s-t+1)\,t\phi_{r,s,t-1} \end{aligned}\right\},$$

$$\left.\begin{aligned} \{M_3\}\,\phi_{r,s,t} &= s\phi_{r+1,s-1,t} \\ \{M_4\}\,\phi_{r,s,t} &= t\phi_{r+1,s,t-1} \end{aligned}\right\}, \qquad \left.\begin{aligned} \{N_2\}\,\phi_{r,s,t} &= r\phi_{r-1,s+1,t} \\ \{N_4\}\,\phi_{r,s,t} &= t\phi_{r,s+1,t-1} \end{aligned}\right\}, \qquad \left.\begin{aligned} \{K_2\}\,\phi_{r,s,t} &= r\phi_{r-1,s,t+1} \\ \{K_3\}\,\phi_{r,s,t} &= s\phi_{r,s-1,t+1} \end{aligned}\right\}.$$

Hence $\psi = \phi_{0,0,0}$ satisfies the nine equations

$$\{L_2\} = 0 = \{L_3\} = \{L_4\};$$

$$\{M_3\} = 0 = \{M_4\}, \quad \{N_2\} = 0 = \{N_4\}, \quad \{K_2\} = 0 = \{K_3\},$$

each of these being of the form

$$(J) + u_\sigma \frac{\partial}{\partial u_\rho} = 0;$$

and the only variables which can enter into ψ are u and p.

Now let ψ be arranged in powers of u, so that we may write

$$\psi = u_1^l \psi_{0,0,0} + \ldots\ldots + \frac{u_1^{l-r-s-t}\, u_2^r\, x_3^s\, u_4^t}{r!\; s!\; t!} \psi_{r,s,t} + \ldots\ldots.$$

When this value is substituted in the foregoing nine equations, we have

$$\begin{aligned} (L_2)\,\psi_{r,s,t} + \psi_{r+1,s,t} &= 0, \\ (L_3)\,\psi_{r,s,t} + \psi_{r,s+1,t} &= 0, \\ (L_4)\,\psi_{r,s,t} + \psi_{r,s,t+1} &= 0, \end{aligned}$$

from three of them. Hence if $\psi_{0,0,0}$ and l be known the full development of $\psi_{0,0,0}$ can be obtained for

$$\psi_{r,s,t} = (-1)^{r+s+t} (L_2)^r (L_3)^s (L_4)^t \psi_{0,0,0};$$

and the value of l can be obtained from any of the equations

$$(L_2)^{l+1} \psi_{0,0,0} = 0 = (L_3)^{l+1} \psi_{0,0,0} = (L_4)^{l+1} \psi_{0,0,0}.$$

Digitized by Google

The remaining six equations give

$$\left.\begin{aligned}(M_3)\,\psi_{r,s,t}+r\psi_{r-1,s+1,t}&=0\\(M_4)\,\psi_{r,s,t}+r\psi_{r-1,s,t+1}&=0\end{aligned}\right\},\qquad \left.\begin{aligned}(N_2)\,\psi_{r,s,t}+s\psi_{r+1,s-1,t}&=0\\(N_4)\,\psi_{r,s,t}+s\psi_{r,s-1,t+1}&=0\end{aligned}\right\},$$

$$\left.\begin{aligned}(K_2)\,\psi_{r,s,t}+t\psi_{r+1,s,t-1}&=0\\(K_3)\,\psi_{r,s,t}+t\psi_{r,s+1,t-1}&=0\end{aligned}\right\}.$$

Hence $\chi=\psi_{0,0,0}$ satisfies the six equations

$$(M_3)=0=(M_4);\quad (N_2)=0=(N_4);\quad (K_2)=0=(K_3);$$

and the only variables which can occur in χ are the variables p.

Now let χ be arranged in powers of p, so that we may write

$$\chi=\Sigma\chi_{i,j,k,r,s,t}\frac{p_1^i\,p_2^j\,p_3^k\,p_4^r\,p_5^s\,p_6^t}{i!\;j!\;k!\;r!\;s!\;t!},\qquad (i+j+k+r+s+t=m).$$

When this series is substituted in the six equations, they must be satisfied and relations among coefficients will be obtained; but in this regard we must take account of the permanent relation $p_1p_6+p_2p_5+p_3p_4=0$, and therefore instead of putting the coefficient of every term equal to zero we make it equal to the coefficient of the same variable-combination in

$$(p_1p_6+p_2p_5+p_3p_4)\,U_{m-2},$$

where U_{m-2} is an arbitrary homogeneous function of order $m-2$ in the variables p. This equality we shall indicate by the congruence symbol, and shall leave the coefficient unspecified; and it is to be remembered that in some cases, e.g. for the coefficient of p_1^m, the congruence is an equality. We thus have

$$M_3\chi_{i,j,k,r,s,t}\equiv r\chi_{i,j,k,r-1,s+1,t}-j\chi_{i,j-1,k+1,r,s,t},$$

$$M_4\chi_{i,j,k,r,s,t}\equiv t\chi_{i,j,k+1,r,s,t-1}-r\chi_{i+1,j,k,r-1,s,t},$$

$$N_2\chi_{i,j,k,r,s,t}\equiv s\chi_{i,j,k,r+1,s-1,t}-k\chi_{i,j+1,k-1,r,s,t},$$

$$N_4\chi_{i,j,k,r,s,t}\equiv s\chi_{i+1,j,k,r,s-1,t}-t\chi_{i,j+1,k,r,s,t-1},$$

$$K_2\chi_{i,j,k,r,s,t}\equiv k\chi_{i,j,k-1,r,s,t+1}-i\chi_{i-1,j,k,r+1,s,t},$$

$$K_3\chi_{i,j,k,r,s,t}\equiv i\chi_{i-1,j,k,r,s+1,t}-j\chi_{i,j-1,k,r,s,t+1},$$

with the limitation $i+j+k+r+s+t=m$.

These congruences and equations are not, in their present form, sufficient to give the complete development of χ: others would be derived from them for that purpose, but no essentially new equation would (§ 5) be introduced. But our present aim is the derivation of the independent characteristic equations satisfied by the leading coefficient $\chi_{l,0,0,0,0,0}$, say θ; and since all the equations which in form are independent of one

Digitized by Google

another have been retained, the foregoing system is sufficient. Making then j, k, r, s, t all zero (and i therefore equal to l) and bearing in mind that for these values the congruences are equations, we find that θ satisfies the four equations

$$M_2 = 0,\ M_4 = 0,\ N_2 = 0,\ N_4 = 0;$$

while the two remaining equations are easily seen to give the complete expression of that part of χ which involves only p_1, p_4 and p_5.

But of the four equations satisfied by θ only three are independent; for by § 6 we have

$$N_4 = N_2 M_4 - M_4 N_2,$$

and therefore any solution common to $N_2 = 0$ and $M_4 = 0$ necessarily satisfies $N_4 = 0$. Hence we have the result:—

The leading coefficient of every fundamental concomitant, that is the coefficient of the highest power of x_1, the highest power of p_1 and the highest power of u_1, satisfies the three (independent) linear partial differential equations

$$M_2 = 0,\ M_4 = 0,\ N_2 = 0.$$

And as in the corresponding cases of binariants and ternariants, it follows that *every solution, which is common to the foregoing three equations and is homogeneous and isobaric in the coefficients of the quantic, is a leading coefficient of some fundamental concomitant of the quantic.*

13. The following results relative to the leading coefficients of the eight different kinds of fundamental concomitants can easily be verified:—

(i) The independent characteristic equations satisfied by the coefficient of $x_1^n p_1^m u_1^l$ in a point-line-plane covariant are

$$M_2 = M_4 = N_2 = 0.$$

(ii) The independent characteristic equations satisfied by the coefficient of $x_1^n u_1^l$ in a point-plane covariant are

$$M_2 = M_4 = N_2 = K_2 = 0.$$

(iii) The independent characteristic equations satisfied by the coefficient of $p_1^m u_1^l$ in a line-plane covariant are

$$M_2 = M_4 = N_2 = K_1 = 0.$$

(iv) The independent characteristic equations satisfied by the coefficient of $x_1^n p_1^m$ in a point-line covariant are

$$M_2 = M_4 = N_2 = L_4 = 0.$$

Digitized by Google

(v) The independent characteristic equations satisfied by the coefficient of u_1^i in a plane covariant are

$$M_3 = M_4 = N_2 = K_2 = K_1 = 0.$$

(vi) The independent characteristic equations satisfied by the coefficient of p_1^m in a line covariant are

$$M_3 = M_4 = N_2 = K_1 = L_4 = 0.$$

(vii) The independent characteristic equations satisfied by the coefficient of x_1^n in a point covariant are

$$M_3 = M_4 = N_2 = L_4 = K_2 = 0.$$

(viii) The independent characteristic equations satisfied by a pure invariant are

$$M_3 = M_4 = N_2 = L_4 = K_2 = K_1 = 0.$$

14. The foregoing is an outline of the general theory; we now proceed to apply it to a few cases of quantics taken (with one exception) to be unipartite. The sole difference, when the concomitants of bipartite or tripartite quantics are desired, lies in the forms of the literal operators.

The first step necessary is the derivation of common solutions of the three characteristic equations. In this regard a limitation will be assigned so that, for the most part, only those which are *algebraically independent* of one another will be obtained; no attempt will be made to obtain the aggregate of 'irreducible' concomitants.

Since the partial differential equations are all linear, it is a consequence (from the theory of such equations) that all their solutions can be expressed in terms of a finite number of common and algebraically independent solutions. And, as will be seen from the process of solution and from § 18, this finite number is five for a quadratic, is fifteen for a cubic and fifteen for a system of two quadratics, is thirty for a quartic; and for the nth the finite number is $\frac{1}{6}(n+1)(n+2)(n+3)-5$.

15. It is convenient to arrange the notation for quantics of different orders so that the operators of the characteristic equations are the same so far as common letters occur in them. Thus the quadratic will be taken in the form

$$\begin{array}{l} a_0x_1^2 + 2a_1x_1x_3 + a_2x_3^2 + 2a_0'x_1x_4 + 2a_1'x_3x_4 + a_0''x_4^2; \\ + 2b_0x_1x_2 + 2b_1x_2x_3 \qquad\quad + 2b_0'x_2x_4 \\ + c_0x_2^2; \end{array}$$

the cubic in the form

$$\left.\begin{array}{l} a_0x_1^3 + 3a_1x_1^2x_3 + 3a_2x_1x_3^2 + a_3x_3^3 + 3x_4 \\ +3b_0x_1^2x_2 + 6b_1x_1x_2x_3 + 3b_2x_2x_3^2 \\ +3c_0x_1x_2^2 + 3c_1x_2^2x_3 \\ + d_0x_2^3 \end{array}\right| \left.\begin{array}{l} a_0'x_1^2 + 2a_1'x_1x_3 + a_2'x_3^2 \\ + 2b_0'x_1x_2 + 2b_1'x_2x_3 \\ + c_0'x_2^2 \end{array}\right| + 3x_4^2 \left|\begin{array}{l} a_0''x_1 + a_1''x_3 \\ + b_0''x_2 \end{array}\right| + a_0'''x_4^3,$$

Digitized by Google

and so on; the general law being that in the literal coefficient, associated with any variable-combination and the proper multinomial coefficient, the subscript indicates the power of x_2, the number of dashes the power of x_4, and the position in the sequence $a, b, c, \ldots$ the power of x_3. Thus in the quaternary octavic, the literal coefficient of $x_1^2x_2^2x_3^2x_4^2$ is c_2'': the multinomial coefficient associated with it being $8! \div (2!)^4$.

With this notation, the operators in the characteristic equations are:—

$$M_3 = b_0 \frac{\partial}{\partial a_1} + c_0 \frac{\partial}{\partial b_1} + d_0 \frac{\partial}{\partial c_1} + \ldots + b_0' \frac{\partial}{\partial a_1'} + c_0' \frac{\partial}{\partial b_1'} + \ldots + b_0'' \frac{\partial}{\partial a_1''} + \ldots + \ldots$$

$$+ 2\left(b_1 \frac{\partial}{\partial a_2} + c_1 \frac{\partial}{\partial b_2} + \ldots + b_1' \frac{\partial}{\partial a_2'} + \ldots + \ldots\right) + 3\left(b_2 \frac{\partial}{\partial a_3} + \ldots + \ldots\right) + \ldots$$

$$M_4 = b_0 \frac{\partial}{\partial a_0'} + c_0 \frac{\partial}{\partial b_0'} + d_0 \frac{\partial}{\partial c_0'} + \ldots + b_1 \frac{\partial}{\partial a_1'} + c_1 \frac{\partial}{\partial b_1'} + \ldots + b_2 \frac{\partial}{\partial a_2'} + \ldots + \ldots$$

$$+ 2\left(b_0' \frac{\partial}{\partial a_0''} + c_0' \frac{\partial}{\partial b_0''} + \ldots + b_1' \frac{\partial}{\partial a_1''} + \ldots + \ldots\right) + 3\left(b_0'' \frac{\partial}{\partial a_0'''} + \ldots + \ldots\right) + \ldots$$

$$N_2 = a_1 \frac{\partial}{\partial b_0} + a_2 \frac{\partial}{\partial b_1} + a_3 \frac{\partial}{\partial b_2} + \ldots + a_1' \frac{\partial}{\partial b_0'} + a_2' \frac{\partial}{\partial b_1'} + \ldots + a_1'' \frac{\partial}{\partial b_0''} + \ldots + \ldots$$

$$+ 2\left(b_1 \frac{\partial}{\partial c_0} + b_2 \frac{\partial}{\partial c_1} + \ldots + b_1' \frac{\partial}{\partial c_0'} + \ldots + \ldots\right) + 3\left(c_1 \frac{\partial}{\partial d_0} + \ldots + \ldots\right) + \ldots$$

16. When the fundamental quantic is not unipartite the only difference is, as already remarked, in the literal operators.

Taking for example a bipartite quantic, of order μ in x and class ν in u, symbolically representable by $a_x^\mu u_x^\nu$, we may denote the literal coefficient of

$$x_1^q x_2^r x_3^s x_4^t u_1^\kappa u_2^\rho u_3^\sigma u_4^\tau \text{ by } a_{qrst\kappa\rho\sigma\tau}$$

(with the limitations $q+r+s+t=\mu$, $\kappa+\rho+\sigma+\tau=\nu$) and it is assumed that the corresponding multinomial coefficient is associated with it; the literal operators are then of the form

$$L_2 = \Sigma r a_{q+1,\, r-1,\, s,\, t,\, \kappa,\, \rho,\, \sigma,\, \tau} \frac{\partial}{\partial a_{qrst\kappa\rho\sigma\tau}} - \Sigma \kappa a_{q,\, r,\, s,\, t,\, \kappa-1,\, \rho+1,\, \sigma,\, \tau} \frac{\partial}{\partial a_{qrst\kappa\rho\sigma\tau}}.$$

The *same laws hold of these operators as in* § 6; and hence all the foregoing theory applies, and the independent characteristic equations determining leading coefficients are

$$M_3 = \Sigma\left(s a_{q,\, r+1,\, s-1,\, t,\, \kappa,\, \rho,\, \sigma,\, \tau} - \rho a_{q,\, r,\, s,\, t,\, \kappa,\, \rho-1,\, \sigma+1,\, \tau}\right) \frac{\partial}{\partial a_{qrst\kappa\rho\sigma\tau}} = 0,$$

$$M_4 = \Sigma\left(t a_{q,\, r+1,\, s,\, t-1,\, \kappa,\, \rho,\, \sigma,\, \tau} - \rho a_{q,\, r,\, s,\, t,\, \kappa,\, \rho-1,\, \sigma,\, \tau+1}\right) \frac{\partial}{\partial a_{qrst\kappa\rho\sigma\tau}} = 0,$$

$$N_2 = \Sigma\left(r a_{q,\, r-1,\, s+1,\, t,\, \kappa,\, \rho,\, \sigma,\, \tau} - \sigma a_{q,\, r,\, s,\, t,\, \kappa,\, \rho+1,\, \sigma-1,\, t}\right) \frac{\partial}{\partial a_{qrst\kappa\rho\sigma\tau}} = 0.$$

Digitized by Google

Number of Independent Concomitants.

17. The following is a general summary of the *process of solution*. We first find the independent solutions of $M_3 = 0$, being one less in number than the number of coefficients $\frac{1}{6}(n+1)(n+2)(n+3)$ of the quantic: let these solutions be $\lambda, \mu, \nu, \rho, \ldots$.

We now require the functional combinations of $\lambda, \mu, \nu, \rho, \ldots$ which are such as to satisfy $N_2 = 0$: let one of them be $f(\lambda, \mu, \nu, \rho, \ldots)$. Then

$$N_2 f = 0,$$

that is,
$$\frac{\partial f}{\partial \lambda} N_2\lambda + \frac{\partial f}{\partial \mu} N_2\mu + \frac{\partial f}{\partial \nu} N_2\nu + \frac{\partial f}{\partial \rho} N_2\rho + \ldots = 0,$$

the subsidiary equations for which are

$$\frac{d\lambda}{N_2\lambda} = \frac{d\mu}{N_2\mu} = \frac{d\nu}{N_2\nu} = \frac{d\rho}{N_2\rho} = \ldots,$$

being one less in number than the number of the quantities $\lambda, \mu, \nu, \rho, \ldots$. But it does not follow that the number of independent integrals of the equations is the same as the number of the equations; for, as will be seen in the special cases, the quantities $N_2\lambda, N_2\mu, N_2\nu, N_2\rho, \ldots$ are not functions of $\lambda, \mu, \nu, \rho, \ldots$ alone. To obtain integrals it is necessary to take linear combinations of the form $\lambda N_2\sigma - k\sigma N_2\lambda$ (k an integer which varies from one combination to another); and solutions can then be obtained. The effect of this is to diminish the number of equations by one unit; and therefore the number of their independent solutions is less by 2 than the number of functions $\lambda, \mu, \nu, \rho, \ldots$, that is, it is

$$\tfrac{1}{6}(n+1)(n+2)(n+3) - 3.$$

Let these solutions, independent of one another and common to the two equations, be $\lambda', \mu', \nu', \rho', \ldots$.

In finding the functional combinations of these which satisfy $M_4 = 0$ we have a similar process to carry out, it being sometimes convenient for purposes of integration to introduce algebraical combinations of $\lambda', \mu', \nu', \rho', \ldots$; and the result is that the number of solutions is again reduced by 2, so that the number of independent solutions is

$$\tfrac{1}{6}(n+1)(n+2)(n+3) - 5.$$

In terms of these every solution can be expressed, though not necessarily nor generally as an integral function.

For a system of quantics, containing implicitly in their most general form N coefficients in the aggregate, the algebraical system of solutions is similarly indicated to contain $N - 5$ members.

18. That the number of functions, thus indicated as $N - 5$ by the process of solution, is the complete number of algebraically independent solutions, may be seen by the following considerations.

Digitized by Google

(i) When the general linear transformation is effected, the number of equations connecting the new coefficients with the old is N, the number of coefficients occurring in the most general forms of the quantics that are in question; thus for one quadratic the value of N is 10, for two quadratics the value of N is 20, and so on. All these N equations contain the coefficients of transformation.

In addition to these we have the following equations: four connecting the variables x with X, six connecting p with P, and four connecting U with u; viz. fourteen in all.

And there is one more equation, viz.,

$$\Delta = \text{determinant of transformation.}$$

Thus the total number of equations is $N+14+1=N+15$, all of which involve the coefficients of transformation.

(ii) These equations are not all independent; in order to obtain the number of independent equations to which they can be reduced, it is necessary to diminish their number by an integer equal to the number of relations which subsist among the quantities. These are two, viz.,

$$p_1p_6+p_2p_5+p_3p_4=0, \quad P_1P_6+P_2P_5+P_3P_4=0.$$

Hence the number of independent equations in the system is $N+13$.

(iii) These equations involve a certain number of quantities which do not occur in covariantive relations, viz., the 16* coefficients of the transformation.

When, therefore, the 16 quantities are eliminated between the $N+13$ equations, the number of independent equations left is $N-3$. These relations we otherwise know are in general of the type

$$\Phi(A, X, P, U) = \Delta^{\theta}\phi(a, x, p, u);$$

every such relation determining a concomitant.

But there are two of these $N-3$ equations which are independent of the coefficients of the quantic; they are

$$u_x = U_x,$$

and (§ 1) $$p_1p_6+p_2p_5+p_3p_4=\Delta(P_1P_6+P_2P_5+P_3P_4),$$

by the equations of transformation. Hence the number of independent concomitants which involve the coefficients of the quantics is $N-5$.

(The complementary form of the last results can be at once derived from the differential equations. It is easy to prove that every solution of these equations which does not involve the coefficients of the quantic can be expressed as a function of

$$(\alpha) \quad u_1x_1+u_2x_2+u_3x_3+u_4x_4,$$

$$(\beta) \quad p_1p_6+p_2p_5+p_3p_4;$$

and these would, from this point of view, be added to the other $N-5$ solutions of the equations.)

* In the introductory paragraphs l_1, m_2, n_3, k_4 were all taken to be unity, for the purpose of simplifying the formation of the characteristic equations.

Digitized by Google

The general result thus is:—

The number of fundamental concomitants, algebraically independent of one another and involving the coefficients of the quantics, is $N-5$ and every fundamental concomitant can be expressed in terms of those $N-5$ together with the adjoint form u_x and in virtue of the relation $p_1p_6+p_2p_5+p_3p_4=0$; where N is the number of coefficients in the most general forms of the quantics with which the concomitants are associated.

With the fundamental concomitants their polar derivatives must be associated, to form the algebraically complete system of concomitants.

APPLICATIONS TO PARTICULAR QUANTICS.

I. *A single quadratic.*

19. With the notation suggested, the characteristic equations are

$$M_3 = b_0 \frac{\partial}{\partial a_1} + b_0' \frac{\partial}{\partial a_1'} + c_0 \frac{\partial}{\partial b_1} + 2b_1 \frac{\partial}{\partial a_2} = 0,$$

$$N_3 = a_1 \frac{\partial}{\partial b_0} + a_1' \frac{\partial}{\partial b_0'} + a_2 \frac{\partial}{\partial b_1} + 2b_1 \frac{\partial}{\partial c_0} = 0,$$

$$M_4 = b_0 \frac{\partial}{\partial a_0'} + b_1 \frac{\partial}{\partial a_1'} + c_0 \frac{\partial}{\partial b_0'} + 2b_0' \frac{\partial}{\partial a_0''} = 0;$$

and we have to find the simultaneous solutions of these three equations.

The simultaneous solutions of $M_3=0$ and $N_3=0$, which are independent of one another, can at once be obtained; for those equations are the differential equations satisfied by the concomitants, in the two sets of variables a_1, $-b_0$ and a_1', $-b_0'$, of a binary quadratic with c_0, b_1, a_2 as literal coefficients: and every simultaneous solution is one of the system of concomitants. The system of algebraically independent solutions is thus

$$\theta = (c_0,\ b_1,\ a_2 \between a_1,\ -b_0)^2,$$

$$\vartheta = (c_0,\ b_1,\ a_2 \between a_1,\ -b_0 \between a_1',\ -b_0'),$$

$$t = a_2c_0 - b_1^2,$$

$$j = b_0a_1' - a_1b_0';$$

and the irreducible system of binariants is composed of θ, ϑ, t, j and θ' where

$$\theta' = (c_0,\ b_1,\ a_2 \between a_1',\ -b_0')^2;$$

with the algebraical relation

$$\theta\theta' - \vartheta^2 = tj^2.$$

Digitized by Google

To the set of four solutions, independent of one another, must be added those the variables of which do not enter into either $M_2 = 0$ or $N_2 = 0$; these are

$$u = a_0,$$
$$u' = a_0',$$
$$u'' = a_0''.$$

Hence *an algebraically complete system of independent solutions common to* $M_2 = 0$ *and* $N_2 = 0$ *is constituted by* $u, u', u'', \theta, \vartheta, t, j$; *and an irreducible system—that is a system such that every solution of the equations is an integral rational function of its members—is constituted by* $u, u', u'', \theta, \vartheta, \theta', t, j$.

We now require the functional combinations of the independent solutions which are such as to satisfy $M_4 = 0$, say $\Theta = 0$. Carrying out the process indicated in § 17, we have

$$\Theta u = 0, \quad \Theta\theta = 0, \quad \Theta t = 0,$$

so that u, θ, t are simultaneous solutions of all the three equations; and we have

$$\Theta u' = b_0, \quad \Theta u'' = 2b_0', \quad \Theta\vartheta = b_0 t,$$

with either of the equations

$$\Theta j = b_0 b_1 - c_0 a_1, \quad \Theta\theta' = 2b_0' t,$$

for j and θ' can be expressed algebraically, each in terms of the other by means of the other functions. We require two solutions (§ 17) of this set of equations; they are most easily obtained in the forms

$$\Theta\,(u't - \vartheta) = 0,$$
$$\Theta\,(u''t - \theta') = 0.$$

Hence we have *five solutions, viz.*,

$$u, \; t, \; \theta, \; u't - \vartheta, \; u''t - \theta';$$

and in terms of these every simultaneous solution of the three equations can be expressed.

20. Each of these five is a leading coefficient: we proceed to obtain the concomitants. Taking the quadratic in the symbolical form α_x^2, we have

$$u = a_0 = \alpha_1^2.$$

Since this is the coefficient of the highest power of x_1, the highest power of u_1, and the highest power of p_1, it follows that the concomitant is

$$U = \alpha_x^2 = ux_1^2 + \ldots,$$

being in fact the original quadratic.

Next, we have $$t = a_2 c_0 - b_1^2 = \tfrac{1}{2}(\alpha_2\beta_3 - \alpha_3\beta_2)^2;$$

and the concomitant determined by this coefficient is

$$T = \tfrac{1}{2}(\alpha\beta p)^2 = tp_1^2 + \ldots.$$

The value of θ being $c_0 a_1^2 - 2b_1 a_1 b_0 + a_2 b_0^2$, we have

$$\theta = \alpha_2^2 \alpha_1^2 - 2\alpha_2\alpha_3\alpha_1 b_0 + \alpha_3^2 b_0^2 = (\alpha_2\alpha_1 - \alpha_3 b_0)^2,$$

Digitized by Google

or since $$a_2a_1 - a_3b_0 = \alpha_2\beta_1\beta_3 - \alpha_3\beta_1\beta_2 = \beta_1(\alpha_2\beta_3),$$
we have $$\theta = \beta_1\gamma_1(\alpha_2\beta_3)(\alpha_3\gamma_2);$$
and the corresponding concomitant is
$$\Theta = \beta_x\gamma_x(\alpha\beta p)(\alpha\gamma p) = \theta x_1^2 p_1^2 + \dots.$$

Next, denoting $u't - \vartheta$ by ρ, we have
$$\rho = \begin{vmatrix} b_0, & c_0, & b_1 \\ a_1, & b_1, & a_2 \\ a_0', & b_0', & a_1' \end{vmatrix}$$
$$= \alpha_2\beta_3\gamma_4(\alpha_1\beta_2\gamma_3)$$
$$= \tfrac{1}{6}(\alpha_2\beta_3\gamma_4)(\alpha_1\beta_2\gamma_3).$$

This is the leading coefficient: the completed form of $(\alpha_2\beta_3\gamma_4)$ is evidently $(\alpha\beta\gamma u)$; and that of $(\alpha_1\beta_2\gamma_3)$, viz. of
$$\alpha_1(\beta_2\gamma_3) + \beta_1(\gamma_2\alpha_3) + \gamma_1(\alpha_2\beta_3)$$
is evidently $$\alpha_x(\beta\gamma p) + \beta_x(\gamma\alpha p) + \gamma_x(\alpha\beta p),$$
it being remembered that powers of x_1, of u_1, and of p_1 only may be associated with the leading coefficient. Hence the concomitant is
$$P = \tfrac{1}{6}(\alpha\beta\gamma u)\{\alpha_x(\beta\gamma p) + \beta_x(\gamma\alpha p) + \gamma_x(\alpha\beta p)\}$$
$$= \tfrac{1}{2}\alpha_x(\alpha\beta\gamma u)(\beta\gamma p) = \rho x_1 u_1 p_1 + \dots.$$

Lastly, denoting $u''t - \theta'$ by σ, we have
$$\sigma = \begin{vmatrix} c_0, & b_1, & b_0' \\ b_1, & a_2, & a_1' \\ b_0', & a_1', & a_0'' \end{vmatrix}$$
$$= \alpha_2\beta_3\gamma_4(\alpha_2\beta_3\gamma_4)$$
$$= \tfrac{1}{6}(\alpha_2\beta_3\gamma_4)^2;$$
and the corresponding concomitant is evidently
$$\Sigma = \tfrac{1}{6}(\alpha\beta\gamma u)^2 = \sigma u_1^2 + \dots.$$

Hence *all the fundamental concomitants appertaining to the unipartite quaternary quadratic U can be expressed in terms of U, T, Θ, P, Σ; to which must be added the universal concomitant u_x, and* (§ 18) *the polar derivatives of the fundamental concomitants, to form the complete algebraical system.*

21. Since $\theta\theta' - \vartheta^2$ is divisible by t, it follows that if we take $\rho^2 + \sigma\theta$ we have a reducible combination; and by actual substitution it is easy to shew that
$$u\sigma - \frac{1}{t}(\rho^2 + \sigma\theta) = (uu'' - u'^2)t - u\theta' + 2u'\vartheta - u''\theta$$
$$= \begin{vmatrix} a_0, & b_0, & a_1, & a_0' \\ b_0, & c_0, & b_1, & b_0' \\ a_1, & b_1, & a_2, & a_1' \\ a_0', & b_0', & a_1', & a_0'' \end{vmatrix} = \Delta,$$

Digitized by Google

the discriminant of the quadratic. This function will be otherwise obtained immediately; but we meanwhile have an illustration of the result of § 19, for

$$\Delta = \frac{1}{u_x^2 T}(U\Sigma T - P^2 - \Sigma\Theta).$$

If it seemed desirable, this equation could be used to replace P by Δ; for P is determinate, save as to sign, as the square root of

$$U\Sigma T - \Sigma\Theta - u_x^2 T\Delta.$$

And then every fundamental concomitant appertaining to the quadratic could be expressed in terms of $U, T, \Theta, \Sigma, \Delta$.

22. As a single illustration of the equations of § 13 and to shew how the discriminant arises, independently of the possibility of making a reducible combination of the concomitants in § 18, we shall determine those concomitants into which no u-variable may enter. Their leading coefficients must satisfy $K_1 = 0$, where

$$K_1 = a_0'' \frac{\partial}{\partial a_0'} + 2a_0' \frac{\partial}{\partial a_0} + a_1' \frac{\partial}{\partial a_1} + b_0' \frac{\partial}{\partial b_0}.$$

We must take such functional combinations of $u, t, \theta, \rho, \sigma$ as will satisfy $K_1 = 0$. Using the process of § 17 we have

$$\begin{aligned} K_1 u &= 2u'; \\ K_1 t &= 0; \\ K_1 \theta &= 2\vartheta; \\ K_1 u' &= u'', \\ K_1 \vartheta &= \theta', \end{aligned}$$

so that $\quad K_1 \rho = \sigma;$

and $\quad K_1 u'' = 0,$

$\quad K_1 \theta' = 0,$

so that $\quad K_1 \sigma = 0.$

Hence t and σ are solutions; and we must obtain solutions (if any exist) of

$$K_1 u = 2u', \quad K_1 \theta = 2\vartheta, \quad K_1 \rho = \sigma.$$

From the first two we easily have $\quad K_1(ut - \theta) = 2\rho,$

which, when combined with the third, gives

$$K_1\{\sigma(ut - \theta) - \rho^2\} = 0,$$

so that a new solution is $\sigma(ut - \theta) - \rho^2$, being in fact $t\Delta$. That Δ is an invariant (were it not otherwise known) may at once be seen by symbolizing the determinant Δ into the form $\frac{1}{24}(\alpha\beta\gamma\delta)^2$.

Digitized by Google

II. *A system of two quadratics.*

23. One of the quadratics may be taken to be that which has just been discussed; the other will be taken in the form

$$\begin{aligned} U_2' = \quad & A_0x_1^2 + 2A_1x_1x_2 + A_2x_2^2 + \quad 2A_0'x_1x_4 + 2A_1'x_2x_4 + A_0''x_4^2. \\ & + 2B_0x_1x_2 + 2B_1x_2x_3 \qquad\qquad + 2B_0'x_3x_4 \\ & + \ C_0x_3^2. \end{aligned}$$

The characteristic equations which determine leading coefficients are

$$M_3 = b_0\frac{\partial}{\partial a_1} + b_0'\frac{\partial}{\partial a_1'} + c_0\frac{\partial}{\partial b_1} + 2b_1\frac{\partial}{\partial a_2} + B_0\frac{\partial}{\partial A_1} + B_0'\frac{\partial}{\partial A_1'} + C_0\frac{\partial}{\partial B_1} + 2B_1\frac{\partial}{\partial A_2} = 0,$$

$$N_2 = a_1\frac{\partial}{\partial b_0} + a_1'\frac{\partial}{\partial b_0'} + a_2\frac{\partial}{\partial b_1} + 2b_1\frac{\partial}{\partial c_0} + A_1\frac{\partial}{\partial B_0} + A_1'\frac{\partial}{\partial B_0'} + A_2\frac{\partial}{\partial B_1} + 2B_1\frac{\partial}{\partial C_0} = 0,$$

$$\Theta = M_4 = b_0\frac{\partial}{\partial a_0'} + b_1\frac{\partial}{\partial a_1'} + c_0\frac{\partial}{\partial b_0'} + 2b_0'\frac{\partial}{\partial a_0''} + B_0\frac{\partial}{\partial A_0'} + B_1\frac{\partial}{\partial A_1'} + C_0\frac{\partial}{\partial B_0'} + 2B_0'\frac{\partial}{\partial A_0''} = 0;$$

and we have to find the simultaneous solutions of these three equations.

We proceed as in § 19. The equations $M_3 = 0$ and $N_2 = 0$ are the differential equations of the system of concomitants of two binary quadratics with c_0, b_1, a_2 and C_0, B_1, A_2 as literal coefficients and in four sets of variables, viz., a_1, $-b_0$; a_1', $-b_0'$; A_1, $-B_0$; A_1', $-B_0'$. To this set of solutions must be added those, the variables of which do not enter into either $M_3 = 0$ or $N_2 = 0$.

The latter are

$$\begin{aligned} v_1 &= a_0, & v_2 &= A_0, \\ v_1' &= a_0', & v_2' &= A_0', \\ v_1'' &= a_0''; & v_2'' &= A_0''. \end{aligned}$$

For the system of simultaneous binariants, we take

$$\begin{aligned} \theta_1 &= c_0a_1^2 - 2b_1a_1b_0 + a_2b_0^2, \\ \phi_1 &= C_0a_1^2 - 2B_1a_1b_0 + A_2b_0^2, \\ j_1 &= (b_1C_0 - B_1c_0)\,a_1^2 - (a_2C_0 - c_0A_2)\,a_1b_0 + (a_2B_1 - A_2b_1)\,b_0^2, \end{aligned}$$

where j_1 is the Jacobian of θ_1 and ϕ_1; and the polars of these in the different combinations of the binary variables are denoted by

$$\begin{aligned} \theta_2 &= c_0a_1a_1' - b_1(a_1b_0' + a_1'b_0) + a_2b_0b_0', & \theta_7 &= c_0a_1'A_1' - b_1(a_1'B_0' + A_1'b_0') + a_2b_0'B_0', \\ \theta_3 &= c_0a_1A_1 - b_1(a_1B_0 + A_1b_0) + a_2b_0B_0, & \theta_8 &= c_0A_1^2 - 2b_1A_1B_0 + a_2B_0^2, \\ \theta_4 &= c_0a_1A_1' - b_1(a_1B_0' + A_1'b_0) + a_2b_0B_0', & \theta_9 &= c_0A_1A_1' - b_1(A_1B_0' + A_1'B_0) + a_2B_0B_0', \\ \theta_5 &= c_0a_1'^2 - 2b_1a_1'b_0' + a_2b_0'^2, & \theta_{10} &= c_0A_1'^2 - 2b_1A_1'B_0' + a_2B_0'^2, \\ \theta_6 &= c_0a_1'A_1 - b_1(a_1'B_0 + A_1b_0') + a_2b_0'B_0, \end{aligned}$$

with $\phi_2, \ldots, \phi_{10}$, and $j_2, \ldots, j_{10}$, for the similar polars of ϕ_1 and of j_1 respectively.

Digitized by Google

The invariant of θ_1 we denote by

$$t_1 = a_2 c_0 - b_1^2;$$

that of ϕ_1 we denote by $$t_2 = A_2 C_0 - B_1^2;$$

and the intermediate invariant of θ_1 and ϕ_1 we denote by

$$t_{12} = a_2 C_0 + A_2 c_0 - 2 b_1 B_1.$$

Lastly, the determinants of the variables we represent by

$$\begin{aligned} \epsilon_1 &= a_1' B_0 - b_0' A_1, & \epsilon_4 &= A_1 B_0' - B_0 A_1', \\ \epsilon_2 &= a_1 B_0 - b_0 A_1, & \epsilon_5 &= a_1' B_0' - b_0' A_1', \\ \epsilon_3 &= a_1 b_0' - a_1' b^0, & \epsilon_6 &= a_1 B_0' - b_0 A_1', \end{aligned}$$

with the evident relation $$\epsilon_1 \epsilon_6 + \epsilon_3 \epsilon_4 = \epsilon_2 \epsilon_5.$$

These 39 $(= 3.10 + 3 + 6)$ constitute the 'irreducible' system for the two binary quadratics in the four sets of variables.

24. One set, algebraically 'complete', of solutions—that is, such that every solution can be algebraically expressed in terms of the members of the set—is found by the process of § 17 to be constituted by

$$v_1,\ v_1',\ v_1'';\quad v_2,\ v_2',\ v_2'';\quad t_1,\ t_2;$$

$$\epsilon_1,\ \epsilon_2,\ \epsilon_3,\ \epsilon_5,\ \epsilon_6;$$

$$\theta_1,\ \theta_2;\quad \phi_1,\ \phi_2.$$

This set is unsymmetrical; and it can be modified, by means of algebraical relations among the irreducible solutions, so as to be symmetrical. Thus for instance from the equations

$$\theta_1 \theta_5 - \theta_2^2 = t_1 \epsilon_3^2, \qquad \phi_1 \phi_5 - \phi_2^2 = t_2 \epsilon_3^2,$$

we can in the 'complete' system replace θ_2 by θ_5 and ϕ_2 by ϕ_5.

From the equations

$$\left.\begin{aligned} \theta_1 \epsilon_1 + \theta_3 \epsilon_3 &= \theta_2 \epsilon_2 \\ \theta_1 \theta_8 - \theta_3^2 &= t_1 \epsilon_2^2 \end{aligned}\right\}, \qquad \left.\begin{aligned} \phi_1 \epsilon_1 + \phi_3 \epsilon_3 &= \phi_2 \epsilon_2 \\ \phi_1 \phi_8 - \phi_3^2 &= t_2 \epsilon_2^2 \end{aligned}\right\},$$

and, remembering that θ_3 and ϕ_3 have been replaced, we can in the 'complete' system replace ϵ_1 and ϵ_2 by θ_3 and ϕ_3.

From the equations

$$\left.\begin{aligned} \theta_1 \epsilon_5 + \theta_4 \epsilon_3 &= \theta_2 \epsilon_6 \\ \theta_1 \theta_{10} - \theta_4^2 &= t_1 \epsilon_6^2 \end{aligned}\right\}, \qquad \left.\begin{aligned} \phi_1 \epsilon_5 + \phi_4 \epsilon_3 &= \phi_2 \epsilon_6 \\ \phi_1 \phi_{10} - \phi_4^2 &= t_2 \epsilon_6^2 \end{aligned}\right\},$$

and remembering that θ_2, ϕ_2 and ϵ_3 have all been replaced, we can in the 'complete' system replace ϵ_5 and ϵ_6 by θ_{10} and ϕ_{10}.

Digitized by Google

The 'complete' system is now composed of

$$v_1,\ v_1',\ v_1'';\quad v_2,\ v_2',\ v_2'';$$
$$\epsilon_2;$$
$$t_1,\ t_2;$$
$$\theta_1,\ \theta_5,\ \theta_8,\ \theta_{10};\quad \phi_1,\ \phi_5,\ \phi_8,\ \phi_{10};$$

which is symmetrical so far as the two quadratics are concerned. Other symmetrical systems, 'complete' and therefore equivalent to the foregoing, could similarly be deduced.

The remaining solutions may be regarded as subsidiary to the complete system.

25. In order to obtain as many distinct concomitants as possible of the two quaternary quadratics, we proceed to find the simplest combinations of the preceding 'irreducible' system which satisfy $\Theta = 0$; and we shall select from them those which form an algebraically 'complete' system.

We have
$$\Theta t_1 = 0,\quad \Theta t_2 = 0,$$
$$\Theta t_{12} = 0;$$
$$\left.\begin{aligned}\Theta\theta_1 &= 0\\ \Theta\theta_2 &= 0\\ \Theta\theta_3 &= 0\end{aligned}\right\},\qquad \left.\begin{aligned}\Theta\phi_1 &= 0\\ \Theta\phi_2 &= 0\\ \Theta\phi_3 &= 0\end{aligned}\right\};$$
$$\Theta v_1 = 0,\quad \Theta v_2 = 0,$$
$$\Theta\epsilon_2 = 0;$$
$$\left.\begin{aligned}\Theta v_1' &= b_0\\ \Theta\theta_2 &= t_1 b_0\\ \Theta\phi_4 &= t_2 b_0\end{aligned}\right\},\qquad \left.\begin{aligned}\Theta v_2' &= B_0\\ \Theta\theta_6 &= t_1 B_0\\ \Theta\phi_9 &= t_2 B_0\end{aligned}\right\};$$
$$\left.\begin{aligned}\Theta v_1'' &= 2b_0'\\ \Theta\theta_5 &= 2t_1 b_0'\end{aligned}\right\},\qquad \left.\begin{aligned}\Theta v_2'' &= 2B_0'\\ \Theta\phi_{10} &= 2t_2 B_0'\end{aligned}\right\};$$
$$\left\{\begin{aligned}\Theta\theta_9 &= (a_2C_0 - b_1B_1)\,B_0 + (c_0B_1 - b_1C_0)\,A_1,\\ \tfrac{1}{2}\Theta\theta_{10} &= (a_2C_0 - b_1B_1)\,B_0' + (c_0B_1 - b_1C_0)\,A_1',\\ \Theta\theta_4 &= (a_2C_0 - b_1B_1)\,b_0 + (c_0B_1 - b_1C_0)\,a_1,\\ \Theta\theta_7 &= (a_2C_0 - b_1B_1)\,b_0' + (c_0B_1 - b_1C_0)\,a_1' + t_1B_0';\end{aligned}\right.$$
$$\left\{\begin{aligned}\Theta\phi_2 &= (A_2c_0 - B_1b_1)\,b_0 + (C_0b_1 - B_1c_0)\,a_1,\\ \tfrac{1}{2}\Theta\phi_5 &= (A_2c_0 - B_1b_1)\,b_0' + (C_0b_1 - B_1c_0)\,a_1',\\ \Theta\phi_6 &= (A_2c_0 - B_1b_1)\,B_0 + (C_0b_1 - B_1c_0)\,A_1,\\ \Theta\phi_7 &= (A_2c_0 - B_1b_1)\,B_0' + (C_0b_1 - B_1c_0)\,A_1' + t_2b_0'.\end{aligned}\right.$$

From combinations of the last eight equations, we have

$$\left.\begin{aligned}\Theta(\theta_4 + \phi_2) &= t_{12}b_0\\ \Theta(\tfrac{1}{2}\phi_5 + \theta_7) &= t_{12}b_0' + t_1B_0'\end{aligned}\right\},\qquad \left.\begin{aligned}\Theta(\theta_9 + \phi_6) &= t_{12}B_0\\ \Theta(\tfrac{1}{2}\theta_{10} + \phi_7) &= t_{12}B_0' + t_2b_0'\end{aligned}\right\}.$$

Digitized by Google

In addition to the quantities which are at once seen to be solutions of $\Theta = 0$, viz.,

$$t_1,\ t_2,\ t_{12};\quad v_1,\ v_2;\quad \epsilon_2;\quad \theta_1,\ \theta_2,\ \theta_3;\quad \phi_1,\ \phi_2,\ \phi_3;$$

other solutions are given by

$$\left.\begin{aligned}\rho_1 &= -\theta_2 + t_1 v_1'\\ \rho_2 &= -\phi_9 + t_2 v_2'\end{aligned}\right\},\qquad \left.\begin{aligned}\pi_1 &= -\theta_6 + t_1 v_2'\\ \pi_2 &= -\phi_4 + t_2 v_1'\end{aligned}\right\},\qquad \left.\begin{aligned}\sigma_1 &= -\theta_5 + t_1 v_1''\\ \sigma_2 &= -\phi_{10} + t_2 v_2''\end{aligned}\right\},$$

$$\left.\begin{aligned}\lambda_1 &= t_{12} v_1' - \theta_4 - \phi_2\\ \lambda_2 &= t_{12} v_2' - \theta_9 - \phi_6\end{aligned}\right\},$$

$$\left.\begin{aligned}\xi_2 &= t_{12} v_2'' + t_2 v_1'' - 2\phi_7 - \theta_{10}\\ \xi_1 &= t_{12} v_1'' + t_1 v_2'' - 2\theta_7 - \phi_5\end{aligned}\right\}.$$

26. One algebraically complete system for two simultaneous quaternary quadratics is constituted by the fifteen concomitants which have as their leading coefficients the fifteen quantities

$$\begin{array}{ll} v_1, & v_2;\\ t_1, & t_2;\\ & \epsilon_2;\\ \theta_1, & \phi_2;\\ \theta_2, & \phi_1;\\ \rho_1, & \rho_2;\\ \sigma_1, & \sigma_2;\\ \xi_1, & \xi_2. \end{array}$$

This set is symmetrical so far as the two quadratics are concerned; and every leading coefficient of a concomitant of the irreducible system for the two quaternary quadratics can be expressed algebraically in terms of the foregoing fifteen leading coefficients.

In addition to these fifteen, we have obtained other seven, viz.

$$t_{12};\ \theta_3,\ \phi_3;\ \pi_1,\ \pi_2;\ \lambda_1,\ \lambda_2;$$

and we have not yet considered the series of Jacobians $j_1, j_2, \ldots, j_{10}$. Of these we find at once that j_1, j_2, j_3 all satisfy $\Theta = 0$; but the remainder, as well as $\epsilon_1, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6$ do not lead to equations admitting, as do those of § 23, of immediate integration. We thus have ten simple leading coefficients

$$t_{12};\ \theta_3,\ \phi_3;\ \pi_1,\ \pi_2;\ \lambda_1,\ \lambda_2;\ j_1,\ j_2,\ j_3,$$

subsidiary to the algebraically complete system of fifteen.

27. One remark may be made before passing on to give the full symbolical expression of the concomitants determined by the leading coefficients which have been obtained. Denoting by $U_1 (= v_1 x_1^2 + \ldots)$ and $U_2 (= v_2 x_1^2 + \ldots)$ the two quadratics, we have $U_1 + \lambda U_2$ also a quadratic for every value of λ. Hence when we form the five concomitants, which constitute the complete system for $U_1 + \lambda U_2$, the coefficients of the various

Digitized by Google

powers of λ in each of the five must be simultaneous concomitants of the system made up of U_1 and U_2.

Adopting the notation of § 18, it is easy to shew that

$$u(U_1+\lambda U_2) = v_1 + \lambda v_2,$$

$$t(U_1+\lambda U_2) = t_1 + \lambda t_{12} + \lambda^2 t_2,$$

$$\theta(U_1+\lambda U_2) = \theta_1 + \lambda(2\theta_3+\phi_1) + \lambda^2(\theta_2+2\phi_2) + \lambda^3\phi_3,$$

$$\rho(U_1+\lambda U_2) = \rho_1 + \lambda(\pi_1+\lambda_1) + \lambda^2(\pi_2+\lambda_2) + \lambda^3\rho_2,$$

$$\sigma(U_1+\lambda U_2) = \sigma_1 + \lambda\xi_1 + \lambda^2\xi_2 + \lambda^3\sigma_2,$$

thus verifying the statement. And by means of these we can find expressions equivalent to the five simultaneous invariants similar to that which is equivalent to the discriminant of a single quadratic; for we have

$$t\Delta = u\sigma t - \rho^2 - \sigma\theta,$$

and all that is necessary is to replace each of these quantities by the corresponding quantity for $U_1+\lambda U_2$ and then equate coefficients of λ.

28. The full symbolical expressions of which the foregoing are the leading coefficients are as follow:—

$$U_1 = \alpha_x^2 = v_1 x_1^2 + \dots$$

$$U_2 = \alpha_x'^2 = v_2 x_1^2 + \dots$$

$$T_1 = \tfrac{1}{2}(\alpha\beta p)^2 = t_1 p_1^2 + \dots$$

$$T_{12} = (\alpha\alpha' p)^2 = t_{12} p_1^2 + \dots$$

$$T_2 = \tfrac{1}{2}(\alpha'\beta' p)^2 = t_2 p_1^2 + \dots$$

$$\Theta_1 = \beta_x\gamma_x(\alpha\beta p)(\alpha\gamma p) = \theta_1 x_1^2 p_1^2 + \dots$$

$$\Theta_2 = \alpha_x\alpha_x'(\alpha\beta p)(\alpha'\beta p) = \theta_2 x_1^2 p_1^2 + \dots$$

$$\Theta_3 = \alpha_x'\beta_x'(\alpha\alpha' p)(\alpha\beta' p) = \theta_3 x_1^2 p_1^2 + \dots$$

$$\Phi_1 = \alpha_x\beta_x(\alpha\alpha' p)(\beta\alpha' p) = \phi_1 x_1^2 p_1^2 + \dots$$

$$\Phi_2 = \alpha_x\alpha_x'(\alpha\beta' p)(\alpha'\beta' p) = \phi_2 x_1^2 p_1^2 + \dots$$

$$\Phi_3 = \beta_x'\gamma_x'(\alpha'\beta' p)(\alpha'\gamma' p) = \phi_3 x_1^2 p_1^2 + \dots$$

$$E_2 = \alpha_x\alpha_x'(\alpha'\alpha p) = e_2 x_1^2 p_1 + \dots$$

$$P_1 = \tfrac{1}{2}\alpha_x(\alpha\beta\gamma u)(\beta\gamma p) = \rho_1 x_1 u_1 p_1 + \dots$$

$$P_2 = \tfrac{1}{2}\alpha_x'(\alpha'\beta'\gamma' u)(\beta'\gamma' p) = \rho_2 x_1 u_1 p_1 + \dots$$

Digitized by Google

$$\Pi_1 = \tfrac{1}{2}\alpha_x' (\alpha'\alpha\beta u)(\alpha\beta p) = \pi_1 x_1 u_1 p_1 + \dots$$

$$\Pi_2 = \tfrac{1}{2}\alpha_x (\alpha\alpha'\beta' u)(\alpha'\beta' p) = \pi_2 x_1 u_1 p_1 + \dots$$

$$\Sigma_1 = \tfrac{1}{6}(\alpha\beta\gamma u)^2 = \sigma_1 u_1^2 + \dots$$

$$\Sigma_2 = \tfrac{1}{6}(\alpha'\beta'\gamma' u)^2 = \sigma_2 u_1^2 + \dots$$

$$\Lambda_1 = \alpha_x (\alpha'\beta' p)(\alpha\alpha'\beta u) = \lambda_1 x_1 u_1 p_1 + \dots$$

$$\Lambda_2 = \alpha_x' (\alpha\beta' p)(\alpha'\alpha\beta' u) = \lambda_2 x_1 u_1 p_1 + \dots$$

$$\Xi_1 = \tfrac{1}{2}(\alpha'\alpha\beta u)^2 = \xi_1 u_1^2 + \dots$$

$$\Xi_2 = \tfrac{1}{2}(\alpha\alpha'\beta' u)^2 = \xi_2 u_1^2 + \dots$$

$$J_1 = \beta_x\gamma_x (\alpha'\alpha p)(\alpha\beta p)(\alpha'\gamma p) = j_1 x_1^2 p_1^3 + \dots$$

$$J_2 = \tfrac{1}{2}\beta_x\beta_x' (\alpha'\alpha p)\{(\alpha\beta' p)(\alpha\beta p) + (\alpha\beta p)(\alpha'\beta' p)\} = j_2 x_1^2 p_1^3 + \dots$$

$$J_3 = \beta_x'\gamma_x' (\alpha'\alpha p)(\alpha'\beta' p)(\alpha\gamma' p) = j_3 x_1^2 p_1^3 + \dots$$

Further
$$J_2 + \tfrac{1}{2}T_{12}E_2 = \alpha_x\alpha_x' (\alpha\beta p)(\alpha'\beta' p)(\beta\beta' p),$$
$$J_2 - \tfrac{1}{2}T_{12}E_2 = \alpha_x\alpha_x' (\alpha\beta' p)(\alpha'\beta p)(\beta\beta' p);$$

and the symbolical values of the five simultaneous invariants referred to in § 25 are

$$\tfrac{1}{24}(\alpha\beta\gamma\delta)^2,\ \tfrac{1}{6}(\alpha'\alpha\beta\gamma)^2,\ \tfrac{1}{4}(\alpha'\beta'\alpha\beta)^2,\ \tfrac{1}{6}(\alpha'\beta'\gamma'\alpha)^2,\ \tfrac{1}{24}(\alpha'\beta'\gamma'\delta')^2.$$

Hence we have the theorem:—

All the fundamental concomitants appertaining to the system composed of two unipartite quaternary quadratics can be expressed algebraically in terms of the members of the set of fifteen algebraically independent concomitants constituted by

$$U_1,\ U_2;\ T_1,\ T_2;\ E_2;\ \Theta_1,\ \Phi_3,\ \Theta_3,\ \Phi_1;\ P_1,\ P_2;\ \Sigma_1,\ \Sigma_2;\ \Xi_1,\ \Xi_2;$$

and to these must be added the universal concomitant u_x, and (§ 18) *the polar derivatives of the fundamental concomitants, to form the complete algebraical system.*

29. The operators of § 13, annihilation of leading coefficients by which implies the absence of variables of some class, are:—

$$L_4 = a_0\frac{\partial}{\partial a_0'} + b_0\frac{\partial}{\partial b_0'} + a_1\frac{\partial}{\partial a_1'} + 2a_0'\frac{\partial}{\partial a_0''} + A_0\frac{\partial}{\partial A_0'} + B_0\frac{\partial}{\partial B_0'} + A_1\frac{\partial}{\partial A_1'} + 2A_0'\frac{\partial}{\partial A_0''},$$

$$K_2 = a_0'\frac{\partial}{\partial b_0} + a_1'\frac{\partial}{\partial b_1} + a_0''\frac{\partial}{\partial b_0'} + 2b_0'\frac{\partial}{\partial c_0} + A_0'\frac{\partial}{\partial B_0} + A_1'\frac{\partial}{\partial B_1} + A_0''\frac{\partial}{\partial B_0'} + 2B_0'\frac{\partial}{\partial C_0},$$

$$K_1 = a_0''\frac{\partial}{\partial a_0'} + a_1'\frac{\partial}{\partial a_1} + b_0'\frac{\partial}{\partial b_0} + 2a_0'\frac{\partial}{\partial a_0} + A_0''\frac{\partial}{\partial A_0'} + A_1'\frac{\partial}{\partial A_1} + B_0'\frac{\partial}{\partial B_0} + 2A_0'\frac{\partial}{\partial A_0},$$

annihilation by L_4 implying the absence of u, by K_2 of p, by K_1 of x.

Digitized by Google

Thus, to find concomitants which involve at the utmost only x and p, we require to find combinations of $\rho_1, \rho_2, \sigma_1, \sigma_2, \pi_1, \pi_2, \lambda_1, \lambda_2, \xi_1, \xi_2$ which will satisfy $L_4 = 0$; all the other concomitants obtained being evidently such as already satisfy $L_4 = 0$. We have

$$\left.\begin{aligned} L_4\sigma_1 &= 2\rho_1 \\ L_4\sigma_2 &= 2\rho_2 \end{aligned}\right\} \qquad \left.\begin{aligned} L_4\xi_1 &= 2(\pi_1 + \lambda_1) \\ L_4\xi_2 &= 2(\pi_2 + \lambda_2) \end{aligned}\right\}$$

$$\left.\begin{aligned} L_4\rho_1 &= t_1v_1 - \theta_1 \\ L_4\rho_2 &= t_2v_2 - \phi_3 \end{aligned}\right\} \qquad \left.\begin{aligned} L_4\lambda_1 &= t_{12}v_1 - \theta_2 - \phi_1 \\ L_4\lambda_2 &= t_{12}v_2 - \theta_3 - \phi_2 \end{aligned}\right\} \qquad \left.\begin{aligned} L_4\pi_1 &= t_1v_2 - \theta_3 \\ L_4\pi_2 &= t_2v_1 - \phi_3 \end{aligned}\right\}$$

all the quantities on the right-hand sides of the last three sets of equations being solutions of $L_4 = 0$. Thus we have as solutions

$$\begin{aligned} &\sigma_1(t_1v_1 - \theta_1) - \rho_1^2, \\ &\sigma_2(t_1v_1 - \theta_1) + \sigma_1(t_2v_2 - \phi_3) - 2\rho_1\rho_2, \\ &\sigma_2(t_2v_2 - \phi_3) - \rho_2^2, \end{aligned}$$

which are all reducible; there is a similar set connecting the ξ's, π's and λ's; and there is the set of integrals of the last six equations of the same form as

$$(t_1v_1 - \theta)\pi_1 - (t_1v_2 - \theta_2)\rho_1$$

which is reducible, being divisible by t_1. Similarly for the other classes of concomitants.

30. All the preceding investigations have connected the leading coefficients of concomitants with the theory of simultaneous binariants. Some of these leading coefficients may, however, be similarly connected with the theory of simultaneous ternariants; and an easy illustration is afforded by the present system of two quaternary quadratics.

Thus let it be required to find quaternariants of the system of two quadratics which *shall not involve line-coordinates;* it is not difficult to prove that their leading coefficients must satisfy the six equations $M_3 = 0$, $M_4 = 0$, $N_2 = 0$ (already used),

$$N_4 = a_1\frac{\partial}{\partial a_0'} + a_2\frac{\partial}{\partial a_1'} + b_1\frac{\partial}{\partial b_0'} + 2a_1'\frac{\partial}{\partial a_0''} + A_1\frac{\partial}{\partial A_1'} + A_2\frac{\partial}{\partial A_1'} + B_1\frac{\partial}{\partial B_0'} + 2A_1'\frac{\partial}{\partial A_0''} = 0,$$

$$K_2 = a_0'\frac{\partial}{\partial b_0} + a_1'\frac{\partial}{\partial b_1} + a_0''\frac{\partial}{\partial b_0'} + 2b_0'\frac{\partial}{\partial c_0} + A_0'\frac{\partial}{\partial B_0} + A_1'\frac{\partial}{\partial B_1} + A_0''\frac{\partial}{\partial B_0'} + 2B_0'\frac{\partial}{\partial C_0} = 0,$$

and

$$K_3 = a_0'\frac{\partial}{\partial a_1} + b_0'\frac{\partial}{\partial b_1} + a_0''\frac{\partial}{\partial a_1'} + 2a_1'\frac{\partial}{\partial a_2} + A_0'\frac{\partial}{\partial A_1} + B_0'\frac{\partial}{\partial B_1} + A_0''\frac{\partial}{\partial A_1'} + 2A_1'\frac{\partial}{\partial A_2} = 0,$$

of which, as indicated in § 6, N_4 and K_3 are derivable from the other four: they are however retained here in order to make the set of equations complete in form.

Now on comparing these six equations with the six equations which determine ternariants* and associating the six operators $D_1, D_2, D_3, D_4, D_5, D_6$ (*l. c.*) with $K_3, M_4, N_2, M_3, K_2, N_4$ respectively, it appears that the preceding six equations are the differential equations of pure contravariants (including invariants) of a system of the two simultaneous ternary quadratics

* See the memoir on "Ternariants" (§ 1), before referred to.

Digitized by Google

$$c_0\xi^2 + 2b_1\xi\eta + a_2\eta^2 + 2b_0'\xi\zeta + 2a_1'\eta\zeta + a_0''\zeta^2$$
$$C_0\xi^2 + 2B_1\xi\eta + A_2\eta^2 + 2B_0'\xi\zeta + 2A_1'\eta\zeta + A_0''\zeta^2,$$

the variables of the contravariants being two sets, viz. b_0, a_1, a_0'; and B_0, A_1, A_0'. (It must be borne in mind that there are, in addition, the leading coefficients a_0, A_0 of the fundamental quaternary quadratics; they do not enter into any of the six equations.)

Hence then we take *all the pure contravariants in the two sets of variables* b_0, a_1, a_0' *and* B_0, A_1, A_0' *of the two ternary quadratics.* These will consist of

(i) *pure contravariants* (including invariants) of the two ternary quadratics *in* b_0, a_1, a_0' *as contragredient variables:*

(ii) *all polars of pure contravariants,* of the set (i), *with* B_0, A_1, A_0' *as polarising variables:*

(iii) *pure covariants* of the two ternary quadratics in

$$\xi = a_1A_0' - a_0'A_1, \quad \eta = a_0'B_0 - A_0'b_0, \quad \zeta = b_0A_1 - a_1B_0$$

as ordinary variables:

(iv) *mixed concomitants in both sets of contragredient variables and in the ordinary variables of* (iii):

the last two arising from the well-known property that, from two sets of variables cogredient with one another, a third set contragredient with them can be formed. For instance, by means of the equations

$$M_3\xi = -\eta, \quad M_4\xi = -\zeta; \quad N_3\eta = -\xi, \quad N_4\eta = -\zeta; \quad K_3\zeta = -\xi, \quad K_3\zeta = -\eta,$$

(the other combinations of operator and variable giving zero), it is easy to verify that the quadratics satisfy all the six equations.

31. Gordan has given* the complete irreducible system of concomitants of two ternary quadratics in a single set of contragredient variables. If then we take these, bearing in mind the special values of the variables in the present case, and add to the system the concomitants arising from polarisation by the second set of contragredient variables, we shall have *all the leading coefficients which are irreducible without the introduction of* a_0 *and* A_0, the leading coefficients of the two quaternary quadratics.

Taking then Gordan's system and notation for the concomitants and retaining the symbolical notation for the quaternary quadratic, the fundamental ternary quadratics are

$$f = (\alpha_2\xi + \alpha_3\eta + \alpha_4\zeta)^2 = \alpha_\xi^2, \quad f' = (\alpha_2'\xi + \alpha_3'\eta + \alpha_4'\zeta)^2 = \alpha_\xi'^2;$$

we shall denote the ternary contragredient variables by

$$w_1,\ w_2,\ w_3\ (= b_0,\ a_1,\ a_1'),$$

and

$$W_1,\ W_2,\ W_3\ (= B_0,\ A_1,\ A_1'),$$

so that

$$F_{11} = (\alpha\beta w)^2 = w_\theta^2,$$

and similarly for the other functions.

* Clebsch, *Vorlesungen über Geometrie*, p. 291.

Digitized by Google

These functions have to be transformed so as to contain only the umbral elements of the quaternary quadratics; they are then leading coefficients of quaternariants. We have

$$\xi = a_1 A_0' - A_1 a_0' = \beta_1 \beta_1' (\beta_3 \beta_4' - \beta_4 \beta_3'),$$
$$\eta = a_0' B_0 - b_0 A_0' = \beta_1 \beta_1' (\beta_4 \beta_2' - \beta_2 \beta_4'),$$
$$\zeta = b_0 A_1 - a_1 B_0 = \beta_1 \beta_1' (\beta_2 \beta_3' - \beta_3 \beta_2'),$$

so that
$$\alpha_\xi = \beta_1 \beta_1' (\alpha_2 \beta_3 \beta_4'),$$
and therefore
$$f = \beta_1 \beta_1' \gamma_1 \gamma_1' (\alpha_2 \beta_3 \beta_4') (\alpha_2 \gamma_3 \gamma_4').$$

This being a leading coefficient, the concomitant determined by it is

$$\beta_x \beta_x' \gamma_x \gamma_x' (\alpha\beta\beta' u) (\alpha\gamma\gamma' u).$$

Similarly we have, for F_{11},

$$w_\theta = \begin{vmatrix} w_1, & w_2, & w_3 \\ \alpha_2, & \alpha_3, & \alpha_4 \\ \beta_2, & \beta_3, & \beta_4 \end{vmatrix} = \begin{vmatrix} b_0, & a_1, & a_0' \\ \alpha_2, & \alpha_3, & \alpha_4 \\ \beta_2, & \beta_3, & \beta_4 \end{vmatrix} = \gamma_1 (\alpha_2 \beta_3 \gamma_4),$$

so that
$$F_{11} = \gamma_1 \delta_1 (\alpha_2 \beta_3 \gamma_4) (\alpha_2 \beta_3 \delta_4),$$

and the concomitant thence determined is

$$\gamma_x \delta_x (\alpha\beta\gamma u) (\alpha\beta\delta u),$$

which can easily be proved equal to

$$\alpha_x^2 (\beta\gamma\delta u)^2 - \tfrac{1}{4} u_x^2 (\alpha\beta\gamma\delta)^2.$$

The first polar of F_{11} is
$$= w_\theta W_\theta$$
$$= \gamma_1 (\alpha_2 \beta_3 \gamma_4) \gamma_1' (\alpha_2 \beta_3 \gamma_4'),$$

the corresponding concomitant being

$$\gamma_x \gamma_x' (\alpha\beta\gamma u) (\alpha\beta\gamma' u),$$

which can easily be proved equal to

$$\tfrac{1}{3} \gamma_x'^2 (\alpha\beta\gamma u)^2 - \tfrac{1}{3} u_x \gamma_x' (\alpha\beta\gamma u) (\alpha\beta\gamma\gamma').$$

The second polar of F_{11} is
$$= W_\theta^2$$
$$= \alpha_1' \beta_1' (\alpha_2 \beta_3 \alpha_4') (\alpha_2 \beta_3 \beta_4'),$$

and the concomitant thence determined is

$$\alpha_x' \beta_x' (\alpha\beta\alpha' u) (\alpha\beta\beta' u).$$

The other functions are similarly obtained with the following results, the (Gordan) symbol on the left-hand side being used, for convenience, to denote the concomitant as well as its leading coefficient.

32. The aggregate is:—

$$\left.\begin{aligned} f &= \beta_x \beta_x' \gamma_x \gamma_x' (\alpha\beta\beta' u)\ (\alpha\gamma\gamma' u) \\ f' &= \beta_x \beta_x' \gamma_x \gamma_x' (\alpha'\beta\beta' u) (\alpha'\gamma\gamma' u) \end{aligned}\right\};$$

$$\left.\begin{aligned} F_{11} &= \gamma_x \delta_x\ (\alpha\beta\gamma u)\ (\alpha\beta\delta u) \\ \text{*first 'polar' of } F_{11} &= \gamma_x \gamma_x'\ (\alpha\beta\gamma u)\ (\alpha\beta\gamma' u) \\ \text{second 'polar' of } F_{11} &= \gamma_x' \delta_x' (\alpha\beta\gamma' u) (\alpha\beta\delta' u) \end{aligned}\right\};$$

* These 'polars' are polars of the leading coefficients by means of the literal elements in § 30, (ii); there are, in addition, all the concomitants given as polars of the concomitants, here determined, by means of the variables in § 9.

Digitized by Google

and the combination F_{11} (second polar) $-$ (first polar)^2, being in fact $(\theta\phi\xi)^2$, produces no new concomitant;

$$\left.\begin{aligned} F_{22} &= \gamma_x\delta_x\ (\alpha'\beta'\gamma u)\ (\alpha'\beta'\delta u) \\ \text{first 'polar' of } F_{22} &= \gamma_x\gamma_x'\ (\alpha'\beta'\gamma u)\ (\alpha'\beta'\gamma' u) \\ \text{second 'polar' of } F_{22} &= \gamma_x'\delta_x'\ (\alpha'\beta'\gamma' u)\ (\alpha'\beta'\delta' u) \end{aligned}\right\};$$

$$\left.\begin{aligned} A_{111} &= (\alpha\beta\gamma u)^2 \\ A_{112} &= (\alpha'\beta\gamma u)^2 \\ A_{122} &= (\alpha'\beta'\gamma u)^2 \\ A_{222} &= (\alpha'\beta'\gamma' u)^2 \end{aligned}\right\};$$

$$\left.\begin{aligned} B_1 &= \gamma_x\delta_x\beta_x'\ (\alpha\beta\alpha' u)\ (\alpha\beta\gamma u)\ (\alpha'\delta\beta' u) \\ \text{'polar' of } B_1 &= \delta_x\alpha_x'\delta_x'\ (\alpha\beta\gamma' u)\ (\alpha\beta\alpha' u)\ (\gamma'\delta\delta' u) \end{aligned}\right\};$$

$$\left.\begin{aligned} B_2 &= \delta_x\alpha_x\delta_x'\ (\alpha'\beta'\gamma u)\ (\alpha'\beta'\alpha u)\ (\gamma\delta'\delta u) \\ \text{'polar' of } B_2 &= \gamma_x'\delta_x'\beta_x\ (\alpha'\beta'\gamma' u)(\alpha'\beta'\alpha u)(\alpha\delta'\beta u) \end{aligned}\right\};$$

$$\left.\begin{aligned} N &= \beta_x\gamma_x\delta_x\gamma_x'\delta_x'\ (\alpha\alpha'\beta u)(\alpha\gamma\gamma' u)(\alpha'\delta\delta' u) \\ \text{'polar' of } N &= \beta_x'\gamma_x\delta_x\gamma_x'\delta_x'(\alpha\alpha'\beta' u)(\alpha\gamma\gamma' u)(\alpha'\delta\delta' u) \end{aligned}\right\};$$

$$\left.\begin{aligned} C_1 &= \beta_x\epsilon_x\lambda_x\beta_x'\ (\alpha\alpha'\beta u)(\alpha'\gamma\delta u)(\gamma\delta\epsilon u)\ (\alpha\lambda\beta' u) \\ \text{first 'polar' of } C_1 \text{ is congruent to} \quad & \beta_x\epsilon_x\beta_x'\gamma_x'\ (\alpha\alpha'\beta' u)(\alpha'\gamma\delta u)(\gamma\delta\epsilon u)\ (\alpha\beta\gamma' u) \\ \text{second 'polar' of } C_1 &= \beta_x\beta_x'\gamma_x'\delta_x'(\alpha\alpha'\beta' u)(\alpha'\gamma\delta u)(\gamma\delta\delta' u)(\alpha\beta\gamma' u) \end{aligned}\right\};$$

$$\left.\begin{aligned} C_2 &= \beta_x\gamma_x\epsilon_x\beta_x'\ (\alpha\alpha'\beta u)(\alpha\gamma'\delta' u)(\gamma'\delta'\epsilon u)(\alpha'\gamma\beta' u) \\ \text{first 'polar' of } C_2 \text{ is congruent to} \quad & \gamma_x\epsilon_x\beta_x'\epsilon_x'\ (\alpha\alpha'\beta' u)(\alpha\gamma'\delta' u)(\gamma'\delta'\epsilon u)(\alpha'\gamma\epsilon' u) \\ \text{second 'polar' of } C_2 &= \gamma_x\beta_x'\epsilon_x'\lambda_x'(\alpha\alpha'\beta' u)(\alpha\gamma'\delta' u)(\gamma'\delta'\epsilon' u)(\alpha'\gamma\lambda' u) \end{aligned}\right\};$$

$$\left.\begin{aligned} D &= \beta_x\epsilon_x\lambda_x\ (\alpha\alpha'\beta u)(\alpha\gamma'\delta' u)(\alpha'\gamma\delta u)(\gamma\delta\epsilon u)\ (\gamma'\delta'\lambda u) \\ \text{first 'polar' of } D \text{ is congruent to} \quad & \beta_x'\epsilon_x\lambda_x(\alpha\alpha'\beta' u)(\alpha\gamma'\delta' u)(\alpha'\gamma\delta u)(\gamma\delta\epsilon u)(\gamma'\delta'\lambda u) \\ \text{second 'polar' of } D \text{ is congruent to} \quad & \beta_x\epsilon_x'\lambda_x'(\alpha\alpha'\beta u)(\alpha\gamma'\delta' u)(\alpha'\gamma\delta u)(\gamma\delta\epsilon' u)(\gamma'\delta'\lambda' u) \\ \text{third 'polar' of } D \text{ is} \quad & = \beta_x'\epsilon_x'\lambda_x'(\alpha\alpha'\beta' u)(\alpha\gamma'\delta' u)(\alpha'\gamma\delta u)(\gamma\delta\epsilon' u)(\gamma'\delta'\lambda' u) \end{aligned}\right\};$$

$$\left.\begin{aligned} F_{12} &= \beta_x\gamma_x\ (\alpha\alpha'\beta u)\ (\alpha\alpha'\gamma u) \\ \text{first 'polar' of } F_{12} &= \beta_x\beta_x'\ (\alpha\alpha'\beta u)\ (\alpha\alpha'\beta' u) \\ \text{second 'polar' of } F_{12} &= \beta_x'\gamma_x'\ (\alpha\alpha'\beta' u)\ (\alpha\alpha'\gamma' u) \end{aligned}\right\};$$

$$\left.\begin{aligned} N &= \Omega_{\alpha\beta\gamma}\gamma_x\gamma_x'\delta_x\epsilon_x\ (\alpha\beta\delta u)\ (\alpha\beta\epsilon u) \\ \text{first 'polar' of } N \text{ is congruent to} \quad & \Omega_{\alpha\beta\gamma}\gamma_x\gamma_x'\delta_x\delta_x'\ (\alpha\beta\delta u)\ (\alpha\beta\delta' u) \\ \text{second 'polar' of } N \quad & = \Omega_{\alpha\beta\gamma}\gamma_x\gamma_x'\delta_x'\epsilon_x'(\alpha\beta\delta' u)\ (\alpha\beta\epsilon' u) \end{aligned}\right\};$$

Digitized by Google

$$\left.\begin{aligned}\Gamma_1 &= \Omega_{\alpha\beta\gamma}\gamma_x\gamma_x'\epsilon_x\lambda_x\lambda_x'\,(\alpha'\beta'\delta u)(\alpha\beta\epsilon u)(\delta\lambda\lambda' u)\\ \text{'polar' of } \Gamma_1 &= \Omega_{\alpha\beta\gamma}\gamma_x\gamma_x'\epsilon_x'\lambda_x\lambda_x'\,(\alpha'\beta'\delta u)(\alpha\beta\epsilon' u)(\delta\lambda\lambda' u)\end{aligned}\right\};$$

$$\left.\begin{aligned}\Gamma_2 &= \Omega_{\alpha\beta\gamma}\gamma_x\gamma_x'\epsilon_x\lambda_x\lambda_x'\,(\alpha\beta\delta' u)(\alpha'\beta'\epsilon u)(\delta'\lambda\lambda' u)\\ \text{'polar' of } \Gamma_2 &= \Omega_{\alpha\beta\gamma}\gamma_x\gamma_x'\epsilon_x'\lambda_x\lambda_x'\,(\alpha\beta\delta' u)(\alpha'\beta'\epsilon' u)(\delta'\lambda\lambda' u)\end{aligned}\right\};$$

$$\Delta = \Omega_{\alpha\beta\gamma}\gamma_x\gamma_x'\epsilon_x\epsilon_x'\lambda_x\lambda_x'\,(\alpha\beta\delta' u)(\alpha'\beta'\delta u)(\delta\epsilon\epsilon' u)(\delta'\lambda\lambda' u);$$

$$\Phi_{12} = \Omega_{\alpha\beta\gamma}\Omega_{\delta\epsilon\lambda}\gamma_x\gamma_x'\lambda_x\lambda_x';$$

where $$\Omega_{\alpha\beta\gamma} = (\alpha\beta\gamma u)(\alpha'\beta'\gamma' u) - (\alpha\beta\gamma' u)(\alpha'\beta'\gamma u),$$

the value of Gordan's quantity $(\alpha\alpha' x)$ being the product of $\gamma_1\gamma_1'$ by the coefficient of u_1^2 in $\Omega_{\alpha\beta\gamma}$.

This is the system of fundamental concomitants of the two quaternary quadratics; with them must be associated (§ 18) *their polar derivatives.*

Many of the concomitants just obtained might be reduced; the reduction however is not an essential part of the present purpose which has been (§ 31) to connect the class of quaternariants involving x and u with the concomitants of ternary quantics.

33. (*a*) In regard to the general unipartite quantic of order n in point-variables the process of solution of the three characteristic equations of § 15 is simplified by the following theorem, evident from the forms of those equations:—

The simultaneous solutions of the two equations $M_3 = 0$, $N_2 = 0$ are binariants of the simultaneous binary quantics

$$\begin{array}{lll}(c_0, b_1, a_2 \between)^2, & (c_0', b_1', a_2' \between)^2, & (c_0'', b_1'', a_2'' \between)^2, \ldots\ldots\\ (d_0, c_1, b_2, a_3 \between)^3, & (d_0', c_1', b_2', a_3' \between)^3, & \ldots\ldots\ldots\ldots\ldots\ldots\\ (e_0, d_1, c_2, b_3, a_4 \between)^4, & \ldots\ldots\ldots\ldots\ldots\ldots & \\ \ldots\ldots\ldots\ldots\ldots\ldots & & \end{array}$$

in sets of variables $a_1, -b_0$; $a_1', -b_0'$; $a_1'', -b_0''$; $a_1''', -b_0'''$;... To these must be added $a_0, a_0', a_0'', a_0''', a_0'''',\ldots$ which do not enter into either of the two equations mentioned.

To have leading coefficients, which satisfy $M_4 = 0$ in addition to the two preceding equations, it is therefore necessary merely to form such functional combinations of the preceding binariant system (with $a_0, a_0', a_0'', a_0''', a_0'''',\ldots$) as satisfy $M_4 = 0$.

(*b*) To obtain leading coefficients of quaternariants, which do not involve line-variables, associated with the unipartite quantic of order n, the following theorem is useful being derivable from the characteristic equations satisfied. These equations, as in § 30, are, in addition to $M_3 = 0$, $M_4 = 0$, $N_2 = 0$, the three

$$N_4 = a_1\frac{\partial}{\partial a_0'} + b_1\frac{\partial}{\partial b_0'} + a_2\frac{\partial}{\partial a_1'} + 2a_1'\frac{\partial}{\partial a_0''} + c_1\frac{\partial}{\partial c_0'} + b_2\frac{\partial}{\partial b_1'} + 2b_1'\frac{\partial}{\partial b_0''} + a_3\frac{\partial}{\partial a_2'} + 2a_2'\frac{\partial}{\partial a_1''} + 3a_1''\frac{\partial}{\partial a_0'''} + \ldots$$

$$K_2 = a_0'\frac{\partial}{\partial b_0} + a_1'\frac{\partial}{\partial b_1} + a_0''\frac{\partial}{\partial b_0'} + 2b_0'\frac{\partial}{\partial c_0} + a_2'\frac{\partial}{\partial b_2} + a_1''\frac{\partial}{\partial b_1'} + 2b_1'\frac{\partial}{\partial c_1} + a_0'''\frac{\partial}{\partial b_0''} + 2b_0''\frac{\partial}{\partial c_0'} + 3c_0'\frac{\partial}{\partial d_0} + \ldots$$

$$K_3 = a_0'\frac{\partial}{\partial a_1} + b_0'\frac{\partial}{\partial b_1} + b_0''\frac{\partial}{\partial a_1'} + 2a_1'\frac{\partial}{\partial a_2} + c_0'\frac{\partial}{\partial c_1} + b_0''\frac{\partial}{\partial b_1'} + 2b_1'\frac{\partial}{\partial b_2} + a_0'''\frac{\partial}{\partial a_1''} + 2a_1''\frac{\partial}{\partial a_2'} + 3a_2'\frac{\partial}{\partial a_3} + \ldots$$

Digitized by Google

and, as in § 30, it follows that the concomitants, which do not involve line variables, have as their leading coefficients the pure contravariants (including invariants) in b_0, a_1, a_0' $(=\xi, \eta, \zeta)$ as variables of the simultaneous ternary system

$$(c_0, b_1, a_2 \between X, Y)^2 + 2(b_0', a_1' \between X, Y) Z + a_0'' Z^2,$$

$$(d_0, c_1, b_2, a_3 \between X, Y)^3 + 3(c_0', b_1', a_2' \between X, Y)^2 Z + 3(b_0'', a_1'' \between X, Y) Z^2 + a_0''' Z^3,$$

$$(e_0, d_1, c_2, b_3, a_4 \between X, Y)^4 + 4(d_0', c_1', b_2', a_3' \between X, Y)^3 Z + 6(c_0'', b_1'', a_2'' \between X, Y)^2 Z^2 + 4(b_0''', a_1''' \between X, Y) Z^3 + a_0'''' Z^4,$$

.....

A bipartite lineo-linear quantic.

34. Taking the quantic in the form

$$U = \sum_{r=1}^{r=4} (a_r x_1 + b_r x_2 + c_r x_3 + d_r x_4) u_r,$$

the characteristic equations satisfied by the leading coefficients are, by § 16,

$$M_3 = b_1 \frac{\partial}{\partial c_1} + b_2 \frac{\partial}{\partial c_2} + b_3 \frac{\partial}{\partial c_3} + b_4 \frac{\partial}{\partial c_4} - a_2 \frac{\partial}{\partial a_2} - b_3 \frac{\partial}{\partial b_2} - c_3 \frac{\partial}{\partial c_2} - d_3 \frac{\partial}{\partial d_2} = 0,$$

$$\Delta = N_2 = c_1 \frac{\partial}{\partial b_1} + c_2 \frac{\partial}{\partial b_2} + c_3 \frac{\partial}{\partial b_3} + c_4 \frac{\partial}{\partial b_4} - a_2 \frac{\partial}{\partial a_3} - b_2 \frac{\partial}{\partial b_3} - c_2 \frac{\partial}{\partial c_3} - d_2 \frac{\partial}{\partial d_3} = 0,$$

$$\Theta = M_4 = b_1 \frac{\partial}{\partial d_1} + b_2 \frac{\partial}{\partial d_2} + b_3 \frac{\partial}{\partial d_3} + b_4 \frac{\partial}{\partial d_4} - a_4 \frac{\partial}{\partial a_2} - b_4 \frac{\partial}{\partial b_2} - c_4 \frac{\partial}{\partial c_2} - d_4 \frac{\partial}{\partial d_2} = 0.$$

35. To obtain the independent simultaneous solutions of these equations, we first form the usual set of equations subsidiary to M_3, which are

$$\frac{da_1}{0} = \frac{db_1}{0} = \frac{dc_1}{b_1} = \frac{dd_1}{0} = \frac{da_2}{-a_3} = \frac{db_2}{-b_3} = \frac{dc_2}{b_2 - c_3} = \frac{dd_2}{-d_3}$$

$$= \frac{da_3}{0} = \frac{db_3}{0} = \frac{dc_3}{b_3} = \frac{dd_3}{0} = \frac{da_4}{0} = \frac{db_4}{0} = \frac{dc_4}{b_4} = \frac{dd_4}{0},$$

sixteen in number; of these, fifteen independent integrals are required.

We adopt the process already used, and we have as immediate integrals

$$u = a_1;$$
$$\theta = b_1,$$

the latter of which is taken as a 'variable of reference';

$$\theta_1 = d_1, \quad \theta_2 = a_3, \quad \theta_3 = b_3, \quad \theta_4 = d_3, \quad \theta_5 = a_4, \quad \theta_6 = b_4, \quad \theta_7 = d_4;$$
$$\phi = b_2 + c_3.$$

Further, we have

$$\theta \Delta \theta_2 - \theta_2 \Delta \theta = -\epsilon_3,$$
$$\theta \Delta \theta_3 - 2\theta_3 \Delta \theta = 2\theta_3,$$
$$\theta \Delta \theta_5 - \theta_5 \Delta \theta = u_1,$$
$$\theta \Delta \theta_4 - \theta_4 \Delta \theta = -\epsilon_2,$$
$$\theta \Delta \theta_6 - \theta_6 \Delta \theta = \epsilon_3,$$

Digitized by Google

where
$$\epsilon_3 = a_2 b_1 + a_3 c_1, \qquad \theta_6 = \tfrac{1}{2}(c_3 - b_2) b_1 - b_3 c_1,$$
$$\epsilon_2 = b_1 d_2 + c_1 d_3, \qquad u_1 = b_3 c_1^2 - (c_3 - b_2) b_1 c_1 - c_2 b_1^2,$$
$$\epsilon_6 = b_1 c_4 - b_4 c_1,$$

all being solutions of the equations subsidiary to M_3; and the fifteen necessary independent solutions are u, θ; θ_1, θ_2, θ_3, θ_4, θ_5, θ_6, θ_7; ϕ; ϵ_3, ϵ_2, ϵ_6; θ_3, u_1.

Of these fifteen solutions, nine, viz. u, θ_1, θ_5, θ_7, ϕ, ϵ_3, ϵ_2, ϵ_6, u_1 all satisfy $\Delta = 0$.

Of the set of five equations as given above, subsidiary to the determination of the combinations of θ, θ_2, θ_3, θ_4, θ_5, θ_6 which satisfy $\Delta = 0$, we require four independent integrals; these we take in the form
$$4t = 4\frac{\theta_6^2 - u_1\theta_3}{\theta^2} = (c_3 - b_2)^2 + 4c_2 b_3,$$
$$\epsilon_5 = \frac{\theta_3\epsilon_6 + \theta_6\epsilon_3}{\theta} = a_3 c_4 + a_2 b_4,$$
$$\epsilon_1 = \frac{\theta_2\epsilon_2 - \theta_4\epsilon_3}{\theta} = a_3 d_2 - a_2 d_3,$$
and
$$u_2 = -\frac{\theta_2 u_1 + \theta_6\epsilon_3}{\theta} = b_3 c_1 a_2 - \tfrac{1}{2}(c_3 - b_2)(b_1 a_2 - c_1 a_3) + c_2 b_1 a_3.$$

The thirteen quantities
$$u;\ \theta_1,\ \theta_5,\ \theta_7;\ \phi;\ \epsilon_1,\ \epsilon_2,\ \epsilon_3,\ \epsilon_5,\ \epsilon_6;\ t;\ u_1,\ u_2$$
are a system of simultaneous solutions of $M_3 = 0$ and $N_2 = 0$; and every simultaneous solution of those two equations can be algebraically expressed in terms of these thirteen.

36. If we write
$$c_3 - b_2 = 2k,$$
we have
$$\frac{\partial}{\partial b_2} = \frac{\partial}{\partial \phi} - \tfrac{1}{2}\frac{\partial}{\partial k},$$
$$\frac{\partial}{\partial c_3} = \frac{\partial}{\partial \phi} + \tfrac{1}{2}\frac{\partial}{\partial k};$$
and then the equations $M_3 = 0$ and $N_2 = 0$ become
$$M_3 = b_1\frac{\partial}{\partial c_1} + b_4\frac{\partial}{\partial c_4} - a_3\frac{\partial}{\partial a_2} - d_3\frac{\partial}{\partial d_2} + b_3\frac{\partial}{\partial k} - 2k\frac{\partial}{\partial c_2} = 0,$$
$$N_2 = c_1\frac{\partial}{\partial b_1} + c_4\frac{\partial}{\partial b_4} - a_2\frac{\partial}{\partial a_3} - d_2\frac{\partial}{\partial d_3} - c_2\frac{\partial}{\partial k} + 2k\frac{\partial}{\partial c_3} = 0,$$
that is, *they are the differential equations of the concomitants of a binary quadratic having literal coefficients* b_3, k, $-c_2$ *and in four sets of variables, viz.* c_1, $-b_1$; a_2, a_3; d_2, d_3; c_4, $-b_4$: *and since* u, θ_1, θ_5, θ_7 *and* ϕ *do not enter into the equations they must be regarded in the complete set of solutions.*

From this result it is easy to see the constitution of the set of thirteen solutions. Five of them—u, θ_1, θ_5, θ_7 and ϕ—are already accounted for; five of the remainder, viz. ϵ_1, ϵ_2, ϵ_3, ϵ_5, ϵ_6, are determinants of variables; one, t, is the discriminant of the quadratic;

Digitized by Google

u_1 is the quadratic in one set of variables c_1 and $-b_1$; and the last one, u_9, is the first polar of u_1 by the variables a_2 and a_3.

In terms of the last eight every binariant of the system can be algebraically expressed; and the remaining binariants, helping to constitute the complete irreducible system of binariants, are easily seen to be

$$\epsilon_4 = b_4 d_2 + c_4 d_3;$$

$$u_3 = (b_3,\ k,\ -c_2 \between c_1,\ -b_1 \between d_2,\ d_3),$$

$$u_4 = (b_3,\ k,\ -c_2 \between c_1,\ -b_1 \between c_4,\ -b_4),$$

$$u_5 = (b_3,\ k,\ -c_2 \between a_2,\ a_3)^2,$$

$$u_6 = (b_3,\ k,\ -c_2 \between a_2,\ a_3 \between d_2,\ d_3),$$

$$u_7 = (b_3,\ k,\ -c_2 \between a_2,\ a_3 \between c_4,\ -b_4),$$

$$u_8 = (b_3,\ k,\ -c_2 \between d_2,\ d_3)^2,$$

$$u_9 = (b_3,\ k,\ -c_2 \between d_2,\ d_3 \between c_4,\ -b_4),$$

$$u_{10} = (b_3,\ k,\ -c_2 \between c_4,\ -b_4)^2.$$

37. It is convenient, for the purpose of finding functional combinations which shall satisfy $\Theta = 0$, to modify the system of thirteen algebraically independent solutions. We have

$$\epsilon_1 \epsilon_6 = \epsilon_2 \epsilon_5 - \epsilon_3 \epsilon_4,$$

so that we may replace ϵ_1 by ϵ_4 in the system. We have

$$u_2 \epsilon_6 = u_4 \epsilon_3 - u_4 \epsilon_5,$$

so that we may replace u_2 by u_4 in the system. We have

$$u_9 \epsilon_6 = u_{10} \epsilon_2 - u_4 \epsilon_4,$$

so that we may replace ϵ_2 by u_9 in the system. And we have

$$u_1 u_{10} - u_4^2 = -t \epsilon_6^2,$$

so that we may replace t by u_{10} in the system. Hence a system of independent solutions is

$$u;\ \theta_1,\ \theta_5,\ \theta_7;\ \phi;\ \epsilon_3,\ \epsilon_4,\ \epsilon_5,\ \epsilon_6;\ u_1,\ u_4,\ u_9,\ u_{10}.$$

We now proceed to obtain the algebraically independent combinations of these thirteen quantities which are such as to satisfy $\Theta = 0$. The process is that indicated in § 17; and there must be obtained eleven such combinations. There will then finally be eleven solutions which are common to the three characteristic equations; they are independent of one another and are such that every solution common to the three equations can be expressed in terms of them.

In order to obtain these eleven solutions it proves to be desirable to consider some of the other solutions of the irreducible binariant system given in § 36 and to introduce a quantity

$$p = t - \tfrac{1}{4}\phi^2 = c_2 b_3 - c_3 b_2.$$

But all the equations used in this connection are subsidiary to the equations in the above thirteen quantities.

Digitized by Google

38. We have
$$\Theta u = 0, \quad \Theta\theta_5 = 0, \quad \Theta\epsilon_6 = 0, \quad \Theta u_{10} = 0;$$
$$\left.\begin{aligned} \Theta\theta_1 &= b_1 \\ \Theta\epsilon_3 &= b_1\theta_5 \\ \Theta u_1 &= b_1\epsilon_6 \end{aligned}\right\}; \qquad \left.\begin{aligned} \Theta\theta_7 &= b_4 \\ \Theta\phi &= -b_4 \\ \Theta\epsilon_5 &= -b_4\theta_5 \\ \Theta u_4 &= \tfrac{1}{2}b_4\epsilon_6 \end{aligned}\right\};$$
$$\Theta\epsilon_4 = b_4(b_3 - d_4) + c_4 b_3,$$
$$\Theta(u_9 - \tfrac{1}{2}\epsilon_4\phi) = b_4(c_3 b_3 - c_3 b_3) + d_4(c_3 b_4 - b_3 c_4),$$
for the thirteen principal quantities; and for a subsidiary equation
$$\Theta p = c_3 b_4 - b_3 c_4,$$
others being useful for other solutions as given afterwards.

As solutions we have
$$u, \ \theta_5, \ \epsilon_6, \ u_{10};$$
$$\epsilon_3 + \theta_1\theta_5;$$
$$u_1 - \theta_1\epsilon_6;$$
$$\theta_7 + \phi;$$
$$\epsilon_5 + \theta_7\theta_5;$$
$$u^4 + \tfrac{1}{2}\epsilon_6\phi,$$
which are immediately obtainable. For the remaining two which are necessary, we have
$$\Theta(p + \epsilon_4) = b_4(\phi - \theta_7),$$
so that
$$4\Theta(p + \epsilon_4) = -2(\phi - \theta_7)\,\Theta(\phi - \theta_7),$$
and therefore an integral is
$$4p + 4\epsilon_4 + (\phi - \theta_7)^2;$$
and lastly
$$\Theta(u_9 - \tfrac{1}{2}\epsilon_4\phi) = b_4 p + \theta_7\Theta p = \Theta(p\theta_7),$$
and therefore an integral is
$$u_9 - \tfrac{1}{2}\epsilon_4\phi - p\theta_7.$$

It follows then from the theory that *every simultaneous solution of the three characteristic equations can be expressed in terms of the eleven solutions,* (*evidently independent from the method of integration*), *given by*
$$u; \quad \epsilon_3 + \theta_1\theta_5, \quad \epsilon_4 + p + \tfrac{1}{4}(\phi - \theta_7)^2, \quad \epsilon_5 + \theta_7\theta_5, \quad \epsilon_6;$$
$$\theta_5, \quad \theta_7 + \phi;$$
$$u_1 - \theta_1\epsilon_6, \quad u_4 + \tfrac{1}{2}\epsilon_6\phi, \quad u_9 - \tfrac{1}{2}\epsilon_4\phi - p\theta_7, \quad u_{10}.$$

39. Other solutions obtainable by the use of other subsidiary equations are
$$X = \tfrac{1}{2}\epsilon_3\phi + u_2 + \epsilon_3\theta_5 + \theta_1(\epsilon_5 + \theta_5\theta_7), \qquad u_7 + \tfrac{1}{2}\theta_7\epsilon_5 - \theta_5 t + \tfrac{1}{4}\theta_5\theta_7^2,$$
and so for others; thus
$$a_\beta' b_\gamma' c_\alpha' = u^3 + 3u(\epsilon_3 + \theta_1\theta_5) + 3X + \tfrac{1}{4}(\theta_7 + \phi)^3 + 3(\theta_7 + \phi)\{\epsilon_4 + p + \tfrac{1}{4}(\phi - \theta_7)^2\} + 3(u_9 - \tfrac{1}{2}\epsilon_4\phi - p\theta_7).$$

Digitized by Google

40. In order to obtain the fundamental concomitants determined by the foregoing leading coefficients, we turn the expressions of these leading coefficients into umbral forms.

The quantic itself we take in the form $a_x'u_\alpha = b_x'u_\beta = \ldots$ (using a', b', ... instead of a, b, ... to avoid confusion with the real coefficients). Then

$$u = a_1 = a_1'\alpha_1,$$

and therefore
$$U = a_x'u_\alpha = ux_1u_1 + \ldots$$

Next, we have $\theta_5 = a_4 = a_1'\alpha_4$. The completed form of a_1' is evidently a_x'. To complete α_4 we notice that α_4 in transformation is congruent with the point variables, so that $\Sigma \pm x_1y_2z_3\alpha_4$ is a covariantive determinant; one term of it is $x_1p_1\alpha_4$, and it may be conveniently denoted by $(xp\alpha)$. Hence
$$\Theta_5 = a_x'(xp\alpha) = a_4x_1^2p_1 + \ldots$$

Next we have
$$\epsilon_6 = b_1c_4 - b_4c_1 = \tfrac{1}{2}(a_2'b_3' - a_3'b_2')(\alpha_1\beta_4 - \alpha_4\beta_1).$$

The completed expression of the first factor is $(a'b'p)$. The second factor is the coefficient in $v_\alpha w_\beta - v_\beta w_\alpha$ of $v_1w_4 - v_4w_1$, that is, of p_1 in plane variables; hence we may write the completed expression in the form $(\alpha\beta p)$, and therefore

$$E_6 = \tfrac{1}{2}(a'b'p)(\alpha\beta p) = \epsilon_6p_1^2 + \ldots$$

The remaining leading coefficients may be modified with proper combinations of the orders so as to diminish the order, the grade, or the class, as the case may be. We take

$$\begin{aligned} u_{10}' &= u_{10} + \epsilon_6\theta_5 \\ &= (a_2'b_3' - a_3'b_2')\beta_4\gamma_4c_\alpha', \end{aligned}$$

and therefore
$$U_{10}' = c_\alpha'(a'b'p)(xp\beta)(xp\gamma) = u_{10}'x_1^2p_1^3 + \ldots$$

We take
$$\begin{aligned} \epsilon_8' &= \epsilon_8 + \theta_1\theta_5 + u^2 \\ &= a_1'\beta_1b_\alpha', \end{aligned}$$

and therefore
$$E_8' = b_\alpha'a_x'u_\beta = \epsilon_8'x_1u_1 + \ldots$$

We take
$$\begin{aligned} \epsilon_5' &= \epsilon_5 + \theta_7\theta_5 + u\theta_5 \\ &= b_\alpha'a_1'\beta_4, \end{aligned}$$

and therefore
$$E_5' = b_\alpha'a_x'(xp\beta) = \epsilon_5'x_1^2p_1 + \ldots$$

We take
$$\begin{aligned} i_1 &= \theta_7 + \phi + u \\ &= a_\alpha', \end{aligned}$$

viz. we have
$$i_1 = a_\alpha = a_1 + b_2 + c_3 + d_4,$$

an invariant. We take
$$\begin{aligned} u_1' &= u_1 - \theta_1\epsilon_6 \\ &= (a_2'b_3' - a_3'b_2')c_\alpha'\beta_1\gamma_1, \end{aligned}$$

and therefore
$$U_1' = c_\alpha'u_\beta u_\gamma(a'b'p) = u_1'p_1u_1^2 + \ldots$$

We take
$$\begin{aligned} u_4' &= u_4 + \tfrac{1}{2}\epsilon_6\phi + u\epsilon_6 \\ &= (a_2'b_3' - a_3'b_2')c_\alpha'\gamma_1\beta_4, \end{aligned}$$

and therefore
$$U_4' = c_\alpha'u_\gamma(a'b'p)(xp\beta) = u_4'x_1p_1^2u_1 + \ldots$$

Digitized by Google

We take
$$u_9' = u_9 - \tfrac{1}{2}\epsilon_4\phi - p\theta_7$$
$$= \begin{vmatrix} b_2, & b_3, & b_4 \\ c_2, & c_3, & c_4 \\ d_2, & d_3, & d_4 \end{vmatrix} = \tfrac{1}{6}(a_2'b_3'c_4')(\alpha_2\beta_3\gamma_4),$$

and therefore
$$U_9' = \tfrac{1}{6}(a'b'c'u)(\alpha\beta\gamma x) = u_9'x_1u_1 + \ldots$$

We take
$$i_2 = 2\{\epsilon_4 + p + \tfrac{1}{4}(\phi - \theta_7)^2\} + \tfrac{1}{2}(\phi + \theta_7)^2 + u^2 + 2\epsilon_3 + 2\theta_1\theta_5$$
$$= a_\beta' b_\alpha';$$

viz., we have
$$i_2 = a_\beta' b_\alpha',$$

an invariant. Hence

A system of fundamental concomitants of the lineo-linear quantic is given by the eleven algebraically independent concomitants U; u_x; i_1, i_2; E_3', E_5', E_6'; Θ_5; U_1', U_4', U_9', U_{10}'; *and, in order to form an algebraically complete system, their polar derivatives* (§ 18) *must be associated with them.*

41. The *canonical form* of U is
$$l_1X_1U_1 + l_2X_2U_2 + l_3X_3U_3 + l_4X_4U_4.$$

To reduce U to this form we first find l_1, l_2, l_3, l_4. We have
$$i_1 = l_1 + l_2 + l_3 + l_4.$$

Next we have an invariant
$$j_2 = \tfrac{1}{2}\left(\frac{\partial^2}{\partial x_1\partial u_1} + \frac{\partial^2}{\partial x_2\partial u_2} + \frac{\partial^2}{\partial x_3\partial u_3} + \frac{\partial^2}{\partial x_4\partial u_4}\right)(4U - E_3')$$

which for the canonical form gives
$$j_2 = l_1l_2 + l_1l_3 + l_1l_4 + l_2l_3 + l_2l_4 + l_3l_4.$$

We have another invariant
$$i_3 = \left(\frac{\partial^2}{\partial x_1\partial u_1} + \frac{\partial^2}{\partial x_2\partial u_2} + \frac{\partial^2}{\partial x_3\partial u_3} + \frac{\partial^2}{\partial x_4\partial u_4}\right)U_9',$$

which for the canonical form gives
$$i_3 = l_1l_2l_3 + l_2l_3l_4 + l_3l_4l_1 + l_4l_1l_2.$$

Lastly we have an invariant
$$i_4 = \begin{vmatrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ a_4, & b_4, & c_4, & d_4 \end{vmatrix} = l_1l_2l_3l_4$$

for the canonical form. Hence l_1, l_2, l_3, l_4 are the roots of the equation
$$l^4 - i_1l^3 + j_2l^2 - i_3l + i_4 = 0,$$

so that the coefficients of the canonical form are known.

Digitized by Google

To find the variables, we have in the canonical form

$$\begin{aligned} l_1X_1U_1 + l_2X_2U_2 + l_3X_3U_3 + l_4X_4U_4 &= U, \\ l_1^2X_1U_1 + l_2^2X_2U_2 + l_3^2X_3U_3 + l_4^2X_4U_4 &= E_3', \\ l_1^{-1}X_1U_1 + l_2^{-1}X_2U_2 + l_3^{-1}X_3U_3 + l_4^{-1}X_4U_4 &= U_3'i_4^{-1}, \\ X_1U_1 + X_2U_2 + X_3U_3 + X_4U_4 &= u_x. \end{aligned}$$

When these are solved to give X_1U_1, X_2U_2, X_3U_3, X_4U_4 all that is necessary is to resolve the right-hand sides, which must be the product of a linear x-factor and a linear u-factor; and the variables for the canonical form are then obtained.

42. As in § 33 we may obtain those concomitants of the quantic which involve *only point- and plane-variables, x and u.* Their leading coefficients satisfy the three equations $M_2=0$, $N_3=0$, $M_4=0$ as given in § 34, and the three

$$N_4 = c_1\frac{\partial}{\partial d_1} + c_2\frac{\partial}{\partial d_2} + c_3\frac{\partial}{\partial d_3} + c_4\frac{\partial}{\partial d_4} - a_4\frac{\partial}{\partial a_3} - b_4\frac{\partial}{\partial b_3} - c_4\frac{\partial}{\partial c_3} - d_4\frac{\partial}{\partial d_3} = 0,$$

$$K_2 = d_1\frac{\partial}{\partial b_1} + d_2\frac{\partial}{\partial b_2} + d_3\frac{\partial}{\partial b_3} + d_4\frac{\partial}{\partial b_4} - a_2\frac{\partial}{\partial a_4} - b_2\frac{\partial}{\partial b_4} - c_2\frac{\partial}{\partial c_4} - d_2\frac{\partial}{\partial d_4} = 0,$$

and

$$K_3 = d_1\frac{\partial}{\partial c_1} + d_2\frac{\partial}{\partial c_2} + d_3\frac{\partial}{\partial c_3} + d_4\frac{\partial}{\partial c_4} - a_3\frac{\partial}{\partial a_4} - b_3\frac{\partial}{\partial b_4} - c_3\frac{\partial}{\partial c_4} - d_3\frac{\partial}{\partial d_4} = 0,$$

of which N_4 is derivable from the former three, and K_3 from K_2 and the earlier equations.

On comparing these with the differential equations of the ternariants of a lineo-linear bipartite ternary quantic, it appears that *they are the differential equations of the concomitants of a ternary quantic*

$$(b_2X + c_2Y + d_2Z)\,U + (b_3X + c_3Y + d_3Z)\,V + (b_4X + c_4Y + d_4Z)\,W,$$

with a_2, a_3, a_4 as point-variables ($= X$, Y, Z) and b_1, c_1, d as line-variables ($= U$, V, W).

To the system we must add a_1, which is a solution of all the equations. When we take the 'irreducible' system in the foregoing variables for the ternary quantic, we have all the concomitants which if 'reducible' can be reduced only by means of the quantic.

Now these are given by Clebsch and Gordan*. First, the universal ternary concomitant $XU + YV + ZW$ is for the present case

$$\begin{aligned} &a_2b_1 + a_3c_1 + a_4d_1 \\ &= \epsilon_3 + \theta_1\theta_3, \end{aligned}$$

so that the quaternariant may be taken as in § 40 to be

$$U_X + a_1^2 = E_3' = b_\alpha' a_\beta' u_\beta.$$

Next for the ternary form itself, we may write it

$$\begin{aligned} &(l_2'X + l_3'Y + l_4'Z)\,(\lambda_2U + \lambda_3V + \lambda_4W) \\ &= l_X'U_\lambda, \end{aligned}$$

(taking the quaternary quantic as $l_x'u_\lambda$).

* *Math. Ann.* t. I. p. 373.

Digitized by Google

Taking then the irreducible system of ternariants as given (*l. c.*) by Clebsch and Gordan, we have for their functions in order

$$i = l_\lambda' = b_2 + c_3 + d_4 = \theta_7 + \phi,$$

so that the corresponding concomitant is

$$u_2 i_1 - U$$

with the notation of § 40 for the quaternariants. Next we have

$$\begin{aligned} f &= l_\lambda' U_\lambda \\ &= a_1'(l_2'\alpha_2 + l_3'\alpha_3 + l_4'\alpha_4) \,.\, \beta_1(\lambda_2 b_2' + \lambda_3 b_3' + \lambda_4 b_4') \\ &= a_1'\beta_1'(l_\alpha' - l_1'\alpha_1)(b_\lambda' - b_1'\lambda_1) \\ &= a_1'\beta_1 l_\alpha' b_\lambda' - 2u a_1' l_\alpha' \lambda_1 + u^2 \\ &= a_1'\beta_1 l_\alpha' b_\lambda' - 2u\epsilon_3' + u^2, \end{aligned}$$

so that we have a new concomitant which we may take

$$F' = c_\alpha' b_\gamma' a_x' u_\beta.$$

Similarly we have

$$\begin{aligned} i_1 &= l_\mu' m_\lambda' \\ &= (l_2'\mu_2 + l_3'\mu_3 + l_4'\mu_4)(m_2'\lambda_2 + m_3'\lambda_3 + m_4'\lambda_4) \\ &= (a_\beta' - a_1'\beta_1)(b_\alpha' - b_1'\alpha_1) \\ &= i_2 - 2\epsilon_3' + u^2 \end{aligned}$$

in the previous notation, so that the ternariant i_1 practically leads to the quaternariant i_2, which is an invariant.

Similarly we find $\quad f_1 = a_1'\beta_1 b_\gamma' c_\delta' d_\alpha' - 3uf' + 3u^2\epsilon_3' - u^4$

(where f' is the leading coefficient of F') and we therefore have a new concomitant

$$F_1 = b_\gamma' c_\delta' d_\alpha' a_x' u_\beta.$$

Instead of i_2 we may conveniently take i'' defined (*l. c.* p. 385) by

$$\begin{aligned} i'' &= \tfrac{1}{6}(i^2 + 2i_2 - 3ii_1) \\ &= \tfrac{1}{6}(l_2'm_3'n_4')(\lambda_2\mu_3\nu_4) \\ &= u_9', \end{aligned}$$

so that the concomitant thus determined is U_9'.

In the same way, it is not difficult to prove that the new concomitant determined by ϕ is

$$\Phi' = d_\alpha' e_\delta' k_\beta' a_x' b_x' c_x' \,(\gamma\epsilon\kappa x),$$

and that the new concomitant determined by ψ is

$$\Psi' = a_\delta' d_\epsilon' b_\kappa' u_\alpha u_\beta u_\gamma \,(ceku).$$

Digitized by Google

Complexes in line-variables.

43. The differential equations of the concomitants of a quaternary quantic in line-variables are similar to those for other quantics, the difference being in the values of the literal operators. If any such complex be denoted by

$$q_p^n = (q_1p_1 + q_2p_2 + q_3p_3 + q_4p_4 + q_5p_5 + q_6p_6)^n$$
$$= \Sigma \frac{n!}{\lambda!\,\mu!\,\nu!\,\rho!\,\sigma!\,\tau!} a_{\lambda\mu\nu\rho\sigma\tau} p_1^\lambda p_2^\mu p_3^\nu p_4^\rho p_5^\sigma p_6^\tau,$$

(with the condition $\lambda+\mu+\nu+\rho+\sigma+\tau=n$), the laws of transformation of the umbral coefficients q are derivable from those in § 1 for the variables p by the relation $Q_P = q_p$; and so far as quantities of the first order in the coefficients of transformation, being all that are necessary (§ 5), are concerned they may be taken in the form

$$\left.\begin{aligned}
Q_1 &= q_1 - q_2l_2 - q_3l_3 - q_4k_2 + q_5k_3 \\
Q_2 &= -q_1m_1 + q_2 - q_3m_3 + q_4k_1 - q_6k_3 \\
Q_3 &= -q_1n_1 + q_2n_2 + q_3 - q_5k_1 + q_6k_2 \\
Q_4 &= -q_1m_4 + q_2l_4 + q_4 + q_5m_3 + q_6l_3 \\
Q_5 &= q_1n_4 - q_3l_4 + q_4n_2 + q_5 + q_6l_2 \\
Q_6 &= -q_2n_4 + q_3m_4 + q_4n_1 + q_5m_1 + q_6
\end{aligned}\right\},$$

from which the variation of $a_{\lambda\mu\nu\rho\sigma\tau}$ $(= q_1^\lambda q_2^\mu q_3^\nu q_4^\rho q_5^\sigma q_6^\tau)$ can be derived. The result is that the characteristic equations $M_3=0$, $N_3=0$, $M_4=0$ are given by

$$\left.\begin{aligned}
M_3 &= \Sigma\left(-\mu a_{\lambda,\,\mu-1,\,\nu+1,\,\rho,\,\sigma,\,\tau} + \rho a_{\lambda,\,\mu,\,\nu,\,\rho-1,\,\sigma+1,\,\tau}\right)\frac{\partial}{\partial a_{\lambda\mu\nu\rho\sigma\tau}} = 0 \\
N_3 &= \Sigma\left(-\nu a_{\lambda,\,\mu+1,\,\nu-1,\,\rho,\,\sigma,\,\tau} + \sigma a_{\lambda,\,\mu,\,\nu,\,\rho+1,\,\sigma-1,\,\tau}\right)\frac{\partial}{\partial a_{\lambda\mu\nu\rho\sigma\tau}} = 0 \\
\text{and}\quad M_4 &= \Sigma\left(-\rho a_{\lambda+1,\,\mu,\,\nu,\,\rho-1,\,\sigma,\,\tau} + \tau a_{\lambda,\,\mu,\,\nu+1,\,\rho,\,\sigma,\,\tau-1}\right)\frac{\partial}{\partial a_{\lambda\mu\nu\rho\sigma\tau}} = 0
\end{aligned}\right\},$$

the summation being in every case for values of λ, μ, ν, ρ, σ, τ such that

$$\lambda+\mu+\nu+\rho+\sigma+\tau$$

is equal to n, the grade of the complex.

When the leading coefficients of concomitants, obtained as simultaneous solutions of these three equations, are to be changed into a symbolical form, with a view to the deduction of the expressions for the concomitant, it will sometimes be convenient to regard the coefficients q as similar to compositions of the umbral elements of two linear powers a_x and b_x, so that we may take

$$\begin{aligned}
q_1 &= a_2b_3 - a_3b_2, & q_6 &= a_1b_4 - a_4b_1, \\
q_2 &= a_3b_1 - a_1b_3, & q_5 &= a_2b_4 - a_4b_2, \\
q_3 &= a_1b_2 - a_2b_1, & q_4 &= a_3b_4 - a_4b_3,
\end{aligned}$$

Digitized by Google

only however as an indication of the completion of the umbral forms and not as necessarily implying the relation $q_1q_6+q_2q_5+q_3q_4=0$. We proceed now to obtain the concomitants of some special complexes.

I. *The linear complex.*

44. We may take the complex in the form

$$\Omega = q_1p_1+q_2p_2+q_3p_3+q_4p_4+q_5p_5+q_6p_6$$
$$= (abvw) = (abp) = (cdp) = \ldots = q_p$$

symbolically. The characteristic equations are

$$M_3 = -q_2\frac{\partial}{\partial q_3}+q_5\frac{\partial}{\partial q_4}=0, \qquad N_2 = -q_3\frac{\partial}{\partial q_2}+q_4\frac{\partial}{\partial q_5}=0;$$

$$\Theta = M_4 = -q_1\frac{\partial}{\partial q_4}+q_3\frac{\partial}{\partial q_6}=0.$$

It is easy to shew that every simultaneous solution of $M_3=0$ and $N_2=0$ can be expressed in terms of the three independent simultaneous solutions

$$q_1, \quad q_6, \quad q_3q_4+q_2q_5.$$

To find the combinations of these which satisfy $\Theta=0$ we have

$$\Theta q_1 = 0,$$
$$\Theta q_6 = q_3,$$
$$\Theta\,(q_3q_4+q_2q_5) = -q_3q_1,$$

so that the solutions can all be expressed in terms of q_1 and $q_1q_6+q_2q_5+q_3q_4$.

The first of these evidently determines Ω; the second is an invariant, being $\frac{1}{2}(abcd)$. The signification of this invariant is given by Plücker*.

II. *A congruence of two linear complexes.*

45. The two complexes we take to be

$$\Omega = q_p, \qquad \Omega' = q_p';$$

the characteristic equations, which determine leading coefficients, are

$$M_3 = -q_3\frac{\partial}{\partial q_2}+q_5\frac{\partial}{\partial q_4}-q_3'\frac{\partial}{\partial q_2'}+q_5'\frac{\partial}{\partial q_4'}=0,$$
$$N_2 = -q_2\frac{\partial}{\partial q_3}+q_4\frac{\partial}{\partial q_5}-q_2'\frac{\partial}{\partial q_3'}+q_4'\frac{\partial}{\partial q_5'}=0,$$
$$\Theta = M_4 = -q_1\frac{\partial}{\partial q_4}+q_3\frac{\partial}{\partial q_6}-q_1'\frac{\partial}{\partial q_4'}+q_3'\frac{\partial}{\partial q_6'}=0.$$

* The quantities $q_1, q_2, q_3, q_4, q_5, q_6$, are the same as Plücker's D, E, F, C, B, A; see his *Neue Geometrie des Raumes*, p. 26.

Digitized by Google

All the simultaneous solutions of the first two of these are expressible in terms of the set of nine simultaneous solutions

$$(\omega=)\,q_1, \qquad (\omega'=)\,q_1', \qquad q_6, \qquad q_6';$$
$$\theta = q_3q_4 + q_2q_5, \qquad \psi = q_3q_4' + q_2q_5', \qquad \phi = q_2q_3' - q_3q_2',$$
$$\theta' = q_3'q_4' + q_2'q_5', \qquad \psi' = q_3'q_4 + q_2'q_5,$$

the only remaining solution necessary to render the set irreducible being

$$\chi = q_4q_5' - q_4'q_5,$$

with the relation $$\phi\chi = \psi\psi' - \theta\theta'.$$

For the combinations of the set of nine which satisfy $\Theta = 0$ we have, by the ordinary method of proceeding, the equations

$$\Theta\omega = 0, \qquad \Theta\omega' = 0, \qquad \Theta\phi = 0;$$
$$\Theta q_6 = q_3, \qquad \Theta q_6' = q_3',$$
$$\Theta\theta = -q_3\omega, \qquad \Theta\psi' = -q_3'\omega,$$
$$\Theta\psi = -q_3\omega', \qquad \Theta\theta' = -q_3'\omega';$$

and therefore the seven independent solutions may be taken in the form

$$\omega, \qquad \omega', \qquad \phi;$$
$$i = \theta + \omega q_6 = q_1q_6 + q_2q_5 + q_3q_4,$$
$$i' = \theta' + \omega' q_6' = q_1'q_6' + q_2'q_5' + q_3'q_4',$$
$$\lambda = \psi + \omega' q_6 = q_1'q_6 + q_2q_5' + q_3q_4',$$
$$\lambda' = \psi' + \omega q_6' = q_1q_6' + q_2'q_5 + q_3'q_4.$$

In terms of these seven solutions every simultaneous solution of the three equations can be expressed.

46. The fundamental concomitants thence determined are

$$\Omega = \omega p_1 + \ldots = (abp);$$
$$\Omega' = \omega' p_1 + \ldots = (a'b'p);$$
$$i = q_1q_6 + q_2q_5 + q_3q_4, \text{ an invariant } = \tfrac{1}{2}(abcd);$$
$$i' = q_1'q_6' + q_2'q_5' + q_3'q_4', \text{ an invariant } = \tfrac{1}{2}(a'b'c'd');$$
$$\Phi = a_xa_x'(bb'p) + b_xb_x'(aa'p) - a_x'b_x(ab'p) - a_xb_x'(ba'p)$$
$$= \phi x_1^2 p_1 + \ldots;$$
$$\Lambda = a_x(ba'b'u) - b_x(aa'b'u) = \lambda x_1u_1 + \ldots;$$
$$\Lambda' = a_x'(b'abu) - b_x'(a'abu) = \lambda' x_1u_1 + \ldots;$$

and it may be remarked that $\Lambda + \Lambda' = (aba'b')\,u_x$,

so that we have an invariant

$$a_1a_6' + a_1'a_6 + a_2a_5' + a_2'a_5 + a_3a_4' + a_3'a_4$$

intermediate between i and i'.

With these fundamental concomitants their polar derivatives (§ 18) must be associated.

Digitized by Google

III. *A regulus of three linear complexes.*

47. We take them in the forms

$$\Omega = q_1p_1 + q_2p_2 + q_3p_3 + q_4p_4 + q_5p_5 + q_6p_6 = q_p,$$
$$\Omega' = q_p', \qquad \Omega'' = q_p'';$$

and then the three characteristic equations are

$$M_3 = -q_3\frac{\partial}{\partial q_2} + q_5\frac{\partial}{\partial q_4} - q_2'\frac{\partial}{\partial q_3'} + q_5'\frac{\partial}{\partial q_4'} - q_2''\frac{\partial}{\partial q_3''} + q_5''\frac{\partial}{\partial q_4''} = 0,$$
$$N_2 = -q_2\frac{\partial}{\partial q_3} + q_4\frac{\partial}{\partial q_5} - q_2'\frac{\partial}{\partial q_3'} + q_4'\frac{\partial}{\partial q_5'} - q_2''\frac{\partial}{\partial q_3''} + q_4''\frac{\partial}{\partial q_5''} = 0,$$
$$\Theta = M_4 = -q_1\frac{\partial}{\partial q_4} + q_3\frac{\partial}{\partial q_6} - q_1'\frac{\partial}{\partial q_4'} + q_3'\frac{\partial}{\partial q_6'} - q_1''\frac{\partial}{\partial q_4''} + q_3''\frac{\partial}{\partial q_6''} = 0.$$

48. Of the complete set of solutions some will be, as in the preceding case, quantities which involve coefficients of only *one* of the three: these are

$$\omega = q_1, \qquad \iota = q_1q_6 + q_2q_5 + q_3q_4;$$
$$\omega' = q_1', \qquad \iota' = q_1'q_6' + q_2'q_5' + q_3'q_4';$$
$$\omega'' = q_1'', \qquad \iota'' = q_1''q_6'' + q_2''q_5'' + q_3''q_4''.$$

Some involve coefficients of *two* of the three: these are

$$\phi_{12} = q_2q_3' - q_3q_2'; \qquad \lambda_{12} = q_1'q_6 + q_5'q_2 + q_4'q_3; \qquad \lambda_{12}' = q_1q_6' + q_5q_2' + q_4q_3';$$
$$\phi_{23} = q_2'q_3'' - q_3'q_2''; \qquad \lambda_{23} = q_1''q_6' + q_5''q_2' + q_4''q_3'; \qquad \lambda_{23}' = q_1'q_6'' + q_5'q_2'' + q_4'q_3'';$$
$$\phi_{31} = q_2''q_3 - q_3''q_2; \qquad \lambda_{31} = q_1q_6'' + q_5q_2'' + q_4q_3''; \qquad \lambda_{31}' = q_1''q_6 + q_5''q_2 + q_4''q_3.$$

And from these we have three invariants, viz.

$$\lambda_{12} + \lambda_{12}';$$
$$\lambda_{23} + \lambda_{23}';$$
$$\lambda_{31} + \lambda_{31}'.$$

For functions which involve the coefficient of *all three* of the congruences we have as simultaneous solutions of $M_3 = 0$, $N_2 = 0$

$$\chi_{12} = q_4q_5' - q_4'q_5;$$
$$\chi_{23} = q_4'q_5'' - q_4''q_5';$$
$$\chi_{31} = q_4''q_5 - q_4q_5''.$$

Now we have $\qquad \Theta\chi_{12} = -q_1q_5' + q_1'q_5,$

and so for the other two, so that

$$q_1''\Theta\chi_{12} + q_1\Theta\chi_{23} + q_1'\Theta\chi_{31} = 0,$$

or

$$\Theta(q_1\chi_{23} + q_1'\chi_{31} + q_1''\chi_{12}) = 0,$$

Digitized by Google

so that we have
$$\delta = \begin{vmatrix} q_1, & q_4, & q_5 \\ q_1', & q_4', & q_5' \\ q_1'', & q_4'', & q_5'' \end{vmatrix},$$

which is a solution new in form. And we also have

$$\Theta q_6 = q_2, \qquad \Theta q_6 = q_3', \qquad \Theta q_6'' = q_3'',$$

so that
$$\Theta(q_6\phi_{23} + q_6'\phi_{31} + q_6''\phi_{12}) = 0,$$

and thus we have
$$\delta' = \begin{vmatrix} q_2, & q_3, & q_6 \\ q_2', & q_3', & q_6' \\ q_2'', & q_3'', & q_6'' \end{vmatrix}$$

a solution new in form. The last two are of course expressible, after our theory, in terms of the earlier solutions: thus

$$\left.\begin{aligned} \iota\phi_{23} + \lambda_{12}'\phi_{31} + \lambda_{31}\phi_{12} &= q_1\delta' \\ \iota'\phi_{31} + \lambda_{23}'\phi_{12} + \lambda_{12}\phi_{23} &= q_1'\delta' \\ \iota''\phi_{12} + \lambda_{31}'\phi_{23} + \lambda_{23}\phi_{31} &= q_1''\delta' \end{aligned}\right\}$$

(which reduce the fifteen solutions $\omega, \omega', \omega''$; ι, ι', ι''; $\phi_{12}, \phi_{23}, \phi_{31}$; $\lambda_{12}, \lambda_{12}'$; $\lambda_{23}, \lambda_{23}'$; $\lambda_{31}, \lambda_{31}'$ to thirteen algebraically independent solutions, being the proper number); and similarly

$$\delta = \frac{q_1}{\phi_{23}}(\lambda_{23}\lambda_{23}' - \iota'\iota'') + \frac{q_1'}{\phi_{31}}(\lambda_{31}\lambda_{31}' - \iota\iota'') + \frac{q_1''}{\phi_{12}}(\lambda_{12}\lambda_{12}' - \iota\iota')$$
$$- \frac{\delta'}{\phi_{12}\phi_{23}\phi_{31}}(q_1q_1'q_1''\delta' - q_1'q_1''\iota\phi_{23} - q_1q_1''\iota'\phi_{31} - q_1q_1'\iota''\phi_{12}).$$

The system of seventeen solutions forms an irreducible system of solutions, that is, every other solution can be expressed as an integral algebraical function of these seventeen. But it does not therefore follow that combinations of the concomitants determined by these are not reducible; reducible examples will actually be found, the reduction taking place owing to a possibility of removal of a factor u_x, which for the present purpose is determined by its leading coefficient 1.

49. It is now necessary to determine variables to be associated with these leading coefficients, so as to have the fundamental concomitants. We take

$$\begin{aligned} \Omega &= q_p = (abvw) = (cdvw) = \ldots \\ \Omega' &= q_p' = (a'b'vw) = (c'd'vw) = \ldots \\ \Omega'' &= q_p'' = (a''b''vw) = (c''d''vw) = \ldots \end{aligned}$$

Then we have

$$\begin{cases} \omega = q_1 = (a_2b_3 - a_3b_2), \text{ so that } \Omega = \omega p_1 + \ldots \\ \omega' = q_1' = (a_2'b_3' - a_3'b_2'), \text{ so that } \Omega' = \omega' p_1 + \ldots \\ \omega'' = q_1'' = (a_2''b_3'' - a_3''b_2''), \text{ so that } \Omega' = \omega'' p_1 + \ldots \end{cases}$$

Digitized by Google

$$\begin{cases} \iota = \tfrac{1}{2}(abcd) & \text{, an invariant,} \\ \iota' = \tfrac{1}{2}(a'b'c'd') & \text{, an invariant,} \\ \iota'' = \tfrac{1}{2}(a''b''c''d''), & \text{an invariant.} \end{cases}$$

$$\begin{cases} \phi_{12} = a_1a_1'(b_2b_3') - a_1'b_1(b_2'a_2) - a_1b_1'(a_2'b_2) + b_1b_1'(a_2a_3') \\ \Phi_{12} = \phi_{12}x_1^2p_1 + \dots \\ \Phi_{23} = \phi_{23}x_1^2p_1 + \dots \\ \Phi_{31} = \phi_{31}x_1^2p_1 + \dots \end{cases}$$

$$\begin{cases} \lambda_{12} = a_1(b_2a_3'b_4') - b_1(a_2a_3'b_4') \\ \Lambda_{12} = \lambda_{12}x_1u_1 + \dots \\ \Lambda_{23} = \lambda_{23}x_1u_1 + \dots \\ \Lambda_{31} = \lambda_{31}x_1u_1 + \dots \end{cases} \qquad \begin{cases} \lambda_{12}' = a_1'(b_2'a_3b_4) - b_1'(a_2'a_3b_4) \\ \Lambda_{12}' = \lambda_{12}'x_1u_1 + \dots \\ \Lambda_{23}' = \lambda_{23}'x_1u_1 + \dots \\ \Lambda_{31}' = \lambda_{31}'x_1u_1 + \dots \end{cases}$$

Also $\lambda_{12} + \lambda_{12}' = (aba'b')$, an invariant; so that $\Lambda_{12} + \Lambda_{12}'$ is reducible, being divisible by u_x. Similarly for $\Lambda_{23} + \Lambda_{23}'$ and $\Lambda_{31} + \Lambda_{31}'$.

Next we have $$\delta = (a_2'a_3b_4)(b_2'a_3''b_4'') - (a_2''a_3''b_4'')(b_2'a_3b_4)$$
and therefore $$\Delta = (a'abu)(b'a''b''u) - (a'a''b''u)(b'abu)$$
$$= \delta u_1^2 + \dots.$$

Lastly we have

$$\begin{aligned} \delta' = a_1a_1'a_1''(b_2b_3'b_4'') - b_1a_1'a_1''(a_2b_3'b_4'') - a_1b_1'a_1''(b_2a_3'b_4'') - a_1a_1'b_1''(b^2b_3'a_4'') \\ + a_1b_1'b_1''(b_2a_3'a_4'') + b_1a_1'b_1''(a_2b_3'a_4'') + b_1b_1'a_1''(a_2a_3'b_4'') \\ - b_1b_1'b_1''(a_2a_3'a_4''), \end{aligned}$$

so that $$\Delta' = \delta' x_1^3u_1 + \dots$$
$$= \Sigma a_x a_x' a_x''(bb'b''u).$$

With these fundamental concomitants their polar derivatives (§ 18) must be associated to form an algebraically complete system.

50. When we consider *more than three linear complexes*, no functions new in form are obtained beyond those already given; and any new functions which occur are reducible to combinations of those which precede. Thus for instance the eliminant of six linear complexes, viz. $$\Sigma \pm q_1q_2'q_3''q_4'''q_5''''q_6'''''$$
can be expressed as a sum of products of the form
$$(q_2'q_3''q_6'''''')(q_1q_4'''q_5'''''),$$
that is, of the form $\delta_{236}'\delta_{145}$. Similarly for others.

Symbolical Expressions of Concomitants of Complexes.

51. We are now in a position to derive an important inference as to the *forms of factors that may enter into the symbolical expressions of concomitants of* q_p^n. These symbolical expressions give, in accordance with the usual forms, concomitants of a set of simultaneous linear complexes q_p, q_p', q_p'',... each being represented n times in the ex-

Digitized by Google

pression. Hence, when the leading coefficients are formed, they are leading coefficients of the concomitants of that simultaneous system.

We have seen the kinds of factors that can occur in these leading coefficients: they are of one or other of the eight types

$$\begin{aligned}
\omega &= q_1, \\
\iota &= q_1q_6 + q_2q_5 + q_3q_4, \\
\lambda' &= q_1q_6' + q_5q_2' + q_4q_3', \\
\lambda &= q_1'q_6 + q_5'q_2 + q_4'q_3, \\
j = \lambda + \lambda' &= q_1q_6' + q_2q_5' + q_3q_4' + q_4q_3' + q_5q_2' + q_6q_1', \\
\phi &= q_2q_3' - q_3q_2', \\
\delta &= (q_1q_4'q_5''), \\
\delta' &= (q_2q_3'q_6'').
\end{aligned}$$

Hence every leading coefficient of a concomitant of a complex of the nth grade is an aggregate of powers and products of factors of the eight types; there being the necessary restriction that each umbral symbol shall occur exactly n times in any product or power, constituting a term in the umbral expression.

To complete the expression of the leading coefficient so as to give the fundamental concomitant we have seen, in the preceding investigation, the combinations of variables that must be associated with the several factors, the respective degrees being as in the following table:—

FACTOR.	ORDER IN x.	GRADE IN p.	CLASS IN u.
ω	0	1	0
ι	0	0	0
λ'	1	0	1
λ	1	0	1
j	0	0	0
ϕ	2	1	0
δ	0	0	2
δ'	3	0	1

In addition to this it must be borne in mind that certain combinations of factors of the types δ and δ' are reducible, that is, the completed expression contains some power of u_x as a factor.

These results will be used in obtaining the concomitants of a complex of the second grade.

Digitized by Google

IV. *A complex of the second grade.*

52. Taking the same coefficients as Plücker (l.c. p. 149), the complex may be written

$$\begin{aligned}\Omega = Dp_1^2 + Ep_2^2 \quad + Fp_3^2 \quad + Cp_4^2 \quad + Bp_5^2 \quad + Ap_6^2 \\ + 2Mp_1p_2 + 2Lp_1p_3 + 2Tp_1p_4 + 2Sp_1p_5 + 2Np_1p_6 \\ + 2Kp_2p_3 + 2Up_2p_4 + 2Op_2p_5 + 2Pp_2p_6 \\ + 2Vp_3p_4 + 2Rp_3p_5 + 2Qp_3p_6 \\ + 2Gp_4p_5 + 2Hp_4p_6 \\ + 2Jp_5p_6 \\ = (q_1p_1 + q_2p_2 + q_3p_3 + q_4p_4 + q_5p_5 + q_6p_6)^2 = q_p^2.\end{aligned}$$

If instead of taking Ω we take

$$\Omega - \tfrac{2}{3}(N + O + V)(p_1p_6 + p_2p_5 + p_3p_4)$$

the complex may be considered as in its 'normal' form, according to Clebsch; we shall however retain the above more general form, for the sake of symmetry.

The three differential equations which determine leading coefficients are:—

$$\left.\begin{aligned}M_3 = B\frac{\partial}{\partial G} + 2G\frac{\partial}{\partial C} - F\frac{\partial}{\partial K} - 2K\frac{\partial}{\partial E} + R\left(\frac{\partial}{\partial V} - \frac{\partial}{\partial O}\right) + (O - V)\frac{\partial}{\partial U} \\ - L\frac{\partial}{\partial M} - Q\frac{\partial}{\partial P} + J\frac{\partial}{\partial H} + S\frac{\partial}{\partial T} \\ N_3 = C\frac{\partial}{\partial G} + 2G\frac{\partial}{\partial B} - E\frac{\partial}{\partial K} - 2K\frac{\partial}{\partial F} + (V - O)\frac{\partial}{\partial R} + U\left(\frac{\partial}{\partial O} - \frac{\partial}{\partial V}\right) \\ - M\frac{\partial}{\partial L} - P\frac{\partial}{\partial Q} + H\frac{\partial}{\partial J} + T\frac{\partial}{\partial S} \\ \Theta = M_4 = F\frac{\partial}{\partial Q} + 2Q\frac{\partial}{\partial A} - D\frac{\partial}{\partial T} - 2T\frac{\partial}{\partial C} + (V - N)\frac{\partial}{\partial H} + L\left(\frac{\partial}{\partial N} - \frac{\partial}{\partial V}\right) \\ - M\frac{\partial}{\partial U} - S\frac{\partial}{\partial G} + K\frac{\partial}{\partial P} + R\frac{\partial}{\partial J}\end{aligned}\right\}.$$

53. The equations $M_3 = 0$, $N_3 = 0$ admit of a transformation the effect of which enables us at once to write down the system of simultaneous solutions. Let new quantities I and W be defined by the relations

$$O - V = 2I, \qquad O + V = 2W,$$

so that

$$\frac{\partial}{\partial O} - \frac{\partial}{\partial V} = \frac{\partial}{\partial I};$$

the first two equations then become

$$\left.\begin{aligned}M_3 = B\frac{\partial}{\partial G} + 2G\frac{\partial}{\partial C} - F\frac{\partial}{\partial K} - 2K\frac{\partial}{\partial E} - R\frac{\partial}{\partial I} + 2I\frac{\partial}{\partial U} - L\frac{\partial}{\partial M} - Q\frac{\partial}{\partial P} + J\frac{\partial}{\partial H} + S\frac{\partial}{\partial T} \\ N_3 = C\frac{\partial}{\partial G} + 2G\frac{\partial}{\partial B} - E\frac{\partial}{\partial K} - 2K\frac{\partial}{\partial F} + U\frac{\partial}{\partial I} - 2I\frac{\partial}{\partial R} - M\frac{\partial}{\partial L} - P\frac{\partial}{\partial Q} + H\frac{\partial}{\partial J} + T\frac{\partial}{\partial S}\end{aligned}\right\}.$$

Digitized by Google

These are the differential equations of the concomitants of a system of three binary quadratics, the literal coefficients of which are

$$\begin{matrix} C, & G, & B: \\ E, & -K, & F: \\ U, & I, & -R; \end{matrix}$$

the variables being the four sets

$$J, \quad -H: \quad S, \quad -T: Q, \quad P: L, \quad M.$$

To this system of solutions must be added those, which are simultaneous solutions in virtue of the fact that certain quantities, viz. D, A, N and W, do not enter into either of the equations; these quantities themselves are the proper solutions to take.

Among these solutions the following occur (they do not constitute the irreducible system, Jacobians for instance being omitted):—

$$\left.\begin{matrix} \epsilon_1 = QT + SP, & \epsilon_4 = PL - QM \\ \epsilon_2 = QH + JP, & \epsilon_5 = LT + MS \\ \epsilon_3 = HS - TJ, & \epsilon_6 = HL + SM \end{matrix}\right\},$$

with the usual relation $\epsilon_1\epsilon_6 = \epsilon_3\epsilon_4 + \epsilon_2\epsilon_5$;

$$\left.\begin{matrix} \xi_1 = (C,\ G,\ B \between J,\ -H)^2, & \xi_6 = (C,\ G,\ B \between S,\ -T \between Q,\ P) \\ \xi_2 = (C,\ G,\ B \between J,\ -H \between S,\ -T), & \xi_7 = (C,\ G,\ B \between S,\ -T \between L,\ M) \\ \xi_3 = (C,\ G,\ B \between J,\ -H \between Q,\ P), & \xi_8 = (C,\ G,\ B \between Q,\ P)^2 \\ \xi_4 = (C,\ G,\ B \between J,\ -H \between L,\ M), & \xi_9 = (C,\ G,\ B \between Q,\ P \between L,\ M) \\ \xi_5 = (C,\ G,\ B \between S,\ -T)^2, & \xi_{10} = (C,\ G,\ B \between L,\ M)^2 \end{matrix}\right\};$$

$$\left.\begin{matrix} \phi_1 = (E,\ -K,\ F \between J,\ -H)^2, \\ \phi_2, \ldots, \phi_{10} \text{ functions the same as } \phi_1 \text{ in} \\ \text{coefficients and the same as } \xi_2, \ldots, \xi_{10} \\ \text{in variables} \end{matrix}\right\};$$

$$\left.\begin{matrix} \theta_1 = (U,\ I,\ -R \between J,\ -H)^2, \\ \theta_2, \ldots, \theta_{10} \text{ functions the same as } \theta_1 \text{ in} \\ \text{coefficients and the same as } \xi_2, \ldots, \xi_{10} \\ \text{in variables.} \end{matrix}\right\};$$

$$\begin{matrix} t_1 = BC - G^2, & t_{12} = CF + BE + 2GK, \\ t_2 = EF - K^2, & t_{13} = CR - BU + 2IG, \\ t_3 = UR + I^2, & t_{23} = FU - ER + 2KI, \end{matrix}$$

$$\rho = \begin{vmatrix} C, & G, & B \\ E, & -K, & F \\ U, & I, & -R \end{vmatrix},$$

Digitized by Google

being the full set of irreducible invariants of the system of three binary quadratics. Among the irreducible members omitted are, as already remarked, all Jacobians; of which there are ten, involving the coefficients of ξ_1 and ϕ_1, ten others, involving the coefficients of ϕ_1 and θ_1, and ten others, involving the coefficients of θ_1 and ξ_1. As is well known, the square of each Jacobian can be expressed as an integral function of the foregoing quantities; and that square is therefore a reducible function.

An algebraically complete set—that is one such that every solution can be expressed as a function of the constituents of the set—is obtainable by the method of § 17 as composed of D, A, N, W; t_1, t_2, t_3; ξ_1, ξ_2; ϕ_1, ϕ_2; θ_1, θ_2; $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_5, \epsilon_6$. By means of the relations which subsist among the irreducible binariants, this set can be changed so as to become the equivalent algebraically complete system composed of

$$D, A, N, W;\ t_1, t_2, t_3;\ \xi_7, \xi_{10};\ \phi_4, \phi_9, \phi_{10};\ \theta_7, \theta_{10};\ \epsilon_1, \epsilon_4, \epsilon_5, \epsilon_6.$$

54. In order to find the functional combinations of these quantities which are such as satisfy $\Theta = 0$ we use the process of § 15; and to obtain the simplest form of solutions it is convenient to retain other members of the system of irreducible binariants, always considered as subsidiary to the algebraically complete system.

I do not reproduce the analytical details; the following are a set of sixteen independent quantities, in terms of which all the simultaneous solutions of the three characteristic equations can be expressed:—

$$\left.\begin{aligned}
\omega &= D \\
\iota &= 2W + N \\
\xi_7' &= \xi_7 - \theta_5 - t_{13}D \\
\xi_{10}' &= \xi_{10} + 2\epsilon_5 N + N^2 D \\
& \phi_{10} \\
\phi_4' &= \phi_4 + Mt_{23} - \theta_9 - W\epsilon_4 \\
\phi_9' &= \phi_9 - Nt_2 \\
& \theta_{10} \\
\theta_7' &= \theta_7 - W\epsilon_5 - Dt_3 + DW^2 \\
t_1' &= t_1 D - \xi_5 \\
& t_2 \\
t_3' &= t_3 + \epsilon_6 + 3W^2 + 2WN + N^2 \\
\epsilon_1' &= 2\epsilon_1 + t_{12} + AD \\
\epsilon_4' &= \epsilon_4 - t_{23} \\
\epsilon_5' &= \epsilon_5 + ND \\
\theta_4' &= \theta_4 - W\epsilon_6 - Nt_3 + NW^2
\end{aligned}\right\}.$$

Other solutions of a simple form are

$$\phi_7' = \phi_7 + \epsilon_4 D,$$
$$\phi_8' = \phi_8 - t_2 A,$$
$$\phi_5' = \phi_5 + 2\epsilon_1 D + AD^2.$$

Digitized by Google

55. In order to obtain the fundamental concomitants determined by these leading coefficients, we turn their explicit expression into one which is umbral, and then complete the latter by means of the variables as in § 51. The results are as follow:—

$$\begin{cases} \omega &= q_1^2 \\ \Omega &= \omega p^2 + \dots \end{cases}$$

$$\iota = q_1 q_6 + q_2 q_5 + q_3 q_4, \text{ an invariant}$$

$$\begin{cases} {}^*\xi_7' &= q_1''(q_1 q_4' q_5'')(q_2 q_5' + q_3 q_4' + q_6 q_1') \\ \Xi_7' &= \xi_7' x_1 u_1^3 p_1 + \dots \end{cases}$$

$$\begin{cases} \xi_{10}' &= q_1 q_1''(q_1 q_6' + q_4 q_3' + q_5 q_2')(q_1 q_6'' + q_4 q_3'' + q_5 q_2'') \\ \Xi_{10}' &= \xi_{10}' x_1^3 p_1^2 u_1^2 + \dots \end{cases}$$

$$\begin{cases} \phi_{10} &= q_1' q_1''(q_2 q_3' - q_3 q_2')(q_2 q_3'' - q_3 q_2'') \\ \Phi_{10} &= \phi_{10} x_1^4 p_1^4 + \dots \end{cases}$$

$$\begin{cases} \phi_4' &= -q_1'(q_2 q_3' q_6'')(q_1 q_6'' + q_4 q_3'' + q_5 q_2'') \\ \Phi_4' &= \phi_4' x_1^4 p_1 u_1^2 + \dots \end{cases}$$

$$\begin{cases} \phi_9' &= (q_1 q_2' q_3'')(q_6 q_2' q_3'') \\ \Phi_9' &= \phi_9' x_1^5 p_1^2 u_1 + \dots \end{cases}$$

$$\begin{cases} \theta_7' &= -(q_1 q_4' q_5'')(q_1 q_2' q_3'') \\ \Theta_7' &= \theta_7' x_1^3 p_1^3 u_1^2 + \dots \end{cases}$$

$$\begin{cases} \theta_{10} &= q_1' q_1''(q_2 q_3' - q_3 q_2')(q_1 q_6'' + q_4 q_3'' + q_5 q_2'') \\ \Theta_{10} &= \theta_{10} x_1^3 p_1^3 u_1 + \dots \end{cases}$$

$$\begin{cases} t_1' &= (q_1 q_4' q_5'')^2 \\ T_1' &= t_1' u_1^4 + \dots \end{cases}$$

$$\begin{cases} t_2 &= (q_2 q_3' - q_3 q_2')^2 \\ T_2 &= t_2 x_1^4 p_1^2 + \dots \end{cases}$$

$$\begin{cases} t_3' &= (q_1 q_6' + q_4 q_3' + q_5 q_2')(q_1' q_6 + q_4' q_3 + q_5' q_2) \\ T_3' &= t_3' x_1^2 u_1^2 + \dots \end{cases}$$

$$\begin{cases} \epsilon_1' &= (q_1 q_6' + q_4 q_3' + q_5 q_2')^2 + (q_1' q_6 + q_4' q_3 + q_5' q_2)^2 \\ \mathrm{E}_1' &= \epsilon_1' x_1^2 u_1^2 + \dots \end{cases}$$

Evidently $2t_3' + \epsilon_1'$ is an invariant; and thus $2T_3' + \mathrm{E}_1'$ is divisible by u_x^2.

$$\begin{cases} \epsilon_4' &= (q_2 q_3' - q_3 q_2')(q_1' q_6 + q_4' q_3 + q_5' q_2) = (q_1 q_6' + q_4 q_3' + q_5 q_2')(q_2' q_3 - q_2 q_3') \\ \mathrm{E}_4' &= \epsilon_4' x_1^3 p_1 u_1 \end{cases}$$

$$\begin{cases} \epsilon_5' &= q_1 q_1'(q_1' q_6 + q_4' q_3 + q_5' q_2 + q_1 q_6' + q_4 q_3' + q_5 q_2') \\ \mathrm{E}_5' &= \epsilon_5' p_1^2 + \dots \end{cases}$$

$$\begin{cases} \theta_4' &= (q_1 q_4' q_5'')(q_2 q_3' q_6'') \\ \Theta_4' &= \theta_4' x_1^3 u_1^2 + \dots \end{cases}$$

* In every case a numerical factor, if it occur on the right-hand side, is omitted.

Digitized by Google

A system of fundamental concomitants of the complex of the second grade is composed of the preceding sixteen concomitants; and with them must be associated their polar derivatives (§ 18) *to form an algebraically complete system.*

56. The expressions for the other three are:—

$$\begin{cases}\phi_7' = q_1'q_1''(q_2q_3' - q_3q_2')(q_1''q_6 + q_4''q_3 + q_5''q_2)\\ \Phi_7' = \phi_7' x_1^2 p_1^2 u_1 + \dots\end{cases}$$

$$\begin{cases}\phi_8' = (q_2q_3'q_6'')^2\\ \Phi_8' = \phi_8' x_1^6 u_1^2 + \dots\end{cases}$$

$$\begin{cases}\phi_5' = q_1'q_1''(q_6q_1' + q_2q_5' + q_3q_4')(q_6q_1'' + q_2q_5'' + q_3q_4'')\\ \Phi_5' = \phi_5' x_1^2 u_1^2 p_1^2 + \dots\end{cases}$$

Two invariants have been obtained, viz.

$$\iota = q_1q_6 + q_2q_5 + q_3q_4,$$

$$2t_3' + \epsilon_1' = \tfrac{1}{2}(q_1q_6' + q_2q_5' + q_3q_4' + q_4q_6' + q_5q_2' + q_6q_1')^2;$$

and others will be obtained explicitly for a canonical form. The discriminant of Ω is (save as to a numerical factor)

$$(q_1q_2'q_3''q_4'''q_5''''q_6'''''')^2$$

$$= \begin{vmatrix} D, & M, & L, & T, & S, & N\\ M, & E, & K, & U, & O, & P\\ L, & K, & F, & V, & R, & Q\\ T, & U, & V, & C, & G, & H\\ S, & O, & R, & G, & B, & J\\ N, & P, & Q, & H, & J, & A \end{vmatrix}.$$

57. The alternative canonical forms of the complex of the second grade, dependent upon relations among its coefficients, are given by Weiler*. The most general of all is the first, in the case of which there is no relation among its coefficients.

When we apply the transformations as given in § 1, there are twelve constants at our disposal; with these it is possible to make twelve of the terms in the complex disappear. Nine terms will thus be left, the aggregate of which may, on account of the symmetry, be in general assumed to be of the form

$$Dp_1^2 + Ep_2^2 + Fp_3^2 + Cp_4^2 + Bp_5^2 + Ap_6^2 + 2Np_1p_6 + 2Op_2p_5 + 2Vp_3p_4,$$

which includes the 1st, 2nd, 4th, 6th, 7th and 8th of Weiler's forms.

The further transformations possible on the introduction of coefficients λ, μ, ν, ρ (instead of the unities of § 1) do not essentially modify this form; they merely give new coefficients of the complex

$$D\mu^2\nu^2,\quad E\nu^2\lambda^2,\quad F\lambda^2\mu^2,\quad C\nu^2\rho^2,\quad B\mu^2\rho^2,\quad A\lambda^2\rho^2,\quad 2N\lambda\mu\nu\rho,\quad 2O\lambda\mu\nu\rho,\quad 2V\lambda\mu\nu\rho.$$

The preceding form will therefore be taken as the canonical form of the most general complex.

* *Math. Ann.* t. VII., pp. 145—207.

Digitized by Google

58. To obtain the *invariants*, we might proceed as follows. We know, from § 51, that every invariant is an aggregate of products of umbral quantities which have the form

$$(m,\ n) = q_1^{(m)}q_6^{(n)} + q_2^{(m)}q_5^{(n)} + q_3^{(m)}q_4^{(n)} + q_4^{(m)}q_3^{(n)} + q_5^{(m)}q_2^{(n)} + q_6^{(m)}q_1^{(n)},$$

each product containing umbral quantities the proper number of times.

We have invariants in umbral and canonical forms as follow:

$$\begin{aligned} I_1 &= \tfrac{1}{2}(0,\ 0) \\ &= N + O + V; \end{aligned}$$

$$\begin{aligned} I_2 &= \tfrac{1}{2}(0,\ 1)^2 \\ &= N^2 + AD + O^2 + BE + V^2 + CF; \end{aligned}$$

$$\begin{aligned} I_3 &= \tfrac{1}{2}(0,\ 1)(0,\ 2)(1,\ 2) \\ &= N^3 + 3ADN + O^3 + 3BEO + V^3 + 3CFV; \end{aligned}$$

$$\begin{aligned} I_4 &= \tfrac{1}{2}(0,\ 1)(0,\ 2)(1,\ 3)(2,\ 3) \\ &= N^4 + 6ADN^2 + A^2D^2 + O^4 + 6BEO^2 + B^2E^2 + V^4 + 6CFV^2 + C^2F^2; \end{aligned}$$

$$\begin{aligned} I_5 &= \tfrac{1}{2}(0,\ 1)(0,\ 2)(1,\ 3)(2,\ 4)(3,\ 4) \\ &= N^5 + 10ADN^3 + 5A^2D^2N + O^5 + 10BEO^3 + 5B^2E^2O + V^5 + 10CFV^3 + 5C^2F^2V; \end{aligned}$$

and
$$\begin{aligned} I_6 &= \tfrac{1}{2}(0,\ 1)(0,\ 2)(1,\ 4)(3,\ 4)(2,\ 5)(3,\ 5) \\ &= N^6 + 15ADN^4 + 15A^2D^2N^2 + A^3D^3 + O^6 + 15BEO^4 + 15B^2E^2O^2 + B^3E^3 \\ &\qquad + V^6 + 15CFV^4 + 15C^2F^2V^2 + C^3F^3. \end{aligned}$$

The expressions for these can be modified. When new quantities $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6$ are introduced, defined by

$$\left.\begin{aligned} \xi_1 &= N + (AD)^{\frac{1}{2}} \\ \xi_2 &= N - (AD)^{\frac{1}{2}} \end{aligned}\right\}, \qquad \left.\begin{aligned} \xi_3 &= O + (BE)^{\frac{1}{2}} \\ \xi_4 &= O - (BE)^{\frac{1}{2}} \end{aligned}\right\}, \qquad \left.\begin{aligned} \xi_5 &= V + (CF)^{\frac{1}{2}} \\ \xi_6 &= V - (CF)^{\frac{1}{2}} \end{aligned}\right\},$$

so that they are the roots of the equation

$$\{(\xi - N)^2 - AD\}\,\{(\xi - O)^2 - BE\}\,\{(\xi - V)^2 - CF\} = 0;$$

then the *invariants are given by*

$$2I_\theta = \sum_{r=1}^{r=6} \xi_r^{\ \theta} \qquad (\theta = 1,\ 2,\ 3,\ 4,\ 5,\ 6).$$

And further the coefficients of the foregoing sextic are rational integral functions of the six invariants.

59. When we take the further transformation indicated in § 57, so as to give the complex the most simple explicit form possible, we may take

$$\lambda\mu\nu\rho = 1$$

$$A\lambda^2\rho^2 = B\mu^2\rho^2 = C\nu^2\rho^2 = 1,$$

Digitized by Google

and then the coefficients of the implicitly general complex are

$$AD,\ BE,\ CF,\ 1,\ 1,\ 1,\ 2N,\ 2O,\ 2V^*.$$

Hence every invariant of the complex is a function of AD, BE, CF, N, O, V.

By means of the substitutions which interchange p_1 and p_2, p_6 and p_5, and which leave p_3 and p_4 unaltered, it follows that BE and O enter into the invariant in a manner similar to that in which AD and N enter. Similarly for CF and V. Since

$$2N = \xi_1 + \xi_2;\quad 4AD = (\xi_1 + \xi_2)^2 - 4\xi_1\xi_2,$$

it therefore follows that when the invariant is expressed in terms of the six quantities ξ it is in the first place a function symmetric in ξ_1 and ξ_2; in ξ_3 and ξ_4; in ξ_5 and ξ_6; that is, any interchange of the two members in each of these three pairs does not affect the function. And secondly, since it is symmetric in the three pairs, it follows that the invariant is a symmetric function of the six quantities ξ; and hence *every invariant is a rational integral function of the six invariants I_1, I_2, I_3, I_4, I_5, I_6, which thus constitute the irreducible system of invariants of the quadratic complex.*

For example, the discriminant is, for the canonical form,

$$\begin{aligned} &= (AD - N^2)(BE - O^2)(CF - V^2) \\ &= -\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6, \end{aligned}$$

the integral expression for which in terms of I_θ is immediately obtainable from well-known formulæ in the theory of equations.

60. Further, if we write

$$Q = p_1p_6 + p_2p_5 + p_3p_4,$$

so that the permanent relation among the line-variables is $Q = 0$ we have

$$I_\theta(\Omega + 2\lambda Q) = (I_\theta,\ I_{\theta-1},\ \ldots,\ I_1,\ 3\between 1,\ \lambda)^\theta.$$

For $\Omega + 2\lambda Q$ differs from Ω only in having $N+\lambda$, $O+\lambda$, $V+\lambda$ instead of N, O, V, i.e. in having $\xi+\lambda$ instead of λ. Hence the theorem.

The two conditions necessary that the complex should reduce to the canonical form adopted by Battaglini† which is without the terms in p_1p_6, p_2p_5, p_3p_4 are evidently

$$I_1(\Omega + 2\lambda Q) = 0,\quad I_3(\Omega + 2\lambda Q) = 0,\quad I_5(\Omega + 2\lambda Q) = 0,$$

equivalent to the two conditions

$$(I_3,\ I_2,\ I_1,\ 3\between 1,\ -\tfrac{1}{3}I_1)^3 = 0,$$
$$(I_5,\ I_4,\ I_3,\ I_2,\ I_1,\ 3\between 1,\ -\tfrac{1}{3}I_1)^5 = 0.$$

* Weiler's (*l.c.*) first and most general canonical form would be obtained by taking

$$D\mu^2\nu^2 = A\lambda^2\rho^2,\ E\nu^2\lambda^2 = B\mu^2\rho^2,\ F\lambda^2\mu^2 = C\nu^2\rho^2.$$

His coefficients λ are then the quantities ξ.

† For references, see Klein, *Math. Ann.* t. XXIII., p. 578, Segre et Loria, *ib.* p. 213.

Digitized by Google

XVI. *On the stresses in rotating spherical shells.* By C. Chree, M.A., Fellow of King's College.

[*Read* Feb. 11, 1889.]

In a paper recently printed in the Society's *Transactions* I considered amongst other problems the case of a rotating spherical shell of isotropic material. A solution was there obtained involving six arbitrary constants which were completely determined. The solution was however given explicitly only for the case of a thin shell, which was examined in detail. The object of the present communication is to deal with the general case in isotropic materials when the ratio of the radii of the bounding surfaces may be any whatever, and specially to examine into the equilibrium form of the bounding surfaces and the tendency to rupture.

In obtaining the solution the surface conditions determining the arbitrary constants were supposed to hold over the truly spherical surfaces existent prior to the strain considered. This introduces a deviation from strict accuracy, and in order that this may be negligible it is essential that the forms of the equilibrium surfaces do not differ much from the spherical. Since in the cases considered here the surfaces are practically spheroids of small eccentricity this condition is secured.

In the paper mentioned above two theories as to tendency to rupture were mainly considered. In one it is supposed that if a value be obtained for the difference between the greatest and least principal stresses which exceeds a certain value found by experiment then the body will sooner or later rupture. On the other theory, if the greatest strain exceed a certain positive value, rupture will ensue.

In both theories the stresses and strains are supposed to be calculated on the hypothesis that the laws of perfect elasticity are strictly followed. This does not however necessarily presuppose that the body will remain truly elastic up to the point of rupture. Doubtless in approaching the point of rupture most, if not all, materials cease to follow the laws of elasticity, but there seems reason to believe that in the case of many solids when the limit of perfect elasticity is passed eventual rupture is merely a question of time. If then the experimental values introduced in the above theories be those answering to the limit of perfect elasticity, it follows that rupture will eventually follow the application of external forces which would call forth, on the theory of perfect elasticity, values for the stress-difference or greatest strain exceeding these critical values.

Digitized by Google

In materials in which the limit of perfect elasticity can be safely passed, the stress-difference and greatest strain merely indicate the limits within which the application of the equations of elasticity should be confined. For brevity the term "tendency to rupture" will be used here, but strictly it must be interpreted to mean "approach to limits of perfect elasticity."

On what Professor Pearson terms the "biconstant" theory of isotropic elasticity, with its two constants m and n, the expressions occurring for the strains and stresses are so complicated that without assigning special values to the ratio $m : n$ it would be well nigh impossible to apply our results practically. To tabulate all the expressions for a variety of values of $m : n$ would be very tedious and seems scarcely worth the labour, as the general features of the case seem pretty much the same for all reasonable values of this ratio. I have thus mainly confined myself to the special case $m = 2n$,—the only case on the "uniconstant" hypothesis,—which at all events represents an average state of things, and is unquestionably nearly true in some of the commoner metals.

A complete analysis of the principal strains and stresses and of the tendency to rupture has been found possible only over the bounding surfaces, which are fortunately the places where the comparison of theory and experiment can best be effected. It has also been found that the tendency to rupture is greatest for a solid sphere at the centre and for a thin shell at the inner surface. This lends considerable support to the view that the bounding surfaces are in general the places most exposed to rupture, and further corroborative evidence will presently be found to present itself. It must be distinctly understood however that this does not amount to a proof. Thus it might not be quite safe to assume that the shell would not rupture when the greatest values we find for the tendency to rupture are less than the critical values, though if they exceed the critical values the shell is certain to break up.

Denote by a and a' the radii of the bounding surfaces of the shell. Let u and v denote the strains along the radius vector and perpendicular to it in the meridian plane through the axis about which rotation takes place, and let ρ and ω denote the density and the angular velocity of the shell.

Then if $P_2, \equiv \dfrac{3\cos^2\theta - 1}{2}$, denote the second zonal harmonic with reference to the axis of rotation, the following are the values for the strains as calculated from the values of the constants given in my former paper, Section VI., pp. 295—6 *,

$$u = \frac{\omega^2 \rho P_2}{nD}\Big[\Big\{24m\,(m-2n)\,(aa')^3(a^2-a'^2)^2 + \tfrac{1}{3}(3m-n)\,(9m+5n)\,(a^3-a'^3)\,(a^7-a'^7)\Big\}\,r^3$$

$$-\tfrac{2}{3}(4m-n)\Big\{(9m+5n)\,(a^5-a'^5)\,(a^7-a'^7)\,r + 5\,(3m+2n)\,(aa')^3\,(a^2-a'^2)\,(a^7-a'^7)\,r^{-2}$$

$$-\,9m\,(aa')^5\,(a^2-a'^2)\,(a^5-a'^5)\,r^{-4}\Big\}\Big]$$

$$+\frac{\omega^2\rho}{15\,(m+n)}\Big\{-r^3 + \frac{5m+n}{3m-n}\,\frac{a^5-a'^5}{a^3-a'^3}\,r + \frac{5m+n}{4n}\,(aa')^3\,\frac{a^2-a'^2}{a^3-a'^3}\,r^{-2}\Big\} \quad \text{.....(1),}$$

* See also footnote on p. 301.

Digitized by Google

$$v = \frac{\omega^2\rho \sin\theta\cos\theta}{nD}\Big[-\Big\{12m(5m+4n)(aa')^3(a^2-a'^2)^2+\tfrac{1}{2}(5m-n)(9m+5n)(a^5-a'^5)(a^7-a'^7)\Big\}r^3$$

$$+(4m-n)\Big\{(9m+5n)(a^5-a'^5)(a^7-a'^7)\,r+10n\,(aa')^3(a^2-a'^2)(a^7-a'^7)\,r^{-2}$$

$$+6m\,(aa')^5(a^2-a'^2)(a^5-a'^5)\,r^{-4}\Big\}\Big] \quad\text{.....(2),}$$

where $$D \equiv (9m+5n)(19m-5n)(a^5-a'^5)(a^7-a'^7)-504m^2(aa')^3(a^2-a'^2)^2 \quad\text{.....(3).}$$

Also the dilatation

$$\delta \equiv \frac{du}{dr}+\frac{2u}{r}+\frac{1}{r}\frac{dv}{d\theta}+\frac{v}{r}\cot\theta \quad\text{.....(4),}$$

is given by

$$\delta = \frac{4\omega^2\rho P_2}{D}\Big[-\Big\{84m\,(aa')^3(a^2-a'^2)^2+\tfrac{1}{6}(9m+5n)(a^5-a'^5)(a^7-a'^7)\Big\}r^2$$

$$+5\,(4m-n)\,(aa')^3(a^2-a'^2)(a^7-a'^7)\,r^{-2}\Big]$$

$$+\frac{\omega^2\rho}{m+n}\Big\{-\tfrac{1}{3}r^2+\frac{5m+n}{5\,(3m-n)}\,\frac{a^5-a'^5}{a^3-a'^3}\Big\} \quad\text{.....(5).}$$

In the special case $m=2n$, writing z for a/a', we have

$$u = \frac{1}{3}\,\frac{\omega^2\rho P_2}{n^{-1}D}\Big[115\,(z^2-1)(z^7-1)\,r^3-322\,(z^5-1)(z^7-1)\,z^{-2}a^2r$$

$$-560\,(z^2-1)(z^7-1)\,z^{-2}a^5r^{-2}+252\,(z^2-1)(z^5-1)\,z^{-2}a^7r^{-4}\Big]$$

$$+\frac{\omega^2\rho}{45n}\Big\{-r^3+\frac{11\,(z^5-1)\,a^2r}{5z^2\,(z^3-1)}+\frac{11\,(z^2-1)\,a^5r^{-2}}{4z^2\,(z^3-1)}\Big\} \quad\text{.....(1 a),}$$

$$v = \frac{\omega^2\rho\sin\theta\cos\theta}{2n^{-1}D}\Big[-\{672z^3(z^2-1)^2+207\,(z^2-1)(z^7-1)\}\,r^3+322\,(z^5-1)(z^7-1)\,z^{-2}a^2r$$

$$+140\,(z^2-1)(z^7-1)\,z^{-2}a^5r^{-2}+168\,(z^2-1)(z^5-1)\,z^{-2}a^7r^{-4}\Big] \quad\text{.....(2 a),}$$

$$n^{-2}D \equiv 759\,(z^2-1)(z^7-1)-2016z^3(z^2-1)^2 \quad\text{.....(3 a),}$$

$$\delta = \frac{\omega^2\rho P_2}{n^{-1}D}\Big[-\tfrac{2}{3}\{1008z^3(z^2-1)^2+23\,(z^2-1)(z^7-1)\}\,r^2+140\,(z^2-1)(z^7-1)\,z^{-2}a^5r^{-2}\Big]$$

$$+\frac{\omega^2\rho}{3n}\Big[-\tfrac{1}{3}r^2+\tfrac{11}{25}z^{-2}\frac{z^5-1}{z^3-1}\,a^2\Big] \quad\text{.....(5 a).}$$

The equilibrium surfaces of the shell being of the form

$$r = c + c_1P_2,$$

where c and c_1 are constants and c_1/c is small, we may treat each surface as very approximately a spheroid, whose ellipticity may be regarded as the ratio of the difference between the equatorial and polar radii after displacement to the radius of the undisturbed spherical surface. As a clearer idea of the nature and magnitude of the

Digitized by Google

phenomena may be obtained from the consideration of a particular case, we shall apply our results to a shell whose outer radius is 4000 miles and which rotates with the Earth's angular velocity. Supposing we take $\omega^2\rho a^2/E = \cdot 006$, where E denotes Young's modulus, then if ρ were the Earth's mean density the material would have the same elasticity as an average specimen of iron.

If r_e and r_p denote equatorial and polar radii, the ellipticities of the two bounding surfaces may be taken as

$$\left.\begin{aligned} \epsilon &= (r_e - r_p)/a \\ \epsilon' &= (r_e - r_p)/a' \end{aligned}\right\} \quad \text{.....(6)},$$

putting $r = a$ in the first expression and $r = a'$ in the second.

The following, Table I., is calculated from (1), noticing that

$$r_e - r_p = -\tfrac{3}{2}\,[\text{coefficient of } P_2 \text{ in (1)}].$$

We have replaced n by $\frac{2}{5}E$ when $m = 2n$, and by $\frac{1}{3}E$ when n/m is negligible; and ϵ is supposed so small that ϵ^2 is negligible.

TABLE I.

Value of a'/a	Value of $\epsilon \Big/ \frac{\omega^2\rho a^2}{E}$ on the outer surface: when $\frac{n}{m}$ negligible	Value of $\epsilon \Big/ \frac{\omega^2\rho a^2}{E}$ on the outer surface: when $m=2n$	Value of $\epsilon' \Big/ \frac{\omega^2\rho a^2}{E}$ on the inner surface: when $\frac{n}{m}$ negligible	Value of $\epsilon' \Big/ \frac{\omega^2\rho a^2}{E}$ on the inner surface: when $m=2n$	Particular case when $a=4000$ miles, and $\omega^2\rho a^2/E = \cdot 006$: Difference in miles between the equatorial and polar diameters: outer surface	Difference in miles between the equatorial and polar diameters: inner surface	Ellipticity: outer surface	Ellipticity: inner surface
0	·39	·341			16·36		·00205	
·01		·341		1·04	16·36	·5	·00205	·00624
·1	·39	·343	1·05	1·034	16·47	4·96	·00206	·00621
·2		·359		1·038	17·25	9·97	·00216	·00623
·25		·376		1·050	18·06	12·60	·00226	·00630
·3	·47	·423	1·12	1·085	20·31	17·36	·00254	·0065
·4		·482		1·13	23·16	21·78	·00289	·0068
·5	·67	·62	1·31	1·25	29·70	30·06	·0037	·0075
·6		·82		1·43	39·33	41·10	·0049	·0086
·6		1·01		1·57	48·57	50·32	·0061	·0094
·8		1·48		1·90	71·12	72·81	·0089	·0114
·9	2·12	1·88	2·34	2·11	90·06	91·11	·0113	·0127
$1-\epsilon$	2·5	2·25			108·		·0135	

In the calculations on which the tables are based a closer degree of approximation was usually aimed at than appears in the tables. In each case in the tables the last figure is selected so as to make the result stated as near the closer approximation as possible, but the number of figures retained has been varied according as the rate of change with a'/a of the quantity considered is more or less rapid.

It will be observed how very similar are the conclusions derived from the widely different hypotheses as to the value of m/n; thus the results deduced from the hypo-

Digitized by Google

thesis $m = 2n$ apply doubtless with considerable accuracy to the case of any ordinary isotropic material.

Another important conclusion is that the existence of a concentric spherical cavity has extremely little influence on the form of the outer surface so long as the radius of the inner surface does not exceed one quarter that of the outer. The practical constancy in the ellipticity of the inner surface until this limit is reached is also worthy of notice. Some interest may also attach to the fact that in the particular case, even when the sphere is solid to the centre, the difference between the equatorial and polar diameters exceeds three-fifths of what is found in the case of the Earth.

Besides converting the spherical surfaces into spheroids of small eccentricity, the rotation increases the mean radii of the bounding surfaces. These increases are given in the following table.

TABLE II.

	Part of $\frac{u}{r} \Big/ \frac{\omega^2 \rho a^2}{E}$ which is independent of P_2				Particular case when $a = 4000$ miles, and $\omega^2 \rho a^2 / E = \cdot 006$	
	Outer surface		Inner surface			
Value of a'/a	$\frac{n}{m}$ negligible	$m = 2n$	$\frac{n}{m}$ negligible	$m = 2n$	Increase, in miles, in mean radius of outer surface, Δa	Increase, in miles, in mean radius of inner surface, $\Delta a'$
0	0	$\cdot 0\dot{6}$			1·6	
·01	negligible	$\cdot 0\dot{6}$	·25	·275	1·6	·066
·1	negligible	·067	·2475	·273	1·608	·65
·2	·002	·069	·242	·269	1·65	1·29
$\cdot\dot{3}$		·076		·261	1·83	2·09
·4		·0825		·257	1·98	2·47
·5	·027	·096	·214	·252	2·31	3·03
$\cdot\dot{6}$		·131		·247	3·14	3·95
·8	·094	·171	·183	·246	4·09	4·72
·9		·207		·247	4·97	5·33
$1 - \epsilon$	$\cdot 1\dot{6}$	·25			6·0	

The small effect on the radius of the outer surface produced by the presence of an internal cavity until the diameter of the cavity approaches to one quarter of that of the outer surface calls for remark. A similar phenomenon has been already pointed out in treating of the ellipticity. It may also be noticed that on the inner surface the increase of the mean radius bears to that radius a ratio which is not far from constant.

On the outer surface, especially for small values of a'/a, there is a very decided difference between the results answering to the two values of n/m. This may seem to contradict previous statements as to the representative character of the hypothesis $m = 2n$; but a comparison of Tables II. and I. will shew that the displacement dealt with in Table II. is a comparatively small one. This will be most clearly seen from the columns in each which deal with the particular case. It will be observed that for no value of

Digitized by Google

a'/a does the increase of the mean radius of the outer surface reach a fifth part of the difference between the equatorial and polar radii.

A complete investigation of the state of strain and stress at the bounding surfaces requires the position of the principal axes there to be known. It will thus be best to consider briefly some points connected with the determination of these axes.

Let us take, as in my former paper, the radius vector, the perpendicular to it in the meridian, and the perpendicular to the meridian at every point as fundamental axes. Then the strains and stresses are given by the following scheme.

Components of the strain	Components of the stress	Planes of which relative motion, or across which force is measured	Direction of relative motion, or force
$\dfrac{du}{dr}$	$R=(m-n)\,\delta+2n\dfrac{du}{dr}$	$\theta\phi$	r
$\dfrac{u}{r}+\dfrac{1}{r}\dfrac{dv}{d\theta}$	$\Theta=(m-n)\,\delta+2n\left(\dfrac{u}{r}+\dfrac{1}{r}\dfrac{dv}{d\theta}\right)$	ϕr	θ
$\dfrac{u}{r}+\dfrac{v}{r}\cot\theta$	$\Phi=(m-n)\,\delta+2n\left(\dfrac{u}{r}+\dfrac{v}{r}\cot\theta\right)$	$r\theta$	ϕ
$r\dfrac{d.\frac{v}{r}}{dr}+\dfrac{1}{r}\dfrac{du}{d\theta}$	$R_\theta=n\left(r\dfrac{d.\frac{v}{r}}{dr}+\dfrac{1}{r}\dfrac{du}{d\theta}\right)$	$\left\{\begin{matrix}r\phi\\ \theta\phi\end{matrix}\right.$	$\left.\begin{matrix}r\\ \theta\end{matrix}\right\}$

If
$$S=\{(R-\Theta)^2+4R_\theta^2\}^{\frac{1}{2}}$$
$$=2n\left\{\left(r\frac{d.\frac{u}{r}}{dr}-\frac{1}{r}\frac{dv}{d\theta}\right)^2+\left(r\frac{d.\frac{v}{r}}{dr}+\frac{1}{r}\frac{du}{d\theta}\right)^2\right\}^{\frac{1}{2}}\quad\text{.....(7)},$$

then it is proved in Section III. of my previous paper that the three principal stresses at any point are
$$R_1\equiv\frac{R+\Theta+S}{2},\quad \Theta_1\equiv\frac{R+\Theta-S}{2},\quad \Phi.$$

Similarly the three principal strains are
$$\frac{1}{2}\left\{\frac{du}{dr}+\frac{u}{r}+\frac{1}{r}\frac{dv}{d\theta}+\frac{S}{2n}\right\},\quad \frac{1}{2}\left\{\frac{du}{dr}+\frac{u}{r}+\frac{1}{r}\frac{dv}{d\theta}-\frac{S}{2n}\right\},\quad \frac{u}{r}+\frac{v}{r}\cot\theta.$$

It is supposed that the sign is taken so that S is always a positive quantity, or zero, and so
$$R_1\geqq\Theta_1,$$
$$\frac{1}{2}\left\{\frac{du}{dr}+\frac{u}{r}+\frac{1}{r}\frac{dv}{d\theta}+\frac{S}{2n}\right\}\geqq\frac{1}{2}\left\{\frac{du}{dr}+\frac{u}{r}+\frac{1}{r}\frac{dv}{d\theta}-\frac{S}{2n}\right\}.$$

Digitized by Google

Consequently R_1 is the greatest stress and

$$\frac{1}{2}\left\{\frac{du}{dr}+\frac{u}{r}+\frac{1}{r}\frac{dv}{d\theta}+\frac{S}{2n}\right\}$$

the greatest strain in the meridian plane; and the corresponding principal axis makes with the radius vector drawn outwards the angle

$$\alpha = \tan^{-1}\left\{\frac{S-R+\Theta}{2R_\theta}\right\} \quad\text{.....(8).}$$

Since the expressions for the normal strains and stresses referred to the fundamental axes are all of the general type

$$a_0 + a_2 r^2 + a_3 r^{-3} + a_5 r^{-5} + (a_0' + a_2' r^2 + a_3' r^{-3} + a_5' r^{-5})\, P_2 \quad\text{.....(9),}$$

where a_0, a_0' etc. are constants which differ for the several strains and stresses, it is obvious that a complete determination of the principal strains and stresses throughout the shell would be a difficult matter.

As depending on P_2 all the principal strains and stresses will have maximum or minimum values in the equator and at the poles; and the principal strain

$$\frac{u}{r}+\frac{v}{r}\cot\theta,$$

and the corresponding principal stress Φ, which are linear functions of P_2, can have maximum or minimum values nowhere else.

For values of r for which S vanishes or is a linear function of P_2 the same is true of the other principal strains and stresses; but for other values of r we should get a quadratic equation in $\cos^2\theta$ determining polar distances at which additional maxima or minima might exist. The probability of a quadratic equation having roots which lie between 0 and 1 is of course very small, but as the quadratic in question contains the quantities m and n it may not improbably supply possible values of θ for particular kinds of material for most values of a'/a.

Referring to (7) it will be noticed that S, as given by the square root of the sum of two squares, can be linear in P_2 only when one of the squares—that depending on R_θ—vanishes, and can be zero only when both squares vanish.

Now $R-\Theta$, or its equivalent

$$2n\left\{r\frac{d.\frac{u}{r}}{dr}-\frac{1}{r}\frac{dv}{d\theta}\right\},$$

is given by an expression of the type (9) and so could vanish, for all values of θ, only for values of r satisfying simultaneously two independent equations of the seventh degree. Further, to have any bearing on our problem such common roots would require to lie between a' and a. Except for the special case of a very thin shell when terms in $\frac{a-a'}{a}$ are negligible, in which case R and Θ both vanish to this degree of approxi-

Digitized by Google

mation throughout the entire shell, this contingency is so unlikely to arise except for some chance value of m/n that it need not be further considered.

It is different in the case of

$$R_\theta \equiv n\left\{r\frac{d.\frac{v}{r}}{dr}+\frac{1}{r}\frac{du}{d\theta}\right\}.$$

It is not very difficult to prove from (1) and (2) that

$$\left(r\frac{d.\frac{v}{r}}{dr}+\frac{1}{r}\frac{du}{d\theta}\right)=\frac{2\omega^2\rho\,(4m-n)\sin\theta\cos\theta\,(a-a')(r-a')(a-r)}{nD}$$

$$\times\left[(9m+5n)\frac{a^7-a'^7}{r^3}\left\{r^3(a^2+aa'+a'^2)+r^2(a+a')(a^2+aa'+a'^2)+raa'(a+a')^2+a^2a'^2(a+a')\right\}\right.$$

$$+24m\,(aa')^2(a-a')(a+a')^2r^{-5}\left\{r^5+r^4(a+a')+r^3(a^2+aa'+a'^2)+r^2(a+a')(a^2+a'^2)\right.$$

$$\left.\left.+\left(r+\frac{aa'}{a+a'}\right)(a^4+a^3a'+a^2a'^2+aa'^3+a'^4)\right\}\right] \quad\text{.....(10).}$$

A glance will shew that the expression inside the square bracket is necessarily positive for all values of r whatever may be the value of m/n. Thus R_θ vanishes when $r=a$ and when $r=a'$ for all possible values of θ,—this is in fact one of the surface-conditions;—but for no other value of r can R_θ vanish unless in the polar axis or in the equator, where it vanishes for all values of r. Thus S is a linear function of P_2 over the surfaces of the shell, and can vanish at no other central distance except in the polar axis or the equator. Consequently the greatest strain and the maximum stress-difference over the bounding surfaces must have their absolutely greatest values either at the poles or in the equator; but for intermediate values of r this is not necessarily the case.

It should also be noticed that when R_θ vanishes the fundamental axes are all three principal axes. This is accordingly the case over both surfaces of the shell for all values of θ, and also along the polar axis and in the equatorial plane for all values of r; but it can occur nowhere else.

Referring to (10) it will be seen that R_θ must have the same sign as $\frac{\sin 2\theta}{D}$, for m cannot well be less than n, and so $4m-n$ is positive*. Now if D could vanish for any such value of m/n as occurs in real materials, we should have a shell of such material suffering infinite displacements when rotated however slowly. We may thus regard D as necessarily of one sign for all real materials, and it may in fact be found by trial that D is essentially positive. Also, referring to (7), it is apparent that, excepting the bounding surfaces, the polar diameter and the equatorial plane, $S-R+\Theta$ must be positive; hence it follows from (8) that $\tan\alpha$ has the same sign as $\sin 2\theta$. We are thus in a position to assert that *the only regions where the fundamental axes are all three principal axes are the*

* Poisson's ratio would be negative if m were less than n, and Young's modulus if $3m$ less than n.

Digitized by Google

bounding surfaces, the polar diameter and the equatorial plane; elsewhere the principal axis in the meridian plane answering to the algebraically greater principal strain and stress in that plane lies between the radius vector drawn outwards and the equatorial plane.

The two following tables contain an analysis of the principal strains over the bounding surfaces of a shell for various values of a'/a, the material in all cases being supposed such that $m = 2n$. The directions of the strains will be found in our scheme of strain and stress.

TABLE III. Principal strains on the outer surface $r = a$ of a shell for which $m = 2n$.

	In equator			At poles	
Value of a'/a	$\frac{E}{\omega^2\rho a^2}\frac{du}{dr}$	$\frac{E}{\omega^2\rho a^2}\left(\frac{u}{r}+\frac{1}{r}\frac{dv}{d\theta}\right)$	$\frac{E}{\omega^2\rho a^2}\left(\frac{u}{r}+\frac{v}{r}\cot\theta\right)$	$\frac{E}{\omega^2\rho a^2}\frac{du}{dr}$	$\frac{E}{\omega^2\rho a^2}\left(\frac{u}{r}+\frac{1}{r}\frac{dv}{d\theta}\right)=\frac{E}{\omega^2\rho a^2}\left(\frac{u}{r}+\frac{v}{r}\cot\theta\right)$
0	$-\cdot 057\dot{0}$	$-\cdot 009$	$+\cdot 180$	$-\cdot 01\dot{9}$	$+\cdot 029$
·01	$\cdot 057\dot{0}$	$-\cdot 009$	$\cdot 180$	$-\cdot 01\dot{9}$	$+\cdot 029$
·1	·058	$-\cdot 008$	·182	$-\cdot 0185$	$+\cdot 028$
·2	·062	$-\cdot 002$	·188	$-\cdot 0135$	$+\cdot 020$
·3	·078	$+\cdot 018$	·217	$+\cdot 0048$	$-\cdot 007$
·4	·093	$+\cdot 034$	·243	$+\cdot 020$	$-\cdot 030$
·5	·121	$+\cdot 062$	·302	$+\cdot 051$	$-\cdot 076$
·6	·185	$+\cdot 088$	·468	$+\cdot 109$	$-\cdot 163$
·8	·232	$+\cdot 032$	·664	$+\cdot 123$	$-\cdot 185$
·9	·250	$-\cdot 082$	·833	$+\cdot 086$	$-\cdot 129$
$1-\epsilon$	·250	$-\cdot 250$	1·00	0	0

TABLE IV. Principal strains on the inner surface $r = a'$ of a shell for which $m = 2n$.

	In equator			At poles	
Value of a'/a	$\frac{E}{\omega^2\rho a^2}\frac{du}{dr}$	$\frac{E}{\omega^2\rho a^2}\left(\frac{u}{r}+\frac{1}{r}\frac{dv}{d\theta}\right)$	$\frac{E}{\omega^2\rho a^2}\left(\frac{u}{r}+\frac{v}{r}\cot\theta\right)$	$\frac{E}{\omega^2\rho a^2}\frac{du}{dr}$	$\frac{E}{\omega^2\rho a^2}\left(\frac{u}{r}+\frac{1}{r}\frac{dv}{d\theta}\right)=\frac{E}{\omega^2\rho a^2}\left(\frac{u}{r}+\frac{v}{r}\cot\theta\right)$
·0	$+\cdot 29\dot{8}$	$-\cdot 23\dot{1}$	$+\cdot 29\dot{8}$	$-\cdot 23\dot{1}$	$+\cdot 29\dot{8}$
·01	$-\cdot 068$	·418	·621	·414	·621
·1	·0676	·415	·618	·411	·616
·2	·066	·417	·615	·406	·609
·3	·062	·437	·623	·398	·598
·4	·060	·455	·635	·395	·592
·5	·061	·488	·670	·384	·575
·6	·083	·521	·771	·327	·490
·8	·134	·475	·878	·223	·334
·9	·190	·380	·950	·114	·171
$1-\epsilon$	·250	·250	1·00	0	0

When no sign is prefixed to a value it is supposed that it has the sign last attached to an element of the column.

The values in Table IV. for $a'/a = 0$ refer to the centre of a solid sphere, and the headings "in equator" and "at poles" must here be interpreted to mean "in an equatorial

Digitized by Google

direction" and "in a polar direction" respectively. The strain here in reality consists of a compression $\cdot 23\dot{1}\omega^2\rho a^2/E$ along the axis of rotation, and an extension $\cdot 29\dot{8}\omega^2\rho a^2/E$ in all directions perpendicular to this axis.

From these tables the principal strains at any point of either surface may be deduced by means of the following formula. Let $f(\theta)$, $f(0)$, $f\left(\frac{\pi}{2}\right)$ represent the values of any one of the principal strains, or stresses, at the angular distances θ, 0, $\frac{\pi}{2}$ respectively from either pole on one of the surfaces: then

$$f(\theta) = f(0) + \sin^2\theta \left\{ f\left(\frac{\pi}{2}\right) - f(0) \right\} \quad \text{.....(11).}$$

On the outer surface the state of strain varies at first extremely slowly and with complete continuity as a'/a increases from zero. Along the radial direction there is compression in the equator for all values of a'/a. There is compression in the same direction at the poles likewise until a'/a is about ·3. As a'/a is further increased this is converted into an extension, at first increasing with a'/a, and then decreasing rapidly and finally vanishing in a very thin shell. The following table gives the polar distance at which there is neither extension nor compression along the radius vector. For smaller polar distances there is extension and for greater distances compression.

TABLE V.

Value of a'/a	·3	·4	·5	·6	·8	·9
Polar distance of locus of no strain in a radial direction on the outer surface	13° 45′	25° 1′	32° 53′	37° 26′	36° 4′	30° 25′

The strain along the tangent to the meridian on the outer surface is for values of a'/a from 0 to ·2 a compression in the equator an extension at the poles, being very small throughout especially in the equator. For values of a'/a from ·3 to ·8 it is an extension in the equator a compression at the poles. As a'/a increases further the compression at the poles diminishes and eventually vanishes in a very thin shell, while the extension in the equator becomes converted into a compression which increases very rapidly as the thickness of the shell is reduced.

Along the perpendicular to the meridian on the outer surface there is in every case extension in the equator, and this increases continually with a'/a. At the poles the strain in this direction is always the same as that along the tangent to the meridian. The following table gives the polar distance at which there is neither extension nor compression at right angles to the meridian. For smaller polar distances there is compression and for greater extension.

Digitized by Google

TABLE VI.

Value of a'/a	·3	·4	·5	·6	·8	·9
Polar distance of locus of no strain in a direction perpendicular to the meridian, on the outer surface	10° 15′	19° 23′	26° 39′	30° 33′	27° 48′	21° 30′

When a'/a is less than ·3 the strain at right angles to the meridian on the outer surface is everywhere an extension.

In the table for the strains on the inner surface the values for the case $a'/a = 0$ are of a wholly distinctive character. There is a complete breach of continuity between this case and that when a'/a has any greater value however small. This is not to be wondered at, because the existence of a cavity completely alters the conditions of the problem. In the solid sphere there is the condition that the strain must contain no negative powers of r, in the shell the condition that $r = a'$ must be a free surface. Leaving out the case $a'/a = 0$ as exceptional, there is in every case compression along the radius vector all over the inner surface. In the equator it at first diminishes slowly as a'/a increases, attaining a minimum when a'/a is slightly less than 1/2, and then increases at a much more rapid rate. The compression in this direction at the poles is very much larger than in the equator for small values of a'/a, and remains nearly constant until a'/a exceeds 1/2. It then decreases comparatively rapidly and becomes less than the compression in the equator before a'/a reaches the value ·9.

Along the tangent to the meridian the strain is in the equator always a compression, having an almost constant value till a'/a exceeds ·2, then increasing slowly to a maximum when a'/a is about ·6, and then again diminishing. At the poles the strain in this direction is the same as that perpendicular to the meridian.

In this latter direction there is in every case extension in the equator. This at first decreases to a trifling extent until a'/a reaches the value ·2, and then increases continually as a'/a is further increased. At the poles there is also extension at right angles to the meridian. For small values of a'/a this is practically the same as in the equator. As a'/a increases this strain continually diminishes, becoming decidedly less than in the equator when a'/a passes the value ·2, and finally approaches zero rapidly as the shell becomes thin.

It has been already pointed out that on one theory the tendency to rupture is measured by the value of the greatest positive strain. On reference to the tables this will be seen to have on either surface the perpendicular to the meridian for its direction; and it is greatest on the inner surface and in the equator. It will be noticed that the existence of a cavity however small more than doubles the tendency to rupture found in the solid sphere. Further slight enlargement of the cavity causes at first a very slight diminution in the tendency to rupture; which has moreover very nearly a constant value

Digitized by Google

over the whole inner surface until a'/a approaches the value ·2. As a'/a is further increased the tendency to rupture continually increases, at first slowly but then more rapidly as a'/a approaches unity. In a very thin shell the tendency to rupture is nearly 5/3 times that existing when the cavity is vanishingly small, and almost exactly 10/3 times that existing in the absolutely solid sphere.

Since the internal cavity has been supposed spherical, we cannot of course expect our numerical results to apply exactly to any other form of cavity; but the following conclusions as to the way in which rupture will take place in a solid sphere, of a material whose limits of perfect elasticity and safety are not far apart, would appear fairly warranted.

A crack will begin at the centre, where the absolutely greatest strain is found, and extend in a direction at right angles to the axis of rotation. On the appearance of the crack the tendency to rupture will suddenly increase, and the crack extend right up to the surface. The exact equatorial radius along which the crack propagates itself will in any practical case be determined by slight variations in the structure of the sphere, or deviations from true sphericity in its surface; and, as it will naturally extend in the direction perpendicular to the greatest strain, the crack may be expected to appear on the outer surface as a split along a meridian most apparent at or close to the equator.

The value of the dilatation δ at any point of either surface can be deduced immediately from Tables III. and IV. One of the surface conditions is in fact

$$(m-n)\,\delta + 2n\frac{du}{dr} = 0,$$

and so when $m = 2n$ we have over either surface of the shell

$$\delta = -2\frac{du}{dr}.$$

Thus to get the dilatation over the surfaces we have only to multiply the values given for $\frac{du}{dr}$ in the tables by -2; and by writing extension for compression and conversely the remarks already made about the dependence of $\frac{du}{dr}$ on a'/a and θ may be at once translated so as to apply to the dilatation.

Further, since δ equals the sum of the three principal strains, it follows that the sum of the values in our tables for $\frac{u}{r} + \frac{1}{r}\frac{dv}{d\theta}$ and $\frac{u}{r} + \frac{v}{r}\cot\theta$ added to thrice the corresponding value of $\frac{du}{dr}$ should be zero. This supplies a very severe test of the accuracy of the tables. It also enables the following tables for the principal stresses over the surfaces to be very simply deduced from the formulæ

$$\Theta = \tfrac{4}{5}E\left\{\left(\frac{u}{r} + \frac{1}{r}\frac{dv}{d\theta}\right) - \frac{du}{dr}\right\},$$

$$\Phi = \tfrac{4}{5}E\left\{\left(\frac{u}{r} + \frac{v}{r}\cot\theta\right) - \frac{du}{dr}\right\},$$

Digitized by Google

noticing that when $m = 2n$ we have $n = \frac{2}{5}E$. The third principal stress, that along the radius, is zero.

TABLE VII. Principal stresses and maximum stress-difference on the outer surface $r = a$ of a shell for which $m = 2n$.

Value of a'/a	Equatorial value of $\Theta/\omega^2\rho a^2$	Equatorial value of $\Phi/\omega^2\rho a^2$	Polar values of $\Theta/\omega^2\rho a^2$ and $\Phi/\omega^2\rho a^2$	Greatest value of maximum stress-difference in tons per square inch on the outer surface of a spherical shell for which $a = 4000$ miles, $\rho = 5\frac{2}{3}$, and ω = earth's angular velocity of rotation
·0	+ ·0384	+ ·1899	+ ·0384	15·21
·01	·0384	·1899	+ ·0384	15·21
·1	·0398	·1914	+ ·0369	15·33
·2	·0478	·200	+ ·027	16·1
·3	·077	·236	+ ·010	18·9
·4	·102	·269	− ·040	21·5
·5	·147	·339	− ·102	27·2
·6	·218	·523	− ·217	41·9
·8	·211	·717	− ·246	57·5
·9	·135	·866	− ·172	69·4
$1 - \epsilon$	0	1·0	0	80·1

Since the maximum stress-difference must have its greatest value either at the poles or in the equator, and Θ is positive in the equator while R vanishes, it is obvious that this greatest value is simply the equatorial value of Φ.

TABLE VIII. Principal stresses and maximum stress-difference over the inner surface $r = a'$ of a shell for which $m = 2n$.

Value of a'/a	Equatorial value of $\Theta/\omega^2\rho a^2$	Equatorial value of $\Phi/\omega^2\rho a^2$	Polar values of $\Theta/\omega^2\rho a^2$ and $\Phi/\omega^2\rho a^2$	Greatest value of maximum stress-difference / $\omega^2\rho a^2$	Greatest value of maximum stress-difference in the special case of last table, measured in tons per square inch
·0				·42	34·0
·01	− ·280	+ ·551	+ ·828	·831	66·60
·1	·278	·548	·822	·826	66·20
·2	·281	·545	·811	·826	66·15
·3	·300	·548	·797	·848	67·9
·4	·316	·556	·789	·873	69·9
·5	·342	·584	·767	·926	74·2
·6	·350	·683	·654	1·034	82·8
·8	·273	·809	·446	1·082	86·7
·9	·152	·912	·228	1·064	85·2
$1 - \epsilon$	0	1·0	0	1·0	80·1

Digitized by Google

In the shell the greatest value of the maximum stress-difference always occurs in the equator, and is $=\Phi-\Theta$. In the solid sphere the greatest value is found at the centre, where the stresses consist of a pressure $\cdot 0\dot{3}\dot{8}\omega^2\rho a^2$ along the axis of rotation and a traction $\cdot 3\dot{8}\dot{5}\omega^2\rho a^2$ along all directions perpendicular to the axis.

The conclusions as to the tendency to rupture at the bounding surfaces deduced from the maximum stress-difference are on the whole closely analogous to those deduced from the greatest strain. The tendency is greatest invariably in the equatorial regions of the inner surface. The formation of a spherical cavity however small almost exactly doubles the tendency to rupture found in a solid sphere. As a'/a increases there is at first a very slight falling off in the value of this expression. After a'/a attains the value 1/3 there is a continual increase until a'/a reaches the value ·8, which is followed by a slight decrease as the shell becomes very thin. This constitutes one rather striking, though numerically small, difference between the conclusions of the two theories; for on the greatest strain theory after a'/a attains the value 1/3 the tendency to rupture increases continually with a'/a, and attains its maximum value in the indefinitely thin shell.

It should be remembered however that we have not *proved* that the absolutely greatest values of either the greatest strain or the maximum stress-difference *must* occur on the bounding surfaces of a shell, except in the case when the shell is very thin. So in every case, except the very thin shell and the solid sphere, there *may* be values for the greatest strain and maximum stress-difference exceeding those of the previous tables. We are consequently only entitled to regard our results as supplying inferior limits to the real values of the tendency to rupture. That there are in reality greater values I think is however on the whole improbable, at least when $m=2n$; otherwise we should not expect to find the values on the inner surface invariably exceeding those on the outer surface.

For practical purposes the following rule may be useful. Let T be the greatest tractional load in tons per square inch that may be safely applied to a beam of a certain isotropic material of density ρ; then a superior limit to the number N of revolutions per second which it is safe to allow in a spherical shell of this material, a feet in external radius, is given on the stress-difference theory by

$$N=\frac{2200}{9a}\left(\frac{17}{3\rho}\right)^{\frac{1}{2}}\left(\frac{T}{Y}\right)^{\frac{1}{2}} \quad\text{.....(12),*}$$

where Y is the number in the last column of Table VIII. answering to the particular value of a'/a occurring in the shell.

Possibly the clearest idea of the state of strain and stress in a rotating spherical shell may be obtained from a study of the accompanying diagrams.

In every case the abscissæ of the curves indicate values of a'/a. The suffix o indicates

* On the greatest strain theory replace Y by 80·12 Z, where Z = corresponding number in the fourth column of Table IV.

Digitized by Google

the outer, and the suffix i the inner surface. Undashed letters refer to the equator and dashed letters to the poles. The strains and stresses at the centre of the solid sphere are not shewn.

In diagram I. the ordinates represent the values of the strains in the equator:

e_o of $\frac{du}{dr}$ on the outer, and e_i on the inner surface;

f_o of $\frac{u}{r}+\frac{1}{r}\frac{dv}{d\theta}$ on the outer, and f_i on the inner surface;

g_o of $\frac{u}{r}+\frac{v}{r}\cot\theta$ on the outer, and g_i on the inner surface.

The curve g_i also represents the value of the absolutely greatest strain found on either surface.

In diagram II. the ordinates represent the values of the strains at the poles:

e_o' of $\frac{du}{dr}$ on the outer, and e_i' on the inner surface;

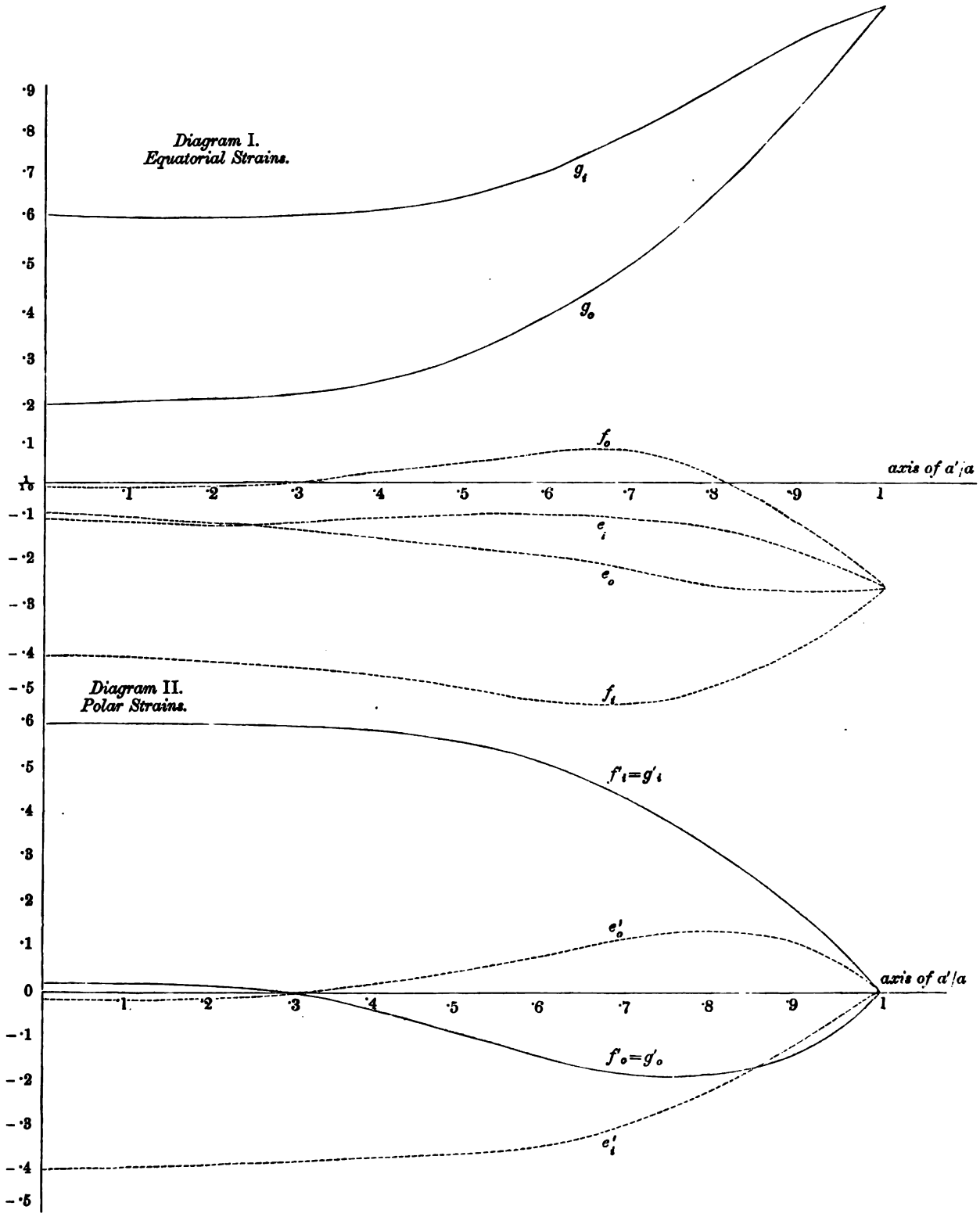
$f_o'=g_o'$, of $\frac{u}{r}+\frac{1}{r}\frac{dv}{d\theta}$ and $\frac{u}{r}+\frac{v}{r}\cot\theta$ on the outer, and $f_i'=g_i'$ on the inner surface.

In diagrams I. and II. the ordinates multiplied by $\omega^2\rho a^2/E$ give the absolute values of the strains.

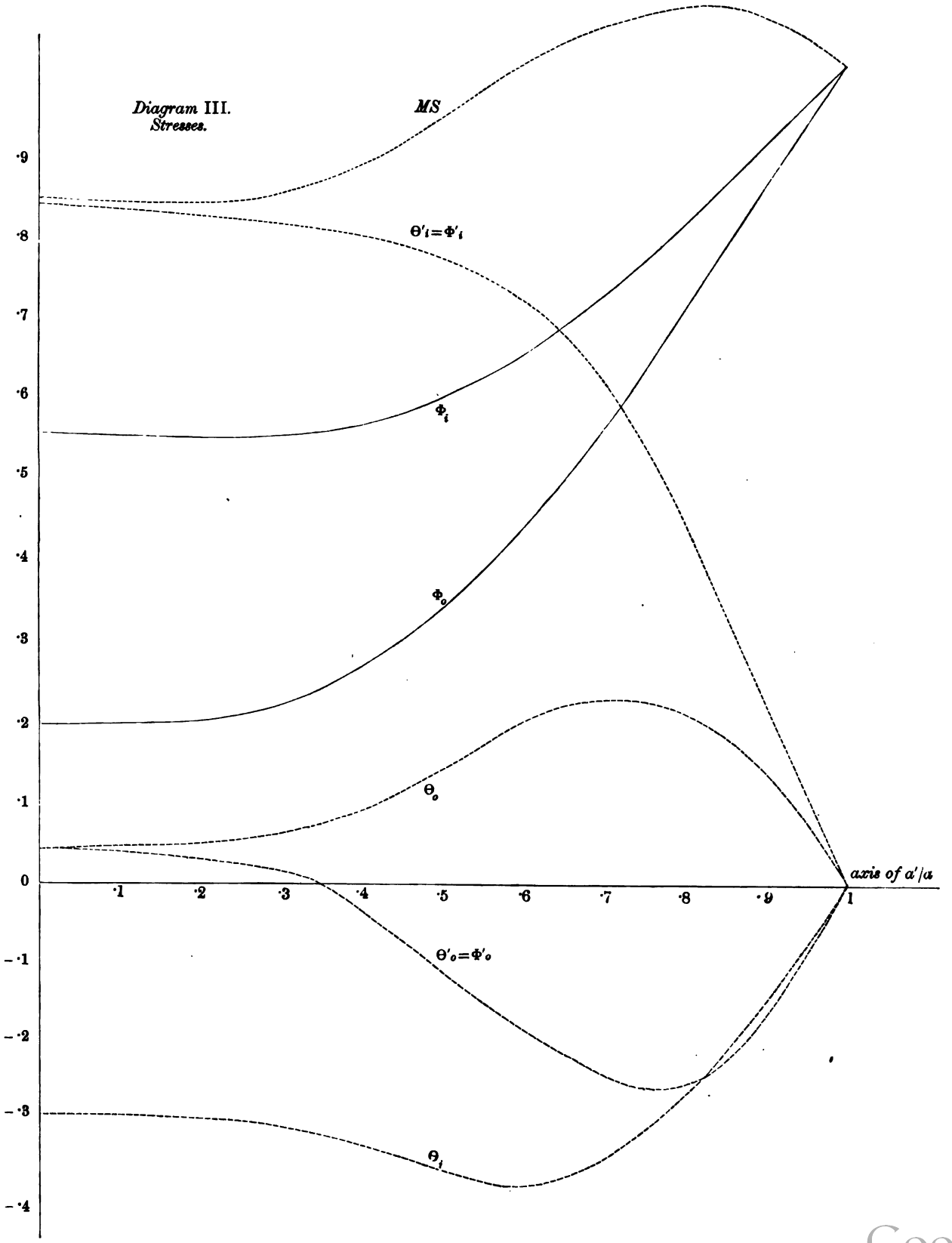
In diagram III. the ordinates represent the values of the stresses Θ and Φ. The above statement as to the suffixes and the dashed letters will sufficiently particularize the separate curves. There is also the curve $M.S$ whose ordinates represent the greatest values of the maximum stress-difference existing on either surface. As already stated these values occur on the inner surface in the equator. The curve Φ_o would in like manner represent the greatest values of the maximum stress-difference existing on the outer surface alone.

Multiplying the ordinates in diagram III. by $\omega^2\rho a^2$ we get absolute stresses.

Digitized by Google



Digitized by Google



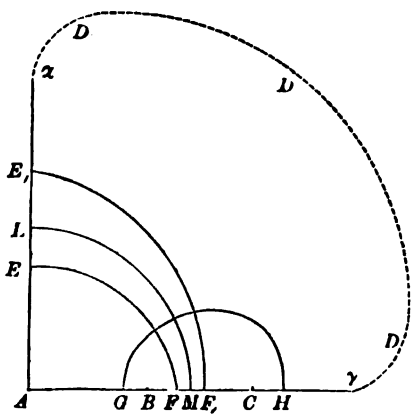
Digitized by Google

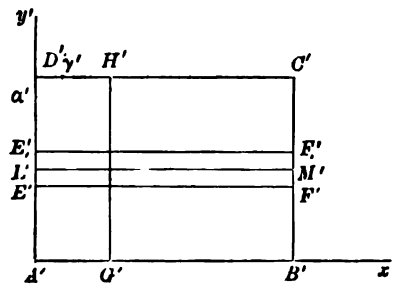
XVII. *On the Binodal Quartic and the Graphical Representation of the Elliptic Functions.* By Prof. Cayley.

[Read *May* 6, 1889.]

I APPROACH the subject from the question of the graphical representation of the elliptic functions: assuming as usual that the modulus is real, positive, and less than unity, and to fix the ideas considering the function sn (but the like considerations are applicable to the functions cn and dn) then the equation $x+iy=\mathrm{sn}(x'+iy')$ establishes a (1, 1) correspondence between the xy infinite quarter plane, and the $x'y'$ rectangle (sides K and K'): viz. to any given point $x+iy$, x and y each positive, there corresponds a single point $x'+iy'$, x', y' each positive and less than K, K' respectively: and conversely to any such point $x'+iy'$, there corresponds a single point $x+iy$, x and y each positive.

I draw in the $x'y'$-figure the rectangle $A'B'C'D'$ (sides K and K'), and in the xy-figure, I take on the axis of x, the points B, C where $AB=1$, $AC=\frac{1}{k}$: and the





point D at infinity. We have thus in the $x'y'$-figure the closed curve or contour $A'B'C'D'A'$: and corresponding hereto we have in the xy-figure the closed curve or contour $ABCDA$, viz. here D is the point at infinity considered as a line always at

Digitized by Google

infinity, extending from the point at infinity on the positive part of the axis of x, to the point at infinity on the positive part of the axis of y, the contour being thus AB, BC, CD (D at infinity on the axis of x); and then D (at infinity on the axis of y) A. And thus to a point P' describing successively the lines $A'B'$, $B'C'$, $C'D'$, $D'A'$ there corresponds a point P describing successively the lines AB, BC, CD, DA: to P' at D' there corresponds P at D, viz. this is any point at infinity from D on the axis of x to D on the axis of y. There is no real breach of continuity: in further illustration suppose that P' instead of actually coming to D' just cuts off the corner, viz. that it passes from a point γ' on $C'D'$ to a point α' on $D'A'$ (γ', α' each of them very near to D'): then P passes from a point γ very near D on the axis of x (that is at a great distance from A) to a point α very near D on the axis of y (that is at a great distance from A): and to the indefinitely small arc $\gamma'\alpha'$ described by P' there corresponds the indefinitely large arc $\gamma\alpha$ described by P.

We thus see that if P' describe any arc $E'F'$ passing from a point E' of $A'D'$ to a point F' of $B'C'$; then P will describe an arc EF passing from a point E of AD to a point F of BC: and similarly if P' describe any arc $G'H$ passing from a point G' of $A'B'$ to a point H' of $C'D'$; then P will describe an arc GH passing from a point G of AB to a point H of CD.

Supposing $E'F'$ is a straight line parallel to $A'x'$, that is cutting $A'D'$ and $B'C$ each at right angles, then EF will be an arc cutting AD and BC each at right angles: and so if $G'H'$ is a straight line parallel to $A'y'$, that is cutting $A'B'$ and $C'D'$ each at right angles, then GH will be an arc cutting AB and CD each at right angles: and moreover since $E'F'$ and $G'H'$ cut each other at right angles, then also EF and GH cut each other at right angles.

Supposing as above that $E'F'$ and $G'H'$ are straight lines, we have EF and GH each of them the arc of a special bicircular quartic: the theory was in fact established in a very elegant manner in a memoir by Siebeck, "Ueber eine Gattung von Curven vierten Grades, welche mit den elliptischen Functionen zusammenhangen," Crelle, t. 57 (1860), pp. 359—370, and t. 59 (1861), pp. 173—184.

In particular if P' describe the line $L'M'$ lying halfway between $A'B'$ and $D'C'$ (that is if $A'L' = \frac{1}{2}K'$) then P will describe the circular quadrant LM, radius $\frac{1}{\sqrt{k}}$, viz. in this case the bicircular quartic degenerates into a circle twice repeated: and so if P' describe successively the lines $E'F'$ and $E_1'F_1'$ equidistant from $L'M'$ ($AE' = \frac{1}{2}K' - \eta$, $AE_1' = \frac{1}{2}K' + \eta$) then P will describe the arcs EF and E_1F_1 which are the images of each other in regard to the centre A and circular quadrant LM, and which together constitute the quadrant of one and the same bicircular quartic.

A bicircular quartic is of course a binodal quartic with the circular points at infinity for the two nodes: there is no real gain of generality in considering the binodal quartic

Digitized by Google

rather than the bicircular quartic, but I have preferred to do so, and I have accordingly introduced the term Binodal Quartic into the title of the present Memoir. I present in a compendious form the properties of the general curve, and I show how the curve is to be particularised so as to obtain from it the special bicircular quartics which present themselves as above in the graphical representation of the elliptic functions.

A binodal quartic has the Plückerian numbers

$$\begin{array}{cccccc} m & n & \delta & \kappa & \tau & \iota \\ =4 & 8 & 2 & 0 & 8 & 12. \end{array}$$

The number of tangents to the curve which can be drawn from either of the nodes is $n-4, =4$; and the pencil of tangents from the one node is homographic with the pencil of tangents from the other node. Call the nodes I and J: and let the tangents from I be called (a, b, c, d) and those from J be called (a', b', c', d'), then if the tangents which correspond to (a', b', c', d') respectively are (a, b, c, d), they may also be taken to be (b, a, d, c), (c, d, a, b) or (d, c, b, a): and considering the intersections of corresponding tangents, we have thus four tetrads of points, say the f-points, such that the points of each tetrad lie in a conic through the two nodes: and we have consequently four conics each passing through the two nodes, say these are the f-conics.

Starting as above with the correspondence (a, b, c, d), (a', b', c', d'), if the intersections of a and a', b and b', c and c', d and d' are called A, B, C, D respectively, then we have A, B, C, D for a tetrad of f-points, lying on the f-conic $(ABCD)$.

Writing AB for the two points, the intersections of IA, JB and of IB, JA respectively, and so in other cases, then the remaining three tetrads of f-points are

$$\begin{array}{llll} AB,\ CD & \text{lying on the } f\text{-conic} & (AB,\ CD), \\ AC,\ BD & \quad\text{,,} \qquad\quad\text{,,} & (AC,\ BD), \\ AD,\ BC & \quad\text{,,} \qquad\quad\text{,,} & (AD,\ BC). \end{array}$$

The two points AB may be spoken of as the antipoints of A, B: and so in other cases.

Any two of the f-conics have in common the nodes I, J, and they therefore besides intersect in two points: at each of these the tangents to the two conics, and the lines to I, J respectively form a harmonic pencil.

Consider the two tangents at I and the two tangents at J: we have a conic touching these four lines and passing through the tetrad of f-points, or what is the same thing, intersecting the f-conic in the four f-points: say this is a c-conic. There are thus four c-conics corresponding to the four f-conics respectively.

We may consider a variable conic, passing through the points I and J; and such that the tangents thereto at these points respectively meet on a point of a c-conic:

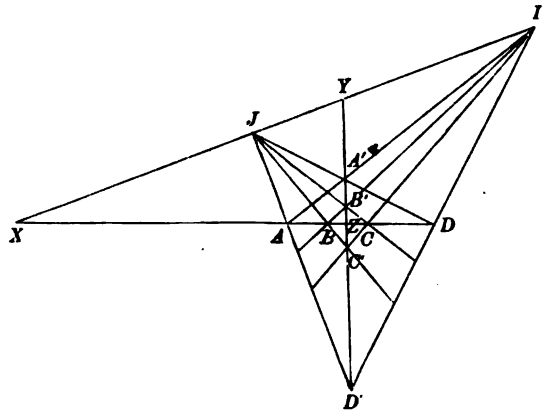
Digitized by Google

the variable conic and the corresponding f-conic each pass through the points I, J and they besides meet in two points: and the variable conic may be such that at each of these the tangents to the variable conic and the f-conic form with the lines drawn to the points I and J a harmonic pencil. The variable conic, as thus defined contains a single variable parameter; and it has for its envelope the binodal quartic: the binodal quartic is thus in four different ways the envelope of a variable conic. [This is of course Casey's Theorem for the four-fold generation of the bicircular quartic as the envelope of a variable circle.]

One of the f-conics say $(ABCD)$ may break up into the line IJ and a line, say the axis $(ABCD)$: we have then the four f-points A, B, C, D on this line. The other f-conics say (AB, CD), (AC, BD), (AD, BC) are as before proper conics: any one of these meets the line $(ABCD)$ in two points: and at each of these the line and the tangent to the conic form with the lines drawn to the points I and J a harmonic pencil. The binodal quartic is in this case said to have an axis.

But a second f-conic, say (AD, BC) may break up into the line IJ and a line, say the axis (AD, BC): we have then the four f-points AD, BC on this line. Writing moreover A', D' for the two points AD, or say $(AI, DJ) = A'$, $(AJ, DI) = D'$; and similarly B', C' for the two points BC, or say $(BI, CJ) = B'$, $(BJ, CI) = C'$: then the four f-points are A', B', C', D', and the axis (AD, BC) may be called $(A'B'C'D')$.

The relation of the f-points A, B, C, D and A', B', C', D' and of the two axes is as shown in the figure: taking X for the intersection of IJ with $(ABCD)$ and Y for



that of IJ with $(A'B'C'D')$, also Z for the intersection of the two axes, then X, Z are the sibiconjugate points of the involution AD, BC and Y, Z the sibiconjugate points of the involution $A'D'$, $B'C'$. The two axes intersect in Z, and form with the lines ZI, ZJ a pencil in involution.

The two remaining f-conics (AB, CD) and (AC, BD), or as they might also be called $(A'B', C'D')$ and $(A'C', B'D')$ are as before proper conics: they touch each other

Digitized by Google

at the points I, J, and have for their common tangents at these points the lines ZI, ZJ respectively. Each conic meets each axis in two points; and at each of these points the axis and the tangent to the conic form with the lines to I, J a harmonic pencil. ~~The binodal quartic is~~ in this case said to be biaxal. The point Z which is the intersection of the two axes may be called the centre.

In the case where an f-conic breaks up into the line IJ, and a line containing four f-points, say an f-line, the corresponding c-conic coincides with the f-conic, viz. it also breaks up into the line IJ and the f-line: the variable conic is a conic through the points I, J such that the tangents thereto at these points respectively meet on the f-line. Moreover the variable conic must be such that at each of its intersections with the f-conic, that is the f-line, the tangent to the variable conic and the f-line must be harmonics in regard to the lines drawn from the point to the points I, J respectively: but this condition is satisfied *ipso facto* for each of the intersections of the variable conic and the f-line. [This depends on the theorem, that taking on a conic any three points P, I, J, then the tangent at P and the line drawn from P to the pole of IJ are harmonics in regard to the lines PI, PJ.] Thus we have only three conditions for the variable conic, or, as above defined, it would in the case in question (of four f-points in a line) depend upon two variable parameters. There is really another condition—but what this in general is I have not ascertained: and this being so the variable conic in the case in question (of the four f-points in a line) depends upon a single variable parameter, and we have as before the bicircular quartic as the envelope. The foregoing is the axial case; in the biaxial case, the same thing happens in regard to the variable conics belonging to the two axes respectively. Thus in every case we have the fourfold generation of the curve as the envelope of a variable conic: only in the axial case, the variable conics belonging to the axis, and in the biaxal case the variable conics belonging to the two axes respectively, are not by the foregoing definitions completely defined. It will be seen further on how, in the case of the biaxal bicircular quartic, we complete the definition of the variable circles belonging to the two axes respectively.

Taking the points I, J to be the circular points at infinity, we have a bicircular quartic. The f-points are the foci, and the f-conics are circles, viz. we have 16 foci situate in fours upon four focal circles. The harmonic relation of two lines to the lines through I, J means of course that the lines cut at right angles: hence the focal circles cut each other at right angles: this must certainly be a known property, but it is not mentioned in Salmon's *Higher Plane Curves*, Ed. 3, Dublin, 1879, and I cannot find it in Darboux or Casey: it is given No. 81 in Lachlan's Memoir "On Systems of Circles and Spheres," *Phil. Trans.*, vol. 177 (1886), and I find it as a question in the *Educational Times*, March 1, 1889, 10034 (Prof. Morley). "Prove that of the four focal circles of a circular cubic or bicircular quartic, any two are orthogonal, and the radii are connected by the relation $\Sigma(\mu^{-2}) = 0$." The theorem is not as well known as it should be.

The c-conics are confocal conics having for their real foci the so-called double-foci of the quartic (more accurately the common foci are the four quadruple foci of the quartic);

Digitized by Google

we have thus four conics corresponding to the four focal circles respectively, each conic intersecting the corresponding circle in the four foci upon this circle. And we have then the quartic as the envelope of a variable circle having its centre upon one of these conics and cutting at right angles the corresponding focal circle: the bicircular quartic is thus generated in four different ways.

Instead of one of the focal circles, we may have a line or axis, and the quartic is then said to be axial: the foci on the axis may be any four points; and for a real curve they may be all real, or two real and two imaginary, or all four imaginary. The remaining focal circles are real or imaginary circles, cut by the axis at right angles, that is having their centres on the axis, and cutting each other at right angles.

But instead of another of the focal circles we may have a line or axis, and the quartic is then said to be biaxal: the two axes cut at right angles at a point which may be called the centre of the curve. The foci on each axis form pairs of points situate symmetrically in regard to the centre. If on one of the axes the foci are real then on the other axis they form two imaginary conjugate pairs; and conversely: but if on one of the axes the foci are two of them real and the other two conjugate imaginaries, then this is so for the other axis also. There are thus only the two cases: 1°, foci on the one axis real, and on the other conjugate imaginaries; 2°, foci on each axis two of them real and the other two conjugate imaginaries: there is however a limiting case where on each axis two foci are united at the centre, the other two foci being real on the one axis and conjugate imaginaries on the other. The remaining two focal circles are real or imaginary circles, cutting each axis at right angles, that is having their centres at the centre; and cutting each other at right angles, that is having the sum of the squares of their radii $=0$.

The biaxal form of bicircular quartic is in fact that which presents itself in the theory of the representation of the elliptic functions.

I consider for a moment the case of a variable circle having its centre upon a given line, and cutting at right angles a given circle. The variable circles pass all of them through two fixed points, the antipoints of the intersections of the given line and circle, and which are thus real or imaginary according as the intersections of the given line and circle are imaginary or real. Hence considering any one variable circle and the consecutive variable circle these intersect in two real points, when the given line does not meet the given circle (meets it in two imaginary points); but when the given line meets the given circle in two real points, then the two variable circles intersect in two imaginary points: if however the given line touches the given circle, then the two variable circles touch each other. Taking now the curve of centres to be any given curve whatever, and considering one of the variable circles, and the consecutive variable circle, it at once appears that if the tangent to the curve of centres at the centre of the variable circle does not meet the given circle, then the two variable circles intersect in two real points (which, if the tangent touch the given circle, unite in a single real point): but if the

Digitized by Google

tangent to the curve of centres meets the given circle, then the two variable circles do not intersect. It hence appears that the real portions of the envelope arise exclusively from those portions of the curve of centres which are such that at any point thereof the tangent to the curve of centres does not meet the given circle. In particular if the given circle be real, and the curve of centres is a real ellipse inclosing the given circle, then the real portion of the envelope arises from the whole ellipse: but if the curve of centres be a real ellipse cutting the given circle in four real points, then drawing the four common tangents of the ellipse and circle, it is at once seen that there are on the ellipse two detached portions, such that at any point of either portion the tangent to the ellipse does not meet the circle: and the real portions of the envelope arise exclusively from these portions of the ellipse.

In the case just referred to there are on the ellipse four portions each lying outside the circle and terminating in the four intersections respectively of the ellipse and circle, such that at a point of any one of these portions the tangent to the ellipse meets the circle in two real points. Starting from the extremity of one of these portions of the ellipse and proceeding to the other extremity on the circle, the corresponding variable circles do not intersect each other, but each of them is a circle lying wholly inside that which immediately precedes it; and the variable circle becomes ultimately a point, viz. this point is a focus of the curve: this agrees with the foregoing statement that the f-conic intersects the circle in the four foci upon this circle. For the two portions of the ellipse which lie inside the circle, the variable circle is of course always imaginary. The like considerations apply to the case where the locus of the centre of the variable circle is a hyperbola or parabola. The foregoing remarks illustrate the actual generation of a bicircular quartic as the envelope of the variable circle.

Starting now from the equation

$$x + iy = \operatorname{sn}(x' + iy')$$

we have
$$x = \frac{\operatorname{sn} x' \operatorname{cn} iy' \operatorname{dn} iy'}{1 - k^2 \operatorname{sn}^2 x' \operatorname{sn}^2 iy'}, \qquad iy = \frac{\operatorname{sn} iy' \operatorname{cn} x' \operatorname{dn} x}{1 - k^2 \operatorname{sn}^2 x' \operatorname{sn}^2 iy'},$$

or putting
$$\operatorname{sn} x' = p, \qquad \operatorname{sn} iy' = iq;$$

these equations are
$$x = \frac{p\sqrt{1+q^2 . 1+k^2q^2}}{1+k^2p^2q^2}, \qquad y = \frac{q\sqrt{1-p^2 . 1-k^2p^2}}{1+k^2p^2q^2};$$

whence also
$$x^2 + y^2 = \frac{p^2+q^2}{1+k^2p^2q^2} = r^2, \quad (\text{if } x^2+y^2 \text{ be put } = r^2).$$

These equations, considering therein q as a given constant, and p as a variable parameter, determine the curve EF: and considering p as a given constant, and q as a variable parameter, they determine the curve GH. But the eliminations are easily effected; we have

$$p^2(1 - k^2q^2r^2) = r^2 - q^2, \qquad q^2(1 - k^2p^2r^2) = r^2 - p^2.$$

Digitized by Google

Hence for EF:

$$p^2 = \frac{r^2 - q^2}{1 - k^2r^2q^2}, \qquad 1 - p^2 = \frac{1 + q^2 - (1 + k^2q^2)\, r^2}{1 - k^2r^2q^2},$$

$$1 - k^2p^2 = \frac{1 + k^2q^2 - k^2(1 + q^2)\, r^2}{1 - k^2r^2q^2}, \quad 1 + k^2p^2q^2 = \frac{1 - k^2q^4}{1 - k^2r^2q^2},$$

and consequently $$x = \frac{\sqrt{1 + q^2 \,.\, 1 + k^2q^2}}{1 - k^2q^4}\sqrt{r^2 - q^2 \,.\, 1 - k^2r^2q^2},$$

$$y = \frac{q\sqrt{1 + q^2 - (1 + k^2q^2)\, r^2}\,\sqrt{1 + k^2q^2 - k^2(1 + q^2)\, r^2}}{1 - k^2q^4},$$

giving x and y each of them in terms of r. And from the first of these we at once derive

$$(x^2 + y^2)^2 - 2Ax^2 - 2By^2 + \frac{1}{k^2} = 0,$$

where $$2A = \frac{1 + q^2}{1 + k^2q^2} + \frac{1}{k^2}\,\frac{1 + k^2q^2}{1 + q^2}, \qquad 2B = q^2 + \frac{1}{k^2q^2}.$$

Similarly for GH:

$$q^2 = \frac{r^2 - p^2}{1 - k^2r^2p^2}, \qquad 1 + q^2 = \frac{(1 - p^2) + r^2(1 - k^2p^2)}{1 - k^2r^2p^2},$$

$$1 + k^2q^2 = \frac{1 - k^2p^2 + k^2(1 - p^2)\, r^2}{1 - k^2r^2p^2}, \quad 1 + k^2p^2q^2 = \frac{1 - k^2p^4}{1 - k^2r^2p^2};$$

and consequently $$x = \frac{p\sqrt{(1 - p^2) + (1 - k^2p^2)\, r^2}\,\sqrt{1 - k^2p^2 + k^2(1 - p^2)\, r^2}}{1 - k^2p^4},$$

$$y = \frac{\sqrt{1 - p^2 \,.\, 1 - k^2p^2}}{1 - k^2p^4}\sqrt{r^2 - p^2 \,.\, 1 - k^2r^2p^2},$$

giving x, y each of them in terms of r. And from the second of them we at once derive

$$(x^2 + y^2)^2 - 2Ax^2 - 2By^2 + \frac{1}{k^2} = 0,$$

where $$2A = p^2 + \frac{1}{k^2p^2}, \qquad 2B = -\frac{1 - p^2}{1 - k^2p^2} - \frac{1}{k^2}\,\frac{1 - k^2p^2}{1 - p^2}.$$

Consider in particular the case where the line EF is the midway line LM: here $y' = \frac{1}{2}K'$, and thence $iq = \operatorname{sn} iy' = \operatorname{sn} \frac{1}{2}iK' = \frac{i}{\sqrt{k}}$, that is $q = \frac{1}{\sqrt{k}}$: and we thence obtain

Digitized by Google

$A = B = \frac{1}{k}$; the equation of the bicircular quartic is

$$(x^2+y^2)^2 - \frac{2}{k}(x^2+y^2) + \frac{1}{k^2} = 0,$$

viz. this is the circle $x^2+y^2-\frac{1}{k}=0$ twice repeated. As a direct verification observe that we have here

$$x+iy = \operatorname{sn}(x'+\tfrac{1}{2}iK') = \frac{\frac{1+k}{\sqrt{k}}.\operatorname{sn}x' + \frac{i}{\sqrt{k}}\operatorname{cn}x'\operatorname{dn}x'}{1+k^2.\frac{1}{k}\operatorname{sn}^2 x'} = \frac{(1+k)p + i\sqrt{1-p^2.1-k^2p^2}}{\sqrt{k}(1+kp^2)},$$

and hence $$x^2+y^2 = \frac{(1+k)^2p^2+1-(1+k^2)p^2+k^2p^4}{k(1+2kp^2+k^2p^4)} = \frac{1}{k}, \text{ as it should be.}$$

Reverting to the equation $$x+iy = \operatorname{sn}(x'+iy'),$$

I write successively $$y' = \tfrac{1}{2}K' - z', \quad \operatorname{sn} iy' = iq_1 = \operatorname{sn} i(\tfrac{1}{2}K' - z'),$$

and $$y' = \tfrac{1}{2}K' + z', \quad \operatorname{sn} iy' = iq_2 = \operatorname{sn} i(\tfrac{1}{2}K' + z');$$

we then have $$iq_1 . iq_2 = \operatorname{sn} i(\tfrac{1}{2}K' - z')\operatorname{sn} i(\tfrac{1}{2}K' + z'), \; = -\frac{1}{k},$$

that is $q_1q_2 = \frac{1}{k}$; hence for q writing q_1 or q_2, we have in each case the same values of A and B; that is we have the same bicircular quartic corresponding to the lines $E'F'$ and $E_1'F_1'$, equidistant from the line $L'M'$: but to one of these lines there corresponds the quadrant lying inside, to the other that lying outside, the circular quadrant

$$x^2+y^2-\frac{1}{k}=0.$$

The curve $$(x^2+y^2) - 2Ax^2 - 2By^2 + \frac{1}{k^2} = 0$$

is in four different ways the envelope of a variable circle: viz. the circle may have its centre on a conic $\alpha x^2+\beta y^2-1=0$, and cut at right angles one of the circles

$$x^2+y^2 \pm \frac{1}{k} = 0;$$

or it may have its centre on the axis of x, or on the axis of y. The circle, centre on either axis, cuts this axis at right angles, but this condition being *ipso facto* satisfied we do not thereby determine the radius of the circle having for its centre a given point on the axis: the expression for the radius must be sought for independently.

Write for shortness $$\lambda^2 = \frac{B-\frac{1}{k^2}}{A-B},$$

and consider the circle $$x^2+y^2-2\alpha x = \lambda\sqrt{A-B-2\alpha^2}+B,$$

Digitized by Google

where α is a variable parameter. Differentiating, we have

$$x = \frac{\lambda\alpha}{\sqrt{A-B-2\alpha^2}}, \quad \text{giving} \quad \alpha = \frac{x\sqrt{A-B}}{\sqrt{2x^2+\lambda^2}},$$

and the equation of the circle then gives

$$x^2+y^2-B = \frac{\alpha}{x}(2x^2+\lambda^2), \quad = \sqrt{A-B}\,\sqrt{2x^2+\lambda^2};$$

that is
$$(x^2+y^2)^2-2Ax^2-2By^2 = B^2+\lambda^2(A-B), \quad = -\frac{1}{k^2};$$

or we have
$$(x^2+y^2)^2-2Ax^2-2By^2+\frac{1}{k^2}=0$$

as the envelope of the variable circle

$$x^2+y^2-2ax = \frac{\sqrt{B-\frac{1}{k^2}}}{\sqrt{A-B}}\sqrt{A-B-2\alpha^2}+B,$$

having its centre on the axis of x.

And similarly the same quartic curve is the envelope of the variable circle

$$x^2+y^2-2\beta y = \frac{\sqrt{A-\frac{1}{k^2}}}{\sqrt{B-A}}\sqrt{B-A-2\beta^2}+A,$$

having its centre on the axis of y.

If we have in like manner the equation $x+iy = \operatorname{cn}(x'+iy')$, then writing as before

$$\operatorname{sn} x' = p, \quad \operatorname{sn} iy' = iq,$$

we find
$$x^2 = \frac{(1-p^2)(1+q^2)}{(1+k^2p^2q^2)^2}, \quad y^2 = \frac{p^2q^2(1-k^2p^2)(1+k^2q^2)}{(1+k^2p^2q^2)^2},$$

and hence
$$r^2 = \frac{1-p^2+q^2-k^2p^2q^2}{1+k^2p^2q^2} \quad (\text{if } r^2 = x^2+y^2).$$

For the curve EF corresponding to a line $E'F'$ parallel to the axis of x', we have from these equations to eliminate p; the expression for r^2 gives

$$p^2 = \frac{1+q^2-r^2}{1+k^2q^2+k^2q^2r^2}, \qquad 1-p^2 = \frac{-(1-k^2)q^2+r^2(1+k^2q^2)}{1+k^2q^2+k^2q^2r^2},$$

$$1-k^2p^2 = \frac{(1-k^2)+k^2r^2(1+q^2)}{1+k^2q^2+k^2q^2r^2}, \qquad 1+k^2p^2q^2 = \frac{1+2k^2q^2+k^2q^4}{1+k^2q^2+k^2q^2r^2},$$

and thence

$$x = \frac{\sqrt{1+q^2}\,\sqrt{-(1-k^2)q^2+r^2(1+k^2q^2)}\,\sqrt{1+k^2q^2+k^2q^2r^2}}{1+2k^2q^2+k^2q^4},$$

$$y = \frac{q\sqrt{1+q^2-r^2}\,\sqrt{1+k^2q^2}\,\sqrt{1-k^2+k^2r^2(1+q^2)}}{1+2k^2q^2+k^2q^4},$$

Digitized by Google

from which we deduce $$(x^2+y^2)^2-2Ax^2-2By^2-\frac{k'^2}{k^2}=0,$$

where $$2A=\frac{-1+2k^2+2k^2q^2+k^2q^4}{k^2(1+q^2)},\quad 2B=\frac{-1-2k^2q^2+k^2(1-2k^2)q^4}{k^2q^2(1+k^2q^2)},$$

which is a bicircular quartic of the foregoing form.

Similarly for the curve GH corresponding to a line $G'H'$ parallel to the axis of y', we have to eliminate q: the expression for r^2 gives

$$q^2=\frac{-1+p^2+r^2}{1-k^2p^2-k^2p^2r^2},\qquad 1+q^2=\frac{(1-k^2)p^2+r^2(1-k^2p^2)}{1-k^2p^2-k^2p^2r^2},$$

$$1+k^2q^2=\frac{(1-k^2)+k^2r^2(1-p^2)}{1-k^2p^2-k^2p^2r^2},\qquad 1+k^2p^2q^2=\frac{1-2k^2p^2+k^2p^4}{1-k^2p^2-k^2p^2r^2}.$$

Hence

$$x=\frac{\sqrt{1-p^2}\,\sqrt{(1-k^2)p^2+r^2(1-k^2p^2)}\,\sqrt{1-k^2p^2-k^2p^2r^2}}{1-2k^2p^2+k^2p^4},$$

$$y=\frac{p\sqrt{-1+p^2+r^2}\,\sqrt{1-k^2p^2}\,\sqrt{1-k^2+k^2r^2(1-p^2)}}{1-2k^2p^2+k^2p^4},$$

from which we deduce $$(x^2+y^2)^2-2Ax^2-2By^2-\frac{k'^2}{k^2}=0,$$

where $$2A=\frac{-1+2k^2-2k^2p^2+k^2p^4}{k^2(1-p^2)},\quad 2B=\frac{1-2k^2p^2-k^2(1-2k^2)p^4}{k^2p^2(1-k^2p^2)},$$

which is again a bicircular quartic of the foregoing form. And we have the like results for the equation $$x+iy=\mathrm{dn}\,(x'+iy');$$

so that, for the sn, cn, and dn, the curve EF or GH is in each case a biaxal bicircular quartic of the form $(x^2+y^2)^2-2Ax^2-2By^2+C=0$.

Digitized by Google

INDEX.

Atomic hypothesis, 71

CAYLEY, Prof. A. On the Binodal Quartic and the Graphical Representation of the Elliptic Functions, 484—494

CHREE, C. The equations of an Isotropic Elastic Solid in polar and cylindrical coordinates, their solution and application, 250—369

Bodily and surface forces, relation between, when n/m negligible: in sphere, 265; in cylinder, 344

Earth, tendency to rupture in: due to its gravitation, 282, 286; due to its rotation, 294

Internal equations: in polar coordinates, 255; in cylindrical coordinates, 324, 325

Maximum stress-difference, 273, also under "Tendency to rupture."

Principal stresses, 274, 327

Strains, expressions for, in terms of displacements: in polar coordinates, 252; in cylindrical coordinates, 322, 323

Strains, expressions for, in terms of arbitrary constants: for solid sphere in general case of equilibrium, 260, 262, 263; for spherical shell in general case of equilibrium, 268; for vibrating solid sphere, general case, 310, 313, radial vibrations, 317; for vibrating spherical shell, general case, 318, 319, radial vibrations, 320; for cylinder in equilibrium, in ascending powers of r and z, 330; for solid cylinder or cylindrical shell in terms of r and θ, 342, 346; for vibrating infinite solid cylinder, general case, 353, radial and tangential vibrations, 355; for vibrating infinite cylindrical shell, radial and tangential vibrations, 356; for finite solid cylinder, general case, 362, 363, 364, 367, case of symmetry round axis, 364, 368; for solid cylinder infinite in one direction, 368

Strains, expressions for, explicitly found in terms of given internal and surface forces, or given surface displacements: for solid sphere, under normal surface forces expressed in terms of spherical harmonics, 264, 265, under uniform normal pressure or tension, 272, under tangential surface forces derivable from a potential expressed in spherical harmonics, 273, rotating with uniform angular velocity, 287, with normal surface displacements expressed in spherical harmonics, 306, with surface displacements derived from a potential expressed in spherical harmonics, 307; for gravitating nearly spherical mass, 280, 281; for spherical shell rotating with uniform angular velocity, in indefinitely thin shell, 298, in shell of small but finite thickness, 301; for finite solid right circular cylinder, under uniform normal pressures or tractions, 331, 332, rotating with uniform angular velocity under various surface conditions, 335, 336, under torsional forces on an end varying as cube of distance from axis, 366; for infinite solid right circular cylinder, rotating with uniform angular velocity, 338, under bodily and normal and tangential surface forces expressed in terms of θ, 344, 346, rotating with varying angular velocity, 359; for infinite right circular cylindrical shell rotating uniformly, 337

Stresses, expressions for: in polar coordinates, 257; in cylindrical coordinates, 326

Stress quadric, in polar coordinates, 274; in cylindrical coordinates, 327

Surface equations: in polar coordinates, 255; in cylindrical coordinates, 324

Tendency to rupture: theories propounded, 273; formulæ for, in polar coordinates, 275, 276; in sphere or spherical shell under uniform normal forces, 276, 277; in gravitating nearly spherical mass, 281, et seq.; in solid rotating sphere, 292, 294; in rotating spherical shell, when indefinitely thin, 300, when of small but finite thickness, 304; formulæ for, in cylindrical coordinates, 327, 328; in infinite right

Digitized by Google

circular solid cylinder or cylindrical shell rotating with uniform angular velocity, 337—339; in infinite right circular solid cylinder under purely normal, or purely tangential, surface forces, 349, 350; in infinite right circular solid cylinder rotating with varying angular velocity, 360

Torsion of finite right circular cylinder by any arbitrary distribution of forces symmetrical about the axis over an end, 366

Vibrations, frequency of free: in solid sphere, general case, 316, 317, transversal vibrations, 316, radial vibrations, 318; in spherical shell of finite thickness or indefinitely thin, transversal vibrations, 319, 320, radial vibrations, 321; in infinite solid cylinder, general case, 354, radial vibrations, 355, tangential vibrations, 356; in infinite cylindrical shell of finite thickness or indefinitely thin, radial vibrations, 356, tangential vibrations, 357

CHREE, C. On the stresses in rotating spherical shells, 467—483

Axes, principal, 474

Earth, difference between equatorial and polar diameters of, 471

Ellipticity of rotating spherical shell, 469, 470

Rupture, tendency to, 467, 477, 478, 480

Strain, greatest, 467, 474, 480

Strains: equatorial and polar, 481, 482; in rotating spherical shell, 468, 469; principal, 468, 472, 475, 476

Stress-difference maximum, 474, 479, 480

Stresses, equatorial and polar, 479, 481, 483; principal, 468, 472, 479

Crystal, uniaxal, 63

Crystallization, 370, 394

Displacement of electromagnetic medium, 170

EDGEWORTH, F. Y. Observations and Statistics, 138—189. *Vide* contents p. 138

Elastic medium, 43, 188, 250, 467

Electricity, vibratory theory of, 188

Elliptic Functions, the Graphical Representation of, 484

Equations of Hydrodynamics, generalized, 1

Exponential function, table of, 237

FORSYTH, A. R. On systems of Quaternariants that are algebraically complete, 409—465

General summary, 409; differential equations of concomitants, 410; number of independent equations reduced to six, 414; symbolical representation and polar variables, 416; isobarism, 421; equations which determine leading coefficients, 422; number of algebraically independent concomitants associated with a quantic, and method of determination, 428. Applications:—a single quadratic, 430; a system of two quadratics, 434; a general inference about special classes of concomitants, 444; a bipartite lineo-linear quantic, 445; complexes in line variables, 453; a linear complex, 454; a congruence of two linear complexes, 454; a regulus of three linear complexes, 456; concomitants of systems of complexes symbolically expressed, 458; a complex of the second grade, 460; invariants of complex of second grade, 465

HILL, M. J. M. On some general equations which include the Equations of Hydrodynamics, 1—29

Treats of the general equations which correspond in n dimensions to the equations of Hydrodynamics in 3 dimensions. Equation of Continuity, Lagrangian form, § 1; component velocities expressed in Weber's form, § 2; analogues of components of molecular notation, n odd, shown to satisfy equations analogous to Helmholtz's, § 3; same components, n even, vanish identically, and this system reduces to a single equation, § 3; construction of the components of a vector, § 4, which include as a case the above components, § 5; analogue of theorem that a vortex line moves with the fluid, § 6; reduction of the Weber velocity components to Clebsch's form, § 7

HOBSON, E. W. On a class of Spherical Harmonics of complex degree with application to Physical Problems, 211—236

Introduction, 211, 212; expressions for the zonal harmonics of degree $-\frac{1}{2}+p\sqrt{-1}$, 213—216; expressions for the tesseral harmonics, 217, 218; addition theorem for the harmonics, 219; application to potential problems for space bounded by coaxial cones and concentric spheres, 220—231; potential problem for figure bounded by two spheres and a spindle joining their common inverse points, 232—234; current function for a spindle moving longitudinally in infinite fluid, 235; two dimensional harmonics, 236

Interference fringes, 170

Isotropic elastic solid, equations of, 250

LARMOR, J. Generalized space-coordinates, 121—137

Coordinates specified by three systems of surfaces. General expression for distance, 122; involves the characteristics of the system, 123. Vector in terms of its components, 124. Reciprocal relation between vector formula and distance formula, 125. Form assumed by Laplace and Poisson's equation for potentials, 126; special case of orthogonal coordinates, 127; simplified forms when the coordinate surfaces

Digitized by Google

form isothermal systems, 127; same expressed in another notation, 128. Equations of an isotropic elastic solid, 128; expression for energy of deformation, 129; alteration of a distance produced by strain, 129; the three principal extensions in generalized coordinates, 130; expression for energy, 131; its variation, 132; resulting equations of equilibrium, 133; special case of orthogonal coordinates, 133—5; other special cases, oblique Cartesian coordinates, 135, columnar, 136, polar, 136—7

LEAHY, A. H. On the pulsations of spheres in an elastic medium, 45—62

Bjerknes' investigation of the mutual action of pulsating spheres in an incompressible fluid, 45; general considerations explaining this action, *ib.*; application of these considerations to the case of pulsations which produce long waves in an elastic medium, 46; other cases of differential action explained by the same considerations, 47; general expressions in polar coordinates for periodic and steady displacements symmetrical about an axis, *ib.*; result, 51; expansion of series and determination of constants, *ib.*; disturbance due to the presence of a small fixed sphere on the axis of symmetry, 53; expression of Legendre's functions of one of the base angles of a triangle in a series of Legendre's functions of the other base angle, *ib.*; transformation of the expression for the disturbance by means of this expansion, 54; displacement produced by simultaneous pulsations of the two spheres, approximate solution in the case of long waves, 57; mutual action of the spheres due to this pulsation, first approximation, 58; second approximation, 59; application of the result to a possible explanation of electrical attractions and repulsions, 61

LEAHY, A. H. On the mutual action of oscillatory twists in an elastic medium, 188—209

Results obtained in the paper, 190; investigation of the displacement if the axis of twist is rectilinear, 191; axis of twist circular, § 4—6; equations of elasticity in toroidal coordinates, 193; approximate solution for long waves, 194; determination of constants so as to satisfy surface conditions on a twist whose axis is circular, 195; case of small ring, 198; solution of the equations of elasticity in the case of long waves to a second approximation, 199; general expressions in cylindrical coordinates for a periodic and steady displacement symmetrical about an axis, § 7; expansion in series, 202, 203; mutual action of two rectilineal twists, § 8; expressions for displacement, 205; expression for force, 207; force on a tube of twist placed in a disturbed elastic medium, *ib.*; application of these results to possible explanation of electromagnetic action, 209; comparison with result already proved for pulsating spheres, *ib.*; modification of result for pulsating spheres so as to give possible explanation of gravitation, 210

LIVEING, Prof. G. D. On Solution and Crystallization, 370—393 and 394—407

Bravais, M., theory of crystallogeny, 405

Cleavage, in crystals, related to surface energy, 376; octahedral, 395; cubic, 402; dodecahedral, *ib*; rhombohedral, 383

Compression determining direction of fracture, 406

Crystal, condensation in surface of a [redacted], 377; pyramidal, 379; rhombohedral, 382, 383; prismatic, 385, 387; oblique, 389; anorthic, 390; physical causes of regular arrangement of molecules in a, 393; cubic, form of molecular volume in a, 395, 401, 402; pyramidal, 401; rhombohedral, *ib.*; cubic, with cubic cleavage, 376, 402; cubic, with dodecahedral cleavage, 402

Curie, P., on the influence of capillarity on growth of crystals, 405

Dana, suggestion that Haüy's parallelopipeds might be replaced by ellipsoids, 407

Dimorphism, due to different aggregations in molecules, 404; relation of, to temperature, 404

Ellipsoids, arrangement of the maximum number in a unit volume, 396; independent of the orientation of the axes, 398; circumscribed dodecahedron, properties of, 400

Energy, surface, comparative, of the faces of different forms of the cubic system, 375, 378; total, a minimum when several forms are combined in a crystal, 376; of different faces of a crystal of the pyramidal system, 380; rhombohedral system, 382; prismatic, 385; oblique, 389; anorthic, 390; equilibrium of, in hemihedral forms of crystals, 391; determines the law of indices in a crystal, 374; the law of symmetry, 376; cleavage related to, *ib.*; a minimum on plane surfaces of crystals, 374

Fracture, surface of, that of compression, 406

Hemihedral crystals, equilibrium of surface energy in, 393; asymmetry of, connected with internal strains, *ib.*

Indices, law of, in a crystal, connected with surface energy, 374

Molecular volume, ellipsoidal, 396; spheroidal, gives pyramidal, rhombohedral, or cubic crystals, 401; spherical, gives cubic crystals, 395

Molecules, arrangement of, in a cubic crystal, 372, 377, 395; in a pyramidal crystal, 379, 401; in a rhombohedral crystal, 382, 401; in a hexagonal, 383, 401; in a prismatic, 385, 387, 401; in an oblique, 389, 401; in an anorthic, 390, 401

Sohncke, on geometrical properties of systems of points regularly disposed in space, 406

Digitized by Google

Solution, saturation of a, determined by surface energy, 371; solid deposited from, when this is attended with immediate increase of entropy, *ib.*; supersaturated, conditions of, *ib.*; crystalloid separating from, will assume such a form that its surface energy is a minimum, 372; partial, of a solid, leaves surface of minimum energy, 393

Solvent, form of surface left by, 393

Spheres, arrangement of the maximum number in a unit volume, 395

Surface, form of, left by a solvent, 393

Surfaces of minimum energy, in a crystalline arrangement of molecules will be planes, 374

Symmetry, law of, in crystallization dependent on surface energy, 376

NEWMAN, F. W. Table of the Exponential Function to twelve places of decimals, 237—249.

PEARSON, Prof. K. On a certain atomic hypothesis, 71—120

Atomic hypothesis, 71 *et seq.*; hydromagnetism, 72; pulsating spheres in a fluid, theory of, 72—78; spectrum analysis, suggested mathematical theory of, 80, 82—90; chemical forces, 90—104; molecular forces, 105—110; gravitational forces, 110—117

Polarized light, intensity of, 63

Pulsation of spheres in elastic medium, 43

Quartic, the Binodal, 484

Rotating spherical shells, strains in, 467

SHAW, W. N. On the measurement of temperature by water-vapour pressure, 30—44

Errors of mercury thermometer, 30; method in use for obtaining comparable readings, 31; correction of thermometers by water-vapour pressure, 33; the repetition of Regnault's experiments, 36; experiments to determine the value of the method proposed, 37—44

Solution and crystallization, 370, 394

Space-coordinates, generalized, 121

Spherical harmonics of complex degree, 211

SPURGE, C. On the curves of constant intensity of polarized light seen in a uniaxal crystal cut at right angles to the optic axes, 63—69

Statistics, and observations, 138

Table of e^x, 237

Temperature, measurement of, by vapour-pressure, 30

Twists, oscillatory, in elastic medium, 188

WILBERFORCE, L. R. On a new method of obtaining interference-fringes, and on their application to determine whether a displacement-current of electricity involves a motion of translation of the electromagnetic medium, 170—187

Connection between electric current and motion of the ether, 171; account of different forms of interference refractometer, 174; new method of obtaining interference fringes, 177; experiment on connection between displacement-current and motion of the ether, 184

CAMBRIDGE: PRINTED BY C. J. CLAY, M.A. AND SONS, AT THE UNIVERSITY PRESS.

Digitized by Google

Digitized by Google

Digitized by Google

Digitized by Google

DO NOT CIRCULATE

~~8 20 AUG 13~~

Digitized by Google

Digitized by Google

Digitized by Google